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FIXED POINTS OF A PAIR OF KANNAN TYPE MAPPINGS ON A CLOSED BALL IN ORDERED PARTIAL METRIC SPACES

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Abstract. Unique common fixed point results for mappings satisfying locally contractive conditions on a closed ball in a complete ordered partial metric space have been established. The notion of dominated mappings of Economics, Finance, Trade and Industry is also been applied to approximate the unique solution to non linear functional equations. Our results improve some well-known, primary and conventional results.

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Keywords: unique common fixed point, Kannan mapping, closed ball, dominated mapping, partial metric spaces

1. Introduction and Preliminaries

Let $T : X \longrightarrow X$ be a mapping. A point $x \in X$ is called a fixed point of T if $x = Tx$. Let x_0 be an arbitrarily chosen point in X. Define a sequence $\{x_n\}$ in X by a simple iterative method given by

$$x_{n+1} = Tx_n, \text{ where } n \in \{0, 1, 2, ...\}.$$

Such a sequence is called a Picard iterative sequence and its convergence plays a very important role in proving an existence of a fixed point of a mapping T. A self mapping T on a metric space X is said to be a Banach contraction mapping if

$$d(Tx, Ty) \leq kd(x, y)$$

holds for all $x, y \in X$ where $0 \leq k < 1$. In 1969, Kannan obtained unique fixed point of a mapping $T : X \longrightarrow X$ satisfying:

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] \tag{1.1}$$

for all $x, y \in X$ where $0 \leq k < \frac{1}{2}$ and X is a complete metric space. The mapping satisfying contractive condition 1.1 is called a Kannan mapping. It is observed that contractive condition 1.1 does not imply continuity of a mapping T on its domain, which differentiates its nature from Banach contraction mapping. It is important to note that Kannan fixed point result has laid down the foundation of modern fixed points theory for contractive type mappings.

Fixed points results of mappings satisfying certain contractive conditions on the entire domain has been at the centre of rigorous research activity, for example (see [6–8, 11, 12]) and it has a wide range of applications in different areas such as nonlinear and adaptive control systems, parameterize estimation problems, fractal image decoding, computing magnetostatic fields in a nonlinear medium, and convergence of recurrent networks, (see [15, 18, 20, 23]).

From the application point of view the situation is not yet completely satisfactory because it frequently happens that a mapping T is a contraction not on the entire space X but merely on a subset Y of X. However, if Y is closed and a Picard iterative sequence $\{x_n\}$ in X converges to some x in X then by imposing a subtle restriction on the choice of x_0, one may force the Picard iterative sequence to stay eventually in Y. In this case, closedness of Y coupled with some suitable contractive condition establish the existence of a fixed point of T. Recently, Arshad et. al. [5] proved a significant result concerning the existence of common fixed point of mappings satisfying a contractive condition on a closed ball in a complete dislocated metric space. Other results on closed ball can be seen in [9, 10, 17, 22].

On the other hand, the notion of a partial metric space was introduced by Matthews in [19]. In partial metric spaces, the distance of a point from itself may not be zero. After the definition of partial metric space, Matthews proved the partial metric version of Banach fixed point theorem. Then, Oltra et al. [21] and Altun et al. [3] gave some generalizations of the result of Matthews (see also [4] , [2], [13] and references mentioned therein). The dominated mapping, which satisfies the condition $fx \preceq x$, occurs very naturally in several practical problems. For example if x denotes the total quantity of food produced over a certain period of time and $f(x)$ gives the quantity of food consumed over the same period in a certain town, then we must have $fx \preceq x$. In this paper, we will exploit this concept for Kannan mappings [16] to generalize, extend and improve some classical fixed point results for two, three and four mappings in the framework of complete ordered partial metric space X. Our results not only extend some primary results to ordered partial metric spaces but also restrict the contractive conditions on a closed ball only. The concept of dominated mapping has been applied to approximate the unique solution of non linear functional equations. We have used weaker contractive condition and weaker restrictions to obtain unique fixed point. Our results do not exist even yet in metric spaces. Consistent with [1, 2, 6, 14] and [19], the following definitions and results will be needed in the sequel.

Definition 1 ([19])**.** Let $p : X \times X \to R^+$ be a function, where X is a nonempty set. Then p is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold.

(P_1) $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$,

(P_2) $p(x,x) \leq p(x,y)$,
(P_3) $p(x,y) = p(y,x)$,
(P_4) $p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$.
The pair (X,p) is then called a partial metric space.

Each partial metric p on X induces a T_0 topology p on X which has as a base the family of open balls $\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x)+\varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Also $\overline{B(x_0,\varepsilon)} = \{y \in X : p(x,y) \leq p(x,x)+\varepsilon\}$ is a closed ball in (X,p).

It is clear that if $p(x,y) = 0$, then from P_1 and P_2, $x = y$. But if $x = y$, $p(x,y)$ may not be 0. A basic example of a partial metric space is the pair (R^+, p) , where $p(x,y) = \max\{x,y\}$ for all $x, y \in R^+$.

Example 1 ([19]). If $X = \{[a,b] : a,b \in R, a \leq b\}$ then $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$ defines a partial metric p on X.

Definition 2 ([19]). Let (X,p) be a partial metric space, then we have the following.
A sequence $\{x_n\}$ in (X,p) converges to a point $x \in X$ if and only if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$.
A sequence $\{x_n\}$ in (X,p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
(X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X, there exist some $z \in X$ such that,

$$\lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,z) = p(z,z).$$

If (X,p) is a partial metric space, then $p_s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$, $x, y \in X$, is a metric on X.

Lemma 1 ([19]). *Let (X,p) be a partial metric space.*
$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p_s).
(X,p) is complete if and only if the metric space (X,p_s) is complete. Furthermore, $\lim_{n\to\infty} p_s(x_n,z) = 0$ if and only if

$$p(z,z) = \lim_{n\to\infty} p(x_n,z) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Definition 3 ([2]). Let X be a nonempty set. Then $(X,\preceq,p)$ is called an ordered partial metric space if:
(i) p is a partial metric on X and (ii) $\preceq$ is a partial order on X.

Definition 4. Let $(X,\preceq)$ be a partial ordered set. Then $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. We define $\nabla = \{(x,y) \in X \times X \mid x$ and y are comparable$\}$.

Definition 5 ([1])**.** Let $(X, \preceq)$ be a partially ordered set. A self mapping f on X is called dominated if $fx \preceq x$ for each x in X.

Example 2 ([1]). Let $X = [0,1]$ be endowed with the usual ordering and $f : X \to X$ be defined by $fx = x^n$ for some $n \in \mathbb{N}$. Since $fx = x^n \leq x$ for all $x \in X$, therefore f is a dominated map.

Definition 6. Let X be a non empty set and $T, f : X \to X$. A point $y \in X$ is called point of coincidence of T and f if there exists a point $x \in X$ such that $y = Tx = fx$. The mappings T, f are said to be weakly compatible if they commute at their coincidence point (i. e. $Tfx = fTx$ whenever $Tx = fx$). We require the following lemmas:

Lemma 2 ([14])**.** *Let X be a non empty set and $f : X \to X$ a function. Then there exists a subset $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one to one.*

Lemma 3 ([6])**.** *Let X be a non empty set and the mappings $S, T, f : X \to X$ have a unique point of coincidence v in X. If (S, f) and (T, f) are weakly compatible, then S, T, f have a unique common fixed point.*

2. Fixed Points of Kannan Mappings

In the following we present common fixed point theorems for the Kannan mappings on a closed ball in an ordered partial metric space.

Theorem 1. *Let $(X, \preceq, p)$ be a complete ordered partial metric space, $x_0, x, y \in X$, $r > 0$ and $S, T : X \to X$ be two dominated mappings. Suppose that there exists $t \in [0, \frac{1}{2})$ such that following conditions hold:*

$$p(Sx, Ty) \leq t[p(x, Sx) + p(y, Ty)], \text{ for all } (x, y) \text{ in } (\overline{B(x_0, r)} \times \overline{B(x_0, r)}) \cap \nabla, \tag{2.1}$$

and

$$p(x_0, Sx_0) \leq (1 - \lambda)[r + p(x_0, x_0)], \tag{2.2}$$

where $\lambda = \dfrac{t}{1-t}$. Then there exists a point x^* such that $p(x^*, x^*) = 0$. Also if, for a nonincreasing sequence $\{x_n\}$ in $\overline{B(x_0, r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$, then $x^* = Sx^* = Tx^*$. Moreover, x^* is unique, if for any two points x, y in $\overline{B(x_0, r)}$ there exists a point $z_0 \in \overline{B(x_0, r)}$ such that $z_0 \preceq x$ and $z_0 \preceq y$ and

$$p(x_0, Sx_0) + p(z, Tz) \leq p(x_0, z) + p(Sx_0, Tz) \tag{2.3}$$

for all $z \in \overline{B(x_0, r)}$ such that $z \preceq Sx_0$.

Proof. Choose a point x_1 in X such that $x_1 = Sx_0$. As $Sx_0 \preceq x_0$ so $(x_1, x_0) \in \nabla$ and let $x_2 = Tx_1$. Now $Tx_1 \preceq x_1$ gives $(x_2, x_1) \in \nabla$. Continuing this process and having chosen x_n in X such that

$$x_{2i+1} = Sx_{2i} \text{ and } x_{2i+2} = Tx_{2i+1}, \text{ where } i = 0, 1, 2, \ldots.$$

we obtain $(x_{n+1}, x_n) \in \nabla$ for all $n \in N$. We will prove that $x_n \in \overline{B(x_0, r)}$ for all $n \in N$ by mathematical induction. By inequality 2.2

$$\begin{aligned} p(x_0, Sx_0) &\leq (1-\lambda)[r + p(x_0, x_0)] \\ &\leq r + p(x_0, x_0). \end{aligned}$$

It follows that $x_1 \in \overline{B(x_0, r)}$. Let $x_2, \cdots, x_j \in \overline{B(x_0, r)}$ for some $j \in N$. If $j = 2i+1$, then $(x_{2i+1}, x_{2i}) \in (\overline{B(x_0, r)} \times \overline{B(x_0, r)}) \cap \nabla$, where $i = 0, 1, 2, \ldots \frac{j-1}{2}$ so using inequality 2.1, we obtain

$$\begin{aligned} p(x_{2i+1}, x_{2i+2}) &= p(Sx_{2i}, Tx_{2i+1}) \\ &\leq t[p(x_{2i}, Sx_{2i}) + p(x_{2i+1}, Tx_{2i+1})] \end{aligned}$$

which implies that

$$\begin{aligned} p(x_{2i+1}, x_{2i+2}) &\leq \lambda p(x_{2i}, x_{2i+1}) \\ &\leq \lambda^2 p(x_{2i-1}, x_{2i}) \leq \ldots \leq \lambda^{2i+1} p(x_0, x_1). \end{aligned} \tag{2.4}$$

If $j = 2i+2$, then as x_1, x_2 ..., $x_j \in \overline{B(x_0, r)}$ and $(x_{2i+2}, x_{2i+1}) \in (\overline{B(x_0, r)} \times \overline{B(x_0, r)}) \cap \nabla$, $(i = 0, 1, 2, \ldots, \frac{j-2}{2})$. We obtain,

$$p(x_{2i+2}, x_{2i+3}) \leq \lambda^{2i+2} p(x_0, x_1). \tag{2.5}$$

Thus from inequality 2.4 and 2.5 , we have

$$p(x_j, x_{j+1}) \leq \lambda^j p(x_0, x_1) \text{ for some } j \in N. \tag{2.6}$$

Now

$$\begin{aligned} p(x_0, x_{j+1}) &\leq p(x_0, x_1) + \ldots + p(x_j, x_{j+1}) - [p(x_1, x_1) + \ldots + p(x_j, x_j)] \\ &\leq p(x_0, x_1) + \ldots + \lambda^j p(x_0, x_1) \qquad \text{(by 2.6)} \\ &= p(x_0, x_1)[1 + \ldots + \lambda^{j-1} + \lambda^j] \\ &\leq (1-\lambda)[r + p(x_0, x_0)] \frac{(1-\lambda^{j+1})}{1-\lambda} \\ &\leq r + p(x_0, x_0) \end{aligned}$$

gives $x_{j+1} \in \overline{B(x_0, r)}$. Hence $x_n \in \overline{B(x_0, r)}$ for all $n \in N$. Also $(x_{n+1}, x_n) \in (\overline{B(x_0, r)} \times \overline{B(x_0, r)}) \cap \nabla$ for all $n \in N$. It implies that

$$p(x_n, x_{n+1}) \leq \lambda^n p(x_0, x_1), \text{ for all } n \in N. \tag{2.7}$$

Also,

$$\begin{aligned} p(x_n, x_n) &\leq p(x_n, x_{n+1}) \\ &\leq \lambda^n p(x_0, x_1) \longrightarrow 0 \text{ as } n \to \infty \end{aligned} \tag{2.8}$$

Moreover,

$$\begin{aligned} p_s(x_{n+1},x_n) &= 2p(x_n,x_{n+1}) - p(x_n,x_n) - p(x_{n+1},x_{n+1}) \\ &\le 2p(x_n,x_{n+1}) \\ &\le 2\lambda^n p(x_0,x_1). \quad \text{(by 2.7)} \end{aligned} \tag{2.9}$$

So we have,

$$\begin{aligned} p_s(x_{n+k},x_n) &\le p_s(x_{n+k},x_{n+k-1}) + \ldots + p_s(x_{n+1},x_n) \\ &\le 2\lambda^{n+k-1} p(x_0,x_1) + \ldots + 2\lambda^n p(x_0,x_1) \quad \text{by 2.9} \\ &= 2\lambda^n p(x_0,x_1)[\lambda^{k-1} + \lambda^{k-2} + \ldots + 1] \\ &\le 2\lambda^n p(x_0,x_1)\frac{(1-\lambda^k)}{1-\lambda} \longrightarrow 0 \text{ as } n \to \infty. \end{aligned}$$

Hence the sequence $\{x_n\}$ is a Cauchy sequence in $(\overline{B(x_0,r)}, p_s)$. By Lemma 1, $\{x_n\}$ is a Cauchy sequence in $(\overline{B(x_0,r)}, p)$. Therefore there exists a point $x^* \in \overline{B(x_0,r)}$ with $\lim\limits_{n\to\infty} x_n = x^*$. Also $\lim\limits_{n\to\infty} p_s(x_n,x^*) = 0$. Now by Lemma 1and inequality 2.8 we have

$$p(x^*,x^*) = \lim_{n\to\infty} p(x_n,x^*) = \lim_{n\to\infty} p(x_n,x_n) = 0. \tag{2.10}$$

Moreover by the given assumption $x_n \to x^*$ as $n \to \infty$ implies that $(x^*,x_n) \in \nabla$. Also, $(x^*,x_{2n+1}) \in (\overline{B(x_0,r)} \times \overline{B(x_0,r)}) \cap \nabla$. Then,

$$\begin{aligned} p(x^*,Sx^*) &\le p(x^*,x_{2n+2}) + p(x_{2n+2},Sx^*) - p(x_{2n+2},x_{2n+2}) \\ &\le p(x^*,x_{2n+2}) + t[p(x_{2n+1},Tx_{2n+1}) + p(x^*,Sx^*)], \\ (1-t)\,p(x^*,Sx^*) &\le p(x^*,x_{2n+2}) + tp(x_{2n+1},Tx_{2n+1}), \\ p(x^*,Sx^*) &\le \frac{1}{1-t} p(x^*,x_{2n+2}) + \lambda p(x_{2n+1},x_{2n+2}). \end{aligned}$$

On taking limit $n \to \infty$ and by using inequality 2.7 and 2.10 we obtain,

$$p(x^*,Sx^*) \le 0,$$

and $x^* = Sx^*$. Similarly, by using

$$p\left(x^*,Tx^*\right) \le p(x^*,x_{2n+1}) + p(x_{2n+1},Tx^*) - p(x_{2n+1},x_{2n+1}),$$

we can show that $x^* = Tx^*$. Hence S and T have a common fixed point in $\overline{B(x_0,r)}$. Let y be another point in $\overline{B(x_0,r)}$ such that $y = Sy = Ty$. If $x^* \preceq y$, then

$$\begin{aligned} p(x^*,y) &= p(Sx^*,Ty) \\ &\le t[p(x^*,Sx^*) + p(y,Ty)] \\ &= t[p(x^*,x^*) + p(y,y)] \\ &\le tp(y,y) \quad \text{(by 2.10)} \end{aligned}$$

$$< p(y,y).$$

Using the fact that $p(y,y) \leq p(x^*,y)$, we have $x^* = y$. Now if $x^* \npreceq y$, then there exists a point $z_0 \in \overline{B(x_0,r)}$ such that $(z_0,x^*) \in \nabla$ and $(z_0,y) \in \nabla$. Choose a point z_1 in X such that $z_1 = Tz_0$. As $Tz_0 \preceq z_0$ so $(z_1,z_0) \in \nabla$. Let $z_2 = Sz_1 \preceq z_1$ gives $(z_2,z_1) \in \nabla$. Continuing this process and having chosen z_n in X such that

$$z_{2i+1} = Tz_{2i},\ z_{2i+2} = Sz_{2i+1} \text{ and } z_{2i+1} = Tz_{2i} \preceq z_{2i} \text{ where } i = 0,1,2,\ldots.$$

It follows that $z_{n+1} \preceq z_n \preceq \ldots \preceq z_0 \preceq x^* \preceq x_n \ldots \preceq x_0$. We will prove that $z_n \in \overline{B(x_0,r)}$ for all $n \in N$ by using mathematical induction. For $n = 1$. Now $(x_0,z_0) \in \overline{B(x_0,r)} \times \overline{B(x_0,r)} \cap \nabla$

$$\begin{aligned} p(Sx_0,Tz_0) &\leq t[p(x_0,x_1) + p(z_0,Tz_0)] \\ &\leq t[p(x_0,z_0) + p(x_1,Tz_0)] \quad \text{(by 2.3)} \\ &\leq \lambda p(x_0,z_0) \text{ and} \end{aligned} \tag{2.11}$$

$$\begin{aligned} p(x_0,z_1) &\leq p(x_0,x_1) + p(x_1,z_1) - p(x_1,x_1) \\ &\leq (1-\lambda)[r + p(x_0,x_0)] + \lambda p(x_0,z_0) \quad \text{(by 2.2 and 2.11)} \\ &\leq (1-\lambda)r + (1-\lambda)p(x_0,x_0) + \lambda[r + p(x_0,x_0)] \quad (\text{as } z_0 \in \overline{B(x_0,r)}) \\ &\leq r + p(x_0,x_0) \end{aligned}$$

implies that $z_1 \in \overline{B(x_0,r)}$. Let $z_2,z_3,\ldots,z_j \in \overline{B(x_0,r)}$ for some $j \in N$. Following similar arguments as we have used to prove inequality 2.6, we have

$$p(z_j,z_{j+1}) \leq \lambda^j p(z_0,z_1) \text{ for some } j \in N. \tag{2.12}$$

Note that if j is odd then, we have

$$\begin{aligned} p(x_{j+1},z_{j+1}) = p(Tx_j,Sz_j) &\leq t[p(x_j,Tx_j) + p(z_j,Sz_j)] \\ &\leq t[\lambda^j p(x_0,x_1) + \lambda^j p(z_0,z_1)] \quad \text{(by 2.6 and 2.12)} \\ &\leq t\lambda^j[p(x_0,z_0) + p(x_1,z_1)] \quad \text{(by 2.3)} \\ &\leq t\lambda^j[p(x_0,z_0) + \lambda p(x_0,z_0)] \\ &= \lambda^{j+1} p(x_0,z_0). \end{aligned} \tag{2.13}$$

Similarly, if j is even then, we obtain

$$p(x_{j+1},z_{j+1}) \leq \lambda^{j+1} p(x_0,z_0). \tag{2.14}$$

Now,

$$\begin{aligned} p(x_0,z_{j+1}) &\leq p(x_0,x_1) + p(x_1,x_2) + \ldots + p(x_{j+1},z_{j+1}) \\ &\leq p(x_0,x_1) + \lambda p(x_0,x_1) + \ldots + \lambda^{j+1} p(x_0,z_0) \quad \text{(by 2.13 and 2.14)} \\ &\leq p(x_0,x_1)[1 + \lambda + \ldots\lambda^j] + \lambda^{j+1}[r + p(x_0,x_0)] \end{aligned}$$

$$\leq (1-\lambda)[r+p(x_0,x_0)]\frac{(1-\lambda^{j+1})}{1-\lambda}+\lambda^{j+1}r+\lambda^{j+1}p(x_0,x_0)$$
$$= r+p(x_0,x_0)$$

gives $z_{j+1} \in \overline{B(x_0,r)}$. Hence $z_n \in \overline{B(x_0,r)}$ for all $n \in N$. Now inequality 2.12 can be written as

$$p(z_n,z_{n+1}) \leq \lambda^n p(z_0,z_1) \text{ for all } n \in N. \tag{2.15}$$

As $(z_0,x^*),(z_0,y) \in (\overline{B(x_0,r)} \times \overline{B(x_0,r)}) \cap \nabla$, so it follows that (z_n,x^*) and (z_n,y) are in $(\overline{B(x_0,r)} \times \overline{B(x_0,r)}) \cap \nabla$ for all $n \in N$. Then, for $i \in N$,

$$\begin{aligned} p(x^*,y) &= p(Tx^*,Ty) \\ &\leq p(Tx^*,Sz_{2i-1})+p(Sz_{2i-1},Ty)-p(Sz_{2i-1},Sz_{2i-1}) \\ &\leq tp(x^*,x^*)+2tp(z_{2i-1},z_{2i})+tp(y,y) \\ &\leq 2t\lambda^{2i-1}d_l(z_0,z_1)+tp(y,y). \quad \text{(by 2.10 and 2.15)} \end{aligned}$$

On taking limit as $i \to \infty$, we obtain,

$$p(x^*,y) \leq tp(y,y) < p(y,y),$$

a contradiction, so $x^* = y$. Hence $x^*$is a unique common fixed point of T and S in $\overline{B(x_0,r)}$. □

Example 3. Let $X = R^+ \cup \{0\}$ be endowed with order $x \preceq y$ if $p(x,x) \leq p(y,y)$ and let $p : X \times X \to R^+ \cup \{0\}$ be the complete ordered partial metric on X defined by $p(x,y) = \max\{x,y\}$.

$$Sx = \begin{cases} \dfrac{x}{17} \text{ if } x \in [0,1] \\ x-\frac{1}{5} \text{ if } x \in (1,\infty) \end{cases}$$

and

$$Tx = \begin{cases} \dfrac{3x}{17} \text{ if } x \in [0,1] \\ x-\frac{1}{6} \text{ if } x \in (1,\infty). \end{cases}$$

Clearly, S and T are dominated mappings. Take, $t = \dfrac{3}{10} \in [0,\frac{1}{2})$, $x_0 = \frac{1}{2}$, $r = \frac{1}{2}$, we have $p(x_0,x_0) = \max\{\frac{1}{2},\frac{1}{2}\} = \frac{1}{2}$, $\lambda = \frac{t}{1-t} = \frac{3}{7}$ and $\overline{B(x_0,r)} = [0,1]$. Also

$$(1-\lambda)[r+p(x_0,x_0)] = (1-\frac{3}{7})[\frac{1}{2}+\frac{1}{2}] = \frac{4}{7}$$

$$p(x_0,Sx_0) = p(\frac{1}{2},S\frac{1}{2}) = p(\frac{1}{2},\frac{1}{17}) = \max\{\frac{1}{2},\frac{1}{17}\} = \frac{1}{2} < \frac{4}{7}$$

$$\text{Also if } x,y \in (1,\infty),\ p(Sx,Ty) = \max\{x-\frac{1}{5},y-\frac{1}{6}\}$$

$$\geq \frac{3}{10}[x+y]$$
$$= \frac{3}{10}[\max\{x, x-\frac{1}{5}\} + \max\{y, y-\frac{1}{6}\}]$$
$$p(Sx,Ty) \geq t[p(x,Sx)+p(y,Ty)].$$

So the contractive condition does not hold on X. Now if $x,y \in \overline{B(x_0,r)}$, then

$$p(Sx,Ty) = \max\{\frac{x}{17}, \frac{3y}{17}\} = \frac{1}{17}\max\{x,3y\}$$
$$\leq \frac{3}{10}[x+y]$$
$$= \frac{3}{10}[\max\{x, \frac{x}{17}\} + \max\{y, \frac{3y}{17}\}]$$
$$= t[p(x,Sx)+p(y,Ty)].$$

Also, for all $z \in \overline{B(x_0,r)}$ such that $z \preceq Sx_0$, then,

$$p(x_0,Sx_0)+p(z,Tz) \leq p(x_0,z)+p(Sx_0,Tz)$$

Therefore, all the conditions of Theorem 1 are satisfied. Moreover, 0 is the unique common fixed point of S and T.

Corollary 1. *Let $(X,\preceq,p)$ be a complete ordered partial metric space, $x_0,x,y \in X$, $r>0$ and $S: X \to X$ be a dominated mapping. Suppose that there exists $t \in [0,\frac{1}{2})$ such that following conditions hold:*

$$p(Sx,Sy) \leq t[p(x,Sx)+p(y,Sy)], \textit{ for all } (x,y) \textit{ in } (\overline{B(x_0,r)} \times \overline{B(x_0,r)}) \cap \nabla,$$

and

$$p(x_0,Sx_0) \leq (1-\lambda)[r+p(x_0,x_0)],$$

where $\lambda = \dfrac{t}{1-t}$. Then there exists a point x^* such that $p(x^*,x^*)=0$. Also if, for a nonincreasing sequence $\{x_n\}$ in $\overline{B(x_0,r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$, then $x^* = Sx^*$. Moreover, x^* is unique, if for any two points x,y in $\overline{B(x_0,r)}$ there exists a point $z_0 \in \overline{B(x_0,r)}$ such that $z_0 \preceq x$ and $z_0 \preceq y$ and

$$p(x_0,Sx_0)+p(z,Sz) \leq p(x_0,z)+p(Sx_0,Sz)$$

for all $z \in \overline{B(x_0,r)}$ such that $z \preceq Sx_0$.

Proof. In Theorem 1 take $T=S$ to get unique point $x^* \in \overline{B(x_0,r)}$ such that $x^* = Sx^*$. □

In Theorem 1, the conditions 2.3 and 2.2 are imposed to restrict the condition 2.1 only for x,y in $\overline{B(x_0,r)}$ and Example 3 explains the utility of these restrictions. However, the following result relax the conditions 2.2 and 2.3 but impose the condition 2.1 for all comparable elements in the whole space X.

Theorem 2. *Let $(X, \preceq, p)$ be a complete ordered partial metric space and $S, T : X \to X$ be two dominated mappings. Suppose that there exists $t \in [0, \frac{1}{2})$ such that following condition holds for $x, y \in X$,*

$$p(Sx, Ty) \leq t[p(x, Sx) + p(y, Ty)], \textit{ for all } (x, y) \textit{ in } \nabla.$$

Then there exists a point x^* such that $p(x^*, x^*) = 0$. Also if, for a nonincreasing sequence $\{x_n\}$ in X, $\{x_n\} \to u$ implies that $u \preceq x_n$, then $x^* = Sx^* = Tx^*$. Moreover, x^* is unique, if for any two points x, y in X there exists a point $z_0 \in X$ such that $z_0 \preceq x$ and $z_0 \preceq y$.

In Theorem 1, the condition "for a nonincreasing sequence $\{x_n\} \to u$ implies that $u \preceq x_n$", the existence of z_0 and the condition 2.3 are imposed to restrict the condition 2.1 only for comparable elements. However, the following result relax these restrictions but impose the condition 2.1 for all elements in $\overline{B(x_0, r)}$.

Theorem 3. *Let (X, p) be a complete partial metric space, $x_0, x, y \in X$, $r > 0$ and $S, T : X \to X$ be two mappings. Suppose that there exists $t \in [0, \frac{1}{2})$ such that following conditions hold*

$$p(Sx, Ty) \leq t[p(x, Sx) + p(y, Ty)], \textit{ for all } x, y \textit{ in } \overline{B(x_0, r)}.$$
$$p(x_0, Sx_0) \leq (1 - \lambda)[r + p(x_0, x_0)].$$

where $\lambda = \dfrac{t}{1-t}$. Then there exists a unique point x^* in $\overline{B(x_0, r)}$ such that $x^* = Sx^* = Tx^*$. Also $p(x^*, x^*) = 0$. Further S and T have no fixed point other than x^*.

Proof. By following the similar steps of Theorem 1, we can obtain $x^* = Sx^* = Tx^*$. Let $y = Ty$. Then y is the fixed point of T and it may not be the fixed point of S. Then,

$$\begin{aligned} p(x^*, y) = p(Sx^*, Ty) &\leq t[p(x^*, Sx^*) + p(y, Ty)] \\ &= t[p(x^*, x^*) + p(y, y)] \leq tp(y, y) \quad \text{(by 2.10)} \end{aligned}$$

This shows that $x^* = y$. Thus T has no fixed point other than x^*. Similarly S has no fixed point other than x^*. □

Now we apply our Theorem 1 to obtain unique common fixed point of three mappings on a closed ball in complete partial ordered metric space. The technique and style of our proof is quite smart and innovative.

Theorem 4. *Let $(X, \preceq, p)$ be a ordered partial metric space, $x_0, x, y \in X$, $r > 0$ and S, T self mapping and f be a dominated mapping on X such that $SX \cup TX \subset fX$, $\overline{B(fx_0, r)} \subseteq fX$ and $(Tx, fx), (Sx, fx) \in \nabla$. Assume that the following conditions hold:*

$$p(Sx, Ty) \leq k[p(fx, Sx) + p(fy, Ty)] \tag{2.16}$$

for all $(fx, fy) \in (\overline{B(fx_0,r)} \times \overline{B(fx_0,r)}) \cap \nabla$; *where* $0 \le k < 1/2$,

$$p(fx_0, Sx_0) + p(fy, Ty) \le p(fx_0, fy) + p(Sx_0, Ty) \tag{2.17}$$

for all $fy \in \overline{B(fx_0,r)}$ *such that* $fy \preceq Sx_0$,

$$p(fx_0, Tx_0) \le (1-\lambda)[r + p(fx_0, fx_0)] \tag{2.18}$$

where $\lambda = \frac{k}{1-k}$. *If, for a nonincreasing sequence* $\{x_n\}$ *in* $\overline{B(fx_0,r)}$, $\{x_n\} \to u$ *implies that* $u \preceq x_n$ *and for any two points* z *and* x *in* $\overline{B(fx_0,r)}$ *there exists a point* $y \in \overline{B(fx_0,r)}$ *such that* $y \preceq z$ *and* $y \preceq x$. *If* fX *is complete subspace of* X *and* (S, f) *and* (T, f) *are weakly compatible, then* S, T *and* f *have a unique common fixed point* fz *in* $\overline{B(fx_0,r)}$. *Also* $p(fz, fz) = 0$.

Proof. By Lemma 2, there exists $E \subset X$ such that $fE = fX$ and $f : E \to X$ is one-to-one. Now since $SX \cup TX \subset fX$, we define two mappings $g, h : fE \to fE$ by $g(fx) = Sx$ and $h(fx) = Tx$ respectively. Since f is one-to-one on E, then g, h are well-defined. As $(Sx, fx) \in \nabla$ implies that $(g(fx), fx) \in \nabla$ and $(Tx, fx) \in \nabla$ implies that $(h(fx), fx) \in \nabla$ therefore g and h are dominated maps. Now $fx_0 \in \overline{B(fx_0,r)} \subseteq fX$, then $fx_0 \in fX$. Let $y_0 = fx_0$, choose a point y_1 in fX such that $y_1 = h(y_0)$. As $h(y_0) \preceq y_0$, so $(y_1, y_0) \in \nabla$ and let $y_2 = g(y_1)$. Now $g(y_1) \preceq y_1$ gives $(y_2, y_1) \in \nabla$. Continuing this process and having chosen y_n in fX such that

$$y_{2i+1} = h(y_{2i}) \text{ and } y_{2i+2} = g(y_{2i+1}), \text{ where } i = 0, 1, 2, ...,$$

then $y_{2i+1} = h(y_{2i}) \preceq y_{2i}$ implies $(y_{2i+1}, y_{2i}) \in \nabla$. Following similar arguments of Theorem 2, $y_n \in \overline{B(fx_0,r)}$. Also by inequality 2.17, we obtain for all $(fy, g(fx_0)) \in (\overline{B(fx_0,r)} \times \overline{B(fx_0,r)}) \cap \nabla$,

$$p(fx_0, g(fx_0)) + p(fy, h(fy)) \le p(fx_0, fy) + p(g(fx_0), h(fy))$$

and by inequality 2.18, we have,

$$p(fx_0, h(fx_0)) \le (1-\lambda)[r + p(fx_0, fx_0)].$$

By using inequality 2.16, $(fx, fy) \in (\overline{B(fx_0,r)} \times \overline{B(fx_0,r)}) \cap \nabla$, we have,

$$p(g(fx), h(fy)) \le k[p(fx, g(fx)) + p(fy, h(fy))]$$

As fX is a complete space, all conditions of Theorem 2 are satisfied, we deduce that there exists a unique common fixed point $fz \in \overline{B(fx_0,r)}$ of g and h. Also $p(fz, fz) = 0$. Now $fz = g(fz) = h(fz)$ or $fz = Sz = Tz = fz$. Thus fz is the point of coincidence of S, T and f. Let $v \in \overline{B(fx_0,r)}$ be another point of coincidence of f, S and T then there exist $u \in \overline{B(fx_0,r)}$ such that is $v = fu = Su = Tu$, which implies that $fu = g(fu) = h(fu)$. A contradiction as $fz \in \overline{B(fx_0,r)}$ is a unique common fixed point of g and h. Hence $v = fz$. Thus S, T and f have a unique point of coincidence $fz \in \overline{B(fx_0,r)}$. Now since (S, f) and (T, f) are weakly compatible, by Lemma 3 fz is a unique common fixed point of S, T and f. □

Now we can apply our Theorem 2 to obtain unique common fixed point result of three mappings in complete partial ordered metric space.

Theorem 5. *Let $(X,\preceq,p)$ be a ordered partial metric space, $x,y\in X$ and S,T self mapping and f be a dominated mapping on X such that $SX\cup TX\subset fX$ and $(Tx,fx),(Sx,fx)\in\nabla$. Assume that the following conditions hold:*

$$p(Sx,Ty)\le k[p(fx,Sx)+p(fy,Ty)]$$

for all $(fx,\ fy)\in\nabla$; where $0\le k<1/2$. If, for a nonincreasing sequence $\{x_n\}$ in fX, $\{x_n\}\to u$ implies that $u\preceq x_n$ and for any two points z and x in fX there exists a point $y\in fX$ such that $y\preceq z$ and $y\preceq x$. If fX is complete subspace of X and (S,f) and (T,f) are weakly compatible, then S,T and f have a unique common fixed point fz in fX. Also $p(fz,fz)=0$.

Now we can apply our Theorem 3 to obtain unique common fixed point of three mappings on closed ball in complete partial metric space.

Theorem 6. *Let (X,p) be a partial metric space, $x_0,x,y\in X$, $r>0$ and S,T and f be the self mappings on X such that $SX\cup TX\subset fX$, $\overline{B(fx_0,r)}\subseteq fX$. Assume that the following conditions hold:*

$$p(Sx,Ty)\le k[p(fx,Sx)+p(fy,Ty)]$$

for all $fx,\ fy\in\overline{B(fx_0,r)}$; where $0\le k<1/2$,

$$p(fx_0,Tx_0)\le(1-\lambda)[r+p(fx_0,fx_0)]$$

where $\lambda=\frac{k}{1-k}$. If fX is complete subspace of X and (S,f) and (T,f) are weakly compatible, then S,T and f have a unique common fixed point fz in $\overline{B(fx_0,r)}$. Also $p(fz,fz)=0$.

In the following theorem we use Theorem 3 to establish the existence of a unique common fixed point of four mappings on closed ball in complete partial metric space.

Theorem 7. *Let (X,p) be a partial metric space, $x_0,x,y\in X$, $r>0$ and S,T,g and f be self mappings on X such that $SX,TX\subset fX=gX$ and $\overline{B(fx_0,r)}\subseteq fX$. Assume that the following condition holds:*

$$p(Sx,Ty)\le k[p(fx,Sx)+p(gy,Ty)] \tag{2.19}$$

for all $fx,fy\in\overline{B(fx_0,r)}$, where $0\le k<1/2$, and

$$p(fx_0,Sx_0)\le(1-\lambda)[r+p(fx_0,fx_0)] \tag{2.20}$$

where $\lambda=\frac{k}{1-k}$. If fX is complete subspace of X and (S,f) and (T,g) are weakly compatible, then S, T, f and g have a unique common fixed point fz in $\overline{B(fx_0,r)}$. Also $p(fz,fz)=0$.

Proof. By Lemma 2, there exists $E_1, E_2 \subset X$ such that $fE_1 = fX = gX = gE_2$, $f : E_1 \to X$, $g : E_2 \to X$ are one to one. Now define the mappings $A, B : fE_1 \to fE_1$ by $A(fx) = Sx$ and $B(gx) = Tx$ respectively. Since f, g are one to one on E_1, and E_2 respectively, then the mappings A, B are well-defined. As $fx_0 \in \overline{B(fx_0, r)} \subseteq fX$, then $fx_0 \in fX$. Let $y_0 = fx_0$, choose a point y_1 in fX such that $y_1 = A(y_0)$ and let $y_2 = B(y_1)$. Continuing this process and having chosen y_n in fX such that

$$y_{2i+1} = A(y_{2i}) \text{ and } y_{2i+2} = B(y_{2i+1}), \text{ where } i = 0, 1, 2, ...,$$

Following similar arguments of Theorem 1, $y_n \in \overline{B(fx_0, r)}$. Also by inequality 2.20, we have,

$$p(fx_0, A(fx_0)) \leq (1-\lambda)[r + p(fx_0, fx_0)]$$

By using inequality 2.19, for $fx, gy \in \overline{B(fx_0, r)}$, we have

$$p(A(fx), B(gy)) \leq k[p(fx, A(fx)) + p(gy, B(gy))]$$

As fX is a complete space, all conditions of Theorem 3 are satisfied, we deduce that there exists a unique common fixed point $fz \in \overline{B(fx_0, r)}$ of A and B. Further A and B have no fixed point other than fz. Also $p(fz, fz) = 0$. Now $fz = A(fz) = B(fz)$ or $fz = Sz = fz$. Thus fz is a point of coincidence of f and S. Let $w \in \overline{B(fx_0, r)}$ be another point of coincidence of S and f then there exist $u \in \overline{B(fx_0, r)}$ such that is $w = fu = Su$, which implies that $fu = A(fu)$. A contradiction as $fz \in \overline{B(fx_0, r)}$ is a unique fixed point of A. Hence $w = fz$. Thus S and f have a unique point of coincidence $fz \in \overline{B(fx_0, r)}$. Since (S, f) are weakly compatible, by Lemma 3 fz is a unique common fixed point of S and f. As $fX = gX$ then there exist $v \in X$ such that $fz = gv$. Now as $A(fz) = B(fz) = fz \Rightarrow A(gv) = B(gv) = gv \Rightarrow Tv = gv$, thus gv is the point of coincidence of T and g. Now if $Tx = gx \Rightarrow B(gx) = gx$. A contradiction. This implies that $gv = gx$. As (T, g) are weakly compatible, we obtain gv, a unique common fixed point for T and g. But $gv = fz$. Thus S, T, g and f have a unique common fixed point $fz \in \overline{B(fx_0, r)}$. □

Theorem 8. *Let (X, p) be a partial metric space, $x, y \in X$ and S, T, g and f be self mappings on X such that $SX, TX \subset fX = gX$. Assume that the following condition holds:*

$$p(Sx, Ty) \leq k[p(fx, Sx) + p(gy, Ty)]$$

for all $fx, fy \in fX$, where $0 \leq k < 1/2$. If fX is complete subspace of X and (S, f) and (T, g) are weakly compatible, then S, T, f and g have a unique common fixed point fz in fX. Also $p(fz, fz) = 0$.

We can obtain the unique point of coincidence results from Theorem 4 to Theorem 8. Unique point of coincidence result for Theorem 4 is given below.

Theorem 9. *Let $(X, \preceq, p)$ be a ordered partial metric space, $x_0, x, y \in X$, $r > 0$ and S, T self mapping and f be a dominated mapping on X such that $SX \cup$*

$TX \subset fX$, $\overline{B(fx_0,r)} \subseteq fX$ and $(Tx, fx), (Sx, fx) \in \nabla$. Assume that the following conditions holds:

$$p(Sx,Ty) \leq k[p(fx,Sx) + p(fy,Ty)]$$

for all $(fx, fy) \in (\overline{B(fx_0,r)} \times \overline{B(fx_0,r)}) \cap \nabla$; where $0 \leq k < 1/2$,

$$p(fx_0,Sx_0) + p(fy,Ty) \leq p(fx_0,fy) + p(Sx_0,Ty)$$

for all $fy \in \overline{B(fx_0,r)}$ such that $fy \preceq Sx_0$,

$$p(fx_0,Tx_0) \leq (1-\lambda)[r + p(fx_0,fx_0)]$$

where $\lambda = \frac{k}{1-k}$. If, for a nonincreasing sequence $\{x_n\}$ in $\overline{B(fx_0,r)}$, $\{x_n\} \to u$ implies that $u \preceq x_n$ and for any two points z and x in $\overline{B(fx_0,r)}$ there exists a point $y \in \overline{B(fx_0,r)}$ such that $y \preceq z$ and $y \preceq x$. If fX is complete subspace of X, then S,T and f have a unique point of coincidence $fz \in \overline{B(fx_0,r)}$. Also $p(fz, fz) = 0$.

Unique point of coincidence result for Theorem 7 is given below.

Theorem 10. *Let (X, p) be a partial metric space, $x_0, x, y \in X$, $r > 0$ and S, T, f and g be self mappings on X such that $SX, TX \subset fX = gX$ and $\overline{B(fx_0,r)} \subseteq fX$. Assume that the following condition holds:*

$$p(Sx,Ty) \leq k[p(fx,Sx) + p(gy,Ty)]$$

for all $fx, fy \in \overline{B(fx_0,r)}$; where $0 \leq k < 1/2$, and

$$p(fx_0,Tx_0) \leq (1-\lambda)[r + p(fx_0,fx_0)]$$

where $\lambda = \dfrac{k}{1-k}$. If fX is complete subspace of X , then S, T, f and g have a unique point of coincidence fz in $\overline{B(fx_0,r)}$. Also $p(fz, fz) = 0$.

Remark 1. We can obtain the metric version of all theorems, which are still not present in the literature.

3. Conclusion

Azam et. al. [9] very recently exploited the idea of fixed points and proved a significant result concerning the existence of fixed points for fuzzy mappings on closed ball in a complete metric space. We continue their investigations and in this paper, some unique common fixed point theorems for dominated mappings under Kannan contractive conditions on a closed ball in a complete ordered partial metric space $(X, \preceq, p)$ have been discussed. Our analysis is based on the simple observation that fixed point results can be deduced from fixed point theory of mappings on closed balls. Practically speaking there are many situations in which the mappings are not contractive on the whole space but instead they are contractive on its subsets. We feel that this aspect of finding the fixed points via closed balls were overlooked and our paper will bring a lot of interest in to this area. Furthermore, we have applied the concept of dominated mappings in the process of investigating the existence of unique fixed point of contractive mappings on closed balls in the settings of ordered partial metric spaces.

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