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# On $\tau$-supplemented subgroups of finite groups

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# ON $\tau$-SUPPLEMENTED SUBGROUPS OF FINITE GROUPS

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*Abstract.* Let $H$ be a subgroup of a finite group $G$. We say that: (1) $H$ is $\tau$-quasinormal in $G$ if $H$ permutes with every Sylow subgroup $Q$ of $G$ such that $(|H|,|Q|)=1$ and $(|H|,|Q^G|)\neq 1$; (2) $H$ is $\tau$-supplemented in $G$ if $G$ has a subgroup $T$ of $G$ such that $G=HT$ and $H\cap T\leq H_{\tau G}$, where $H_{\tau G}$ is the subgroup generated by all those subgroups of $H$ which are $\tau$-quasinormal in $G$. We investigate the influence of $\tau$-supplemented subgroups on the structure of finite groups. Some recent known results are generalized and unified.

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*Keywords:* $\tau$-supplemented, $p$-supersolvable supplement, $p$-nilpotent, saturated formation

## 1. Introduction

This paper deals with finite groups. We use the standard terminology as in [10]. $G$ denotes always a group, $|G|$ is the order of $G$ and the set of distinct primes dividing $|G|$ will be denoted by $\pi(G)$. A group $G$ is called $p$-supersolvable if it is $p$-solvable and all its $G$-chief $p$-factors are cyclic. A group $G$ is called $p$-nilpotent if it is $p$-solvable and all its $G$-chief $p$-factors are central in $G$. Obviously, a $p$-nilpotent group is also a $p$-supersolvable group and $G$ is supersolvable (or nilpotent) if and only if $G$ is $p$-supersolvable (or $p$-nilpotent) for any $p\in\pi(G)$. If $G=HK$ and $K$ is $p$-supersolvable (or supersolvable, $p$-nilpotent), then we call that $H$ has a $p$-supersolvable (or supersolvable, $p$-nilpotent) supplement $K$ in $G$.

A subgroup $H$ of a group $G$ is said to be $S$-quasinormal (or $S$-permutable) in $G$ if $H$ permutes with all Sylow subgroups of $G$, i.e., $HS=SH$ for any Sylow subgroup $S$ of $G$. This concept was first introduced by Kegel in [12]. Later, many authors generalized $S$-quasinormal concept; see, for example, [5, 13, 14, 34]. A subgroup $H$ is said to be $s$-semipermutable in $G$ if $H$ permutes with every Sylow $p$-subgroup of $G$ such that $(p,|H|)=1$. More recently, Lukanenko and Skiba [19] introduced the concept of $\tau$-quasinormal subgroup as follows: A subgroup $H$ of $G$ is said to be $\tau$-quasinormal in $G$ if $H$ permutes with every Sylow subgroup $Q$ of $G$ such that $(|H|,|Q|)=1$ and $(|H|,|Q^G|)\neq 1$. It is clear that $s$-semipermutability implies $\tau$-quasinormality by definition; however, the converse is not true, as seen in [20, Example 1].

On the other hand, some subgroups with supplemented properties were introduced recently. In particular, Ballester-Bolinches, Wang and Guo [7] called that a subgroup $H$ of a group $G$ is $c$-supplemented in $G$ if there is a subgroup $K$ of $G$ such that $G = HK$ and $H \cap K \leq H_G$, where $H_G$ is the normal core of $H$ in $G$. In 2007, Skiba [25] again gave the concept of $S$-supplemented subgroup as follows: A subgroup $H$ of $G$ is called $S$-supplemented in $G$ if there exists a subgroup $K$ such that $G = HK$ and $H \cap K \leq H_{sG}$, where $H_{sG}$ is the subgroup of $H$ generated by all those subgroups of $H$ which are $S$-quasinormal in $G$. If we take $G =< a, b \mid a^{16} = b^4 = 1, ba = a^3b >$, then $< b^2 >$ is an $S$-supplemented subgroup of $G$. However $< b^2 >$ is not $c$-supplemented in $G$. Hence, $S$-supplemented subgroups generalize $c$-supplemented subgroups.

There is no obvious general relationship between $\tau$-quasinormal subgroups and $S$-supplemented subgroups. Hence it is meaningful to unify and generalize above series subgroups. On the basis of these definitions, we now introduce the following new concept:

**Definition 1.** *A subgroup $H$ of a group $G$ is said to be $\tau$-supplemented in $G$ if $G$ has a subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq H_{\tau G}$, where $H_{\tau G}$ is the subgroup generated by all those subgroups of $H$ which are $\tau$-quasinormal in $G$.*

The next two examples show that the class of all $\tau$-supplemented subgroups is wider than the class of all $\tau$-quasinormal subgroups and the class of all $S$-supplemented subgroups.

*Example* 1. Let $G = S_4$ be the symmetric group of degree 4 and $H =< (14) >$. Obviously, $H$ is $\tau$-supplemented in $G$. However, $H$ is not $\tau$-quasinormal in $G$.

*Example* 2. Let $G =< a, b, c \mid a^5 = b^4 = c^5 = 1, b^{-1}ab = a^2, [a, c] = [b, c] = 1 >$ and $H =< b^2 >$. It is easy to see that $H$ is $\tau$-supplemented in $G$, but not $S$-supplemented in $G$.

Let $\mathcal{F}$ be a class of groups. We call $\mathcal{F}$ a formation provided that (i) if $G \in \mathcal{F}$ and $H \trianglelefteq G$, then $G/H \in \mathcal{F}$, and (ii) if $G/M$ and $G/N$ are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for any normal subgroups $M, N$ of $G$. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ and $\mathcal{N}$ will denote the class of all supersolvable groups and the class of all nilpotent groups, respectively. It is well known that both $\mathcal{U}$ and $\mathcal{N}$ are saturated formations. A chief factor $H/K$ of a group $G$ is called $\mathcal{F}$-central provided $[H/K](G/C_G(H/K)) \in \mathcal{F}$. The product of all normal subgroups of $G$ whose $G$-chief factors are $\mathcal{F}$-central in $G$ is called the $\mathcal{F}$-hypercentre of $G$ and denoted by $Z_{\mathcal{F}}(G)$. Noticing that for any $\mathcal{N}$-central chief factor $H/K$ of $G$ we have $C_G(H/K) = G$. Hence the $\mathcal{N}$-hypercentre of $G$ coincides with the hypercentre $Z_\infty(G)$ of $G$. Another fact is that any chief factor of $G$ under $Z_{\mathcal{U}}(G)$ is of prime order. The major aim of the present paper is to find cyclicity conditions for $G$-chief factors of normal subgroups of a group $G$ by some

$\tau$-supplemented subgroups. As their applications, we not only extend some known results in [1, 2, 4, 7, 9, 15, 16, 22, 28, 29, 32], but also give more simple proofs.

## 2. Preliminaries

**Lemma 1** ([19, Lemma 2.3]). *Let $G$ be a group and $E \le K \le G$.*

(1) *If $E$ is $\tau$-quasinormal in $G$, then $E$ is $\tau$-quasinormal in $K$.*

(2) *Suppose that $E$ is normal in $G$ and $\pi(K/E) = \pi(K)$. If $K$ is $\tau$-quasinormal in $G$, then $K/E$ is $\tau$-quasinormal in $G/E$.*

(3) *Suppose that $E$ is normal in $G$. Then $HE/E$ is $\tau$-quasinormal in $G/E$ fore every $\tau$-quasinormal subgroup $E$ in $G$ satisfying $(|H|, |E|) = 1$.*

(4) *If $E$ is $\tau$-quasinormal in $G$ and $E \le O_p(G)$ for some prime $p$, then $E$ is $S$-quasinormal in $G$.*

**Lemma 2.** *Let $H$ be a $\tau$-supplemented subgroup of a group $G$.*

(1) *If $H \le L \le G$, then $H$ is $\tau$-supplemented in $L$.*

(2) *If $E \trianglelefteq G$, $E \le H \le G$ and $H$ is a $p$-group for some prime $p$, then $H/E$ is $\tau$-supplemented in $G/E$.*

(3) *If $H$ is a $\pi$-subgroup and $E$ is a normal $\pi'$-subgroup of $G$, then $HE/E$ is $\tau$-supplemented in $G/E$.*

*Proof.* By the hypothesis, there is a subgroup $K$ of $G$ such that $G = HK$ and $H \cap K \le H_{\tau G}$.

(1) $L = L \cap HK = H(L \cap K)$ and $H \cap (L \cap K) = H \cap K \le H_{\tau G} \le H_{\tau K}$. Hence $H$ is $\tau$-supplemented in $L$.

(2) We have $G/E = HK/E = H/E \cdot EK/E$ and $(H/E) \cap (KE/E) = (H \cap KE)/E = (H \cap K)E/E \le H_{\tau G}E/E = H_{\tau G}/E \le (H/E)_{\tau(G/E)}$ by Lemma 1. Hence $H/E$ is $\tau$-supplemented in $G/E$.

(3) Since $(|G : K|, |E|) = 1$ and $E \trianglelefteq G$, we have $E \le K$. It is easy to see that $G/E = HE/E \cdot KE/E = HE/E \cdot K/E$ and $(HE/E) \cap (K/E) = (HE \cap K)/E = (H \cap K)E/E \le H_{\tau G}E/E \le (HE/E)_{\tau(G/E)}$ by Lemma 1. Hence $HE/E$ is $\tau$-supplemented in $G/E$. □

**Lemma 3** ([33, p.38, Theorem 7.19]). *Let $H$ be a normal subgroup of $G$. Then $H \le Z_{\mathfrak{U}}(G)$ if and only if $H/\Phi(H) \le Z_{\mathfrak{U}}(G/\Phi(H))$.*

**Lemma 4** ([26, Lemma A]). *Let $E$ be a normal subgroup of a group $G$. Suppose that for every non-cyclic Sylow subgroup $P$ of $E$, either all maximal subgroups of $P$ or all cyclic subgroups of $P$ of prime order and order 4 are $S$-supplemented in $G$. Then each $G$-chief factor below $E$ is cyclic.*

**Lemma 5** ([8, p.362, Proposition 3.11]). *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two saturated formations such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $Z_{\mathcal{F}_1}(G) \le Z_{\mathcal{F}_2}(G)$.*

The generalized Fitting subgroup $F^*(G)$ of $G$ is the unique maximal normal quasinilpotent subgroup of $G$. $F^*(G)$ is an important subgroup of $G$ and it is a natural generalization of $F(G)$. The definition and important properties can be found in [11, Chapter X].

**Lemma 6** ([11, X, 13]). *Let $G$ be a group. Then:*
(1) *If $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*
(2) $C_G(F^*(G)) \leq F(G)$.

**Lemma 7** ([27, Theorem C]). *Let $E$ be a normal subgroup of a group $G$. If $F^*(E) \leq Z_{\mathfrak{U}}(G)$, then $E \leq Z_{\mathfrak{U}}(G)$.*

**Lemma 8** ([10, p.434, Satz 5.4 and p.281, Satz 5.2]). *If $G$ is a group which is not $p$-nilpotent but all of its proper subgroups are $p$-nilpotent, then it is a minimal non-nilpotent group (that is, $G$ is not nilpotent but all of its proper subgroups are nilpotent). Then*

(1) *$G$ has a normal Sylow $p$-subgroup $P$ for some prime $p$ and $G = PQ$, where $Q$ is a non-normal cyclic $q$-subgroup for some prime $q \neq p$.*
(2) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*
(3) *If $P$ is non-abelian and $p > 2$, then the exponent of $P$ is $p$; If $P$ is non-abelian and $p = 2$, then the exponent of $P$ is 4.*
(4) *If $P$ is abelian, then the exponent of $P$ is $p$.*

**Lemma 9** ([28, Lemma 2.8]). *Let $M$ be a maximal subgroup of $G$, $P$ a normal $p$-subgroup of $G$ such that $G = PM$, where $p$ is a prime. Then $P \cap M$ is a normal subgroup of $G$.*

**Lemma 10** ([33, p.220, Theorem 6.3]). *Let $P$ be a normal $p$-subgroup of $G$ such that $|G/C_G(P)|$ is a power of $p$. Then $P \leq Z_\infty(G)$.*

## 3. Main results

**Theorem 1.** *Suppose that $p$ is the smallest prime dividing the order of a group $G$ and $G$ has a normal subgroup $E$ such that $G/E$ is $p$-nilpotent. If every cyclic subgroup $H$ of $E$ with prime order $p$ or order 4 (if $p = 2$) having no $p$-supersoluble supplement in $G$ is $\tau$-supplemented in $G$, then $G$ is $p$-nilpotent.*

*Proof.* Suppose that the theorem is false, and let $G$ be a counterexample of minimal order .

(1) The hypotheses are inherited by all proper subgroups of $G$ and $G$ is a group which is not nilpotent but whose proper subgroups are all nilpotent.

In fact, $\forall K < G$, since $G/E$ is $p$-nilpotent, $K/(K \cap E) \cong KE/E$ is also $p$-nilpotent. Let $H$ be any cyclic subgroup of $K \cap E$ with prime order $p$ or order 4 (if $p = 2$). Obviously, $H$ is a cyclic subgroup of $E$ with prime order $p$ or order 4. If $H$ has a $p$-supersoluble supplement $T$ in $G$, then $H$ has a $p$-supersoluble supplement $T \cap K$ in $K$. If $H$ is $\tau$-supplemented in $G$, then $H$ is $\tau$-supplemented in $K$ by Lemma

2(1). Thus $K$ satisfies the hypotheses of the theorem. By the choice of $G$, $K$ is $p$-nilpotent. Then $G$ is a group which is not $p$-nilpotent but whose proper subgroups are all $p$-nilpotent. By Lemma 8, $G = PQ$ and $P \trianglelefteq G$.

(2) $G/(P \cap E)$ is $p$-nilpotent.

Since $G/P \cong Q$ is $p$-nilpotent, $G/E$ is $p$-nilpotent and $G/(P \cap E) \lesssim G/P \times G/E$, we have $G/(P \cap E)$ is $p$-nilpotent.

(3) $P \leq E$.

If $P \nleq E$, then $P \cap E < P$. So $Q(P \cap E) < QP = G$. Thus $Q(P \cap E)$ is nilpotent by Step (1) and so $Q(P \cap E) = Q \times (P \cap E)$. Since $G/(P \cap E) = P/(P \cap E) \cdot Q(P \cap E)/(P \cap E)$, it follows that $Q(P \cap E)/(P \cap E) \trianglelefteq G/(P \cap E)$ by Step (2). Now $Q$ char $Q(P \cap E) \trianglelefteq G$ implies that $G = P \times Q$, a contradiction.

(4) For every cyclic subgroup $L$ of $P$ with order $p$ or 4, if there is a subgroup $T$ of $G$ such that $G = LT$, then $T = G$.

Let $L$ be a cyclic subgroup of prime order $p$ or of order 4 in $P$ and assume that there exists a subgroup $T$ of $G$ such that $G = LT$. Obviously, $P = P \cap G = P \cap LT = L(P \cap T)$. Since $P/\Phi(P)$ is abelian, we have $(P \cap T)\Phi(P)/\Phi(P) \trianglelefteq G/\Phi(P)$. By Step (1), $P \cap T \leq \Phi(P)$ or $P = (P \cap T)\Phi(P) = P \cap T$. If the former holds, then $L = P$ and so $G$ is $p$-nilpotent by [24, P.280, Theorem 10.1.9], a contradiction. Hence $P = P \cap T$ and $T = G$.

(5) Every cyclic subgroup $L$ of $P$ with prime order $p$ or of order 4 is $S$-quasinormal in $G$.

If $L$ has a $p$-supersoluble supplement $T$ in $G$, then $G = T$ is $p$-supersoluble by Step (4) and so $G$ is $p$-nilpotent since $p$ is the smallest prime dividing $|G|$, a contradiction. Thus we may assume all cyclic subgroups of $P$ with order $p$ or 4 are $\tau$-supplemented in $G$. In view of Step (4), all cyclic subgroups of $P$ with order $p$ or 4 are $\tau$-quasinormal in $G$. By Lemma 1(4), all cyclic subgroups of $P$ with order $p$ or 4 are $S$-quasinormal in $G$.

(6) Final contradiction.

For every $x \in P$, we have $| < x > | = p$ or 4 by Step (1), and so $< x >$ is $S$-quasinormal in $G$ by Step (5). By [24, P.280, Theorem 10.1.9], we have $< x > Q$ is a proper subgroup of $G$, and so $< x > Q = < x > \times Q$ by Step (1). Then we conclude that $G = P \times Q$, a contradiction. □

**Theorem 2.** *Suppose that $P$ is a normal $p$-subgroup of a group $G$. If every cyclic subgroup of $P$ with order $p$ or* 4 *(if $p = 2$) having no $p$-supersoluble supplement in $G$ is $\tau$-supplemented in $G$, then $P \leq Z_{\mathfrak{U}}(G)$.*

*Proof.* We distinguish two cases:

Case I. $p = 2$.

Pick an arbitrary Sylow $q$-subgroup $G_q$ of $G$, where $q \neq 2$. Consider the subgroup $W = G_q P$. Let $L$ be a cyclic subgroup of $P$ with order 2 or 4. If $L$ has a 2-supersoluble supplement $T$ in $G$, then $L$ has a 2-supersoluble supplement $T \cap W$ in $W$. If $L$ is $\tau$-supplemented in $G$, then $L$ is also $\tau$-supplemented in $W$ by Lemma

2(1). Applying Theorem 1, we have $W$ is 2-nilpotent. Hence $W = P \times G_q$. This implies that $O^p(G) \le C_G(P)$. In view of Lemma 10, $P \le Z_\infty(G)$. Consequently, $P \le Z_{\mathfrak{U}}(G)$ from Lemma 5.

Case II. $p > 2$.

First suppose that some cyclic subgroup $L$ of $P$ with order $p$ has a $p$-supersoluble supplement $T$ in $G$. So $G = LT = PT$. If $T = G$, then $P \le Z_{\mathfrak{U}}(G)$ and we are done. Hence we may assume that $T < G$, which shows that $L \cap T = 1$ and $T$ is a maximal subgroup of $G$. Clearly, $P = P \cap LT = L(P \cap T)$ and $P \cap T$ is a maximal subgroup of $P$. By Lemma 9, $P \cap T$ is normal in $G$. Since every cyclic subgroup of $P \cap T$ with order $p$ having no $p$-supersoluble supplement in $G$ is $\tau$-supplemented in $G$, $P \cap T \le Z_{\mathfrak{U}}(G)$ by induction. Noticing that $P/(P \cap T)$ is a normal subgroup of $G/(P \cap T)$ with order $p$, we have $P/(P \cap T) \le Z_{\mathfrak{U}}(G/P \cap T)$. It follows that $P \le Z_{\mathfrak{U}}(G)$. Now we may assume that all cyclic subgroups of $P$ with order $p$ are $\tau$-supplemented in $G$. From Lemma 1(4) we have that all cyclic subgroups of $P$ with order $p$ are $S$-supplemented in $G$ since $P \le O_p(G)$. In view of Lemma 4, we have also $P \le Z_{\mathfrak{U}}(G)$. □

**Theorem 3.** *Let $E$ be a normal subgroup of a group $G$. Suppose that for each $p \in \pi(E)$ and non-cyclic Sylow $p$-subgroup $P$ of $E$, all cyclic subgroups of $P$ with order $p$ or 4 (if $p = 2$) having no $p$-supersoluble supplement in $G$ are $S$-supplemented in $G$. Then $E \le Z_{\mathfrak{U}}(G)$.*

*Proof.* Let $q$ be the smallest prime dividing $|E|$ and $L$ a cyclic subgroup of the Sylow $q$-subgroup $Q$ of $E$ with order $q$ or 4 (if $q = 2$). If $L$ has a $q$-supersoluble supplement $T$ in $G$, then $L$ has a $q$-supersoluble supplement $T \cap E$ in $E$. If $L$ is $\tau$-supplemented in $G$, then $L$ is also $\tau$-supplemented in $E$ by Lemma 2(1). In view of Theorem 1, $E$ is $q$-nilpotent. Let $E_{q'}$ be the normal $q'$-complement of $E$. If $E = Q$, then $E \le Z_{\mathfrak{U}}(G)$ by Theorem 2. Hence we may assume that $E_{q'} \neq 1$. Since $E_{q'}$ char $E \trianglelefteq G$, $E_{q'} \trianglelefteq G$. By the hypothesis of the theorem, for each $p \in \pi(E_{q'})$ and non-cyclic Sylow $p$-subgroup $P$ of $E_{q'}$, all cyclic subgroups of $P$ with order $p$ having no $p$-supersoluble supplement in $G$ are $S$-supplemented in $G$. By induction, $E_{q'} \le Z_{\mathfrak{U}}(G)$. By Lemma 2(3), it is easy to see that all cyclic subgroups of $QE_{q'}/E_{q'}$ with order $q$ or 4 (if $q = 2$) having no $q$-supersoluble supplement in $G/E_{q'}$ are $S$-supplemented in $G/E_{q'}$. By induction, we have also $E/E_{q'} \le Z_{\mathfrak{U}}(G/E_{q'})$. It follows that $E \le Z_{\mathfrak{U}}(G)$. □

**Corollary 1.** *Let $E$ be a normal subgroup of a group $G$. Suppose that every cyclic subgroup of each non-cyclic Sylow subgroup of $E$ with prime order or order 4 having no supersoluble supplement in $G$ are $S$-supplemented in $G$. Then $E \le Z_{\mathfrak{U}}(G)$.*

**Theorem 4.** *Let $P$ be a normal $p$-subgroup of a group $G$, where $p$ is a prime dividing the order of $G$. Suppose that every maximal subgroup of $P$ having no $p$-supersoluble supplement in $G$ is $\tau$-supplemented in $G$. Then $P \le Z_{\mathfrak{U}}(G)$.*

*Proof.* We distinguish two cases:

Case I. $\Phi(P) \neq 1$.

Obviously, $P/\Phi(P)$ is a normal $p$-subgroup of $G/\Phi(P)$. Let $P_1/\Phi(P)$ be a maximal subgroup of $P/\Phi(P)$. Then $P_1$ is a maximal subgroup of $P$. If $P_1$ has a $p$-supersoluble supplement $T$ in $G$, then $P_1/\Phi(P)$ has a $p$-supersoluble supplement $T\Phi(P)/\Phi(P)$ in $G/\Phi(P)$. If $P_1$ is $\tau$-supplemented in $G$, then $P_1/\Phi(P)$ is $\tau$-supplemented in $G/\Phi(P)$ by Lemma 2. Therefore, $G/\Phi(P)$ satisfies the hypothesis of the theorem. By induction, $P/\Phi(P) \leq Z_{\mathfrak{U}}(G/\Phi(P))$. In view of Lemma 3, we have $P \leq Z_{\mathfrak{U}}(G)$.

Case II. $\Phi(P) = 1$.

This shows that $P$ is abelian. First suppose that some maximal subgroup $V$ of $P$ has a $p$-supersoluble supplement $T$ in $G$. Then $G = VT = PT$ and $P \cap T \neq 1$. Since $P \cap T \trianglelefteq T$, we may assume that $T$ has a minimal normal subgroup $N$ contained in $P \cap T$. It is clear that $|N| = p$. From $G = PT$, we have $N$ is also normal in $G$. With the similar argument in Case I, the hypothesis of the theorem holds for $(G/N, P/N)$. By induction, we have $P/N \leq Z_{\mathfrak{U}}(G/N)$. It follows that $P \leq Z_{\mathfrak{U}}(G)$. Now we may assume that every maximal subgroup of $P$ is $\tau$-supplemented in $G$. In view of Lemma 1, every maximal subgroup of $P$ is $S$-supplemented in $G$ since $P \leq O_p(G)$. Then we have also $P \leq Z_{\mathfrak{U}}(G)$ by Lemma 4. □

**Corollary 2.** *Let $P$ be a normal $p$-subgroup of $G$. Suppose that every maximal subgroup of $P$ having no supersoluble supplement in $G$ is $\tau$-supplemented in $G$. Then $P \leq Z_{\mathfrak{U}}(G)$.*

In connection with Theorem 3 and Lemma 4 the following natural question arises:

**Question.** *Let $E$ be a normal subgroup of a group $G$. Suppose that for every non-cyclic Sylow subgroup $P$ of $E$ all maximal subgroups of $P$ having no supersoluble supplement in $G$ are $\tau$-supplemented in $G$. Is then $E \leq Z_{\mathfrak{U}}(G)$?*

## 4. Some Applications

**Theorem 5.** *Let $\mathcal{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that $G$ is a group with a solvable normal subgroup $E$ such that $G/E \in \mathcal{F}$. If every maximal subgroup of each non-cyclic Sylow subgroup of $F(E)$ having no supersoluble supplement in $G$ is $\tau$-supplemented in $G$, then $G \in \mathcal{F}$.*

*Proof.* By Theorem 4, $F(E) \leq Z_{\mathfrak{U}}(G)$. Since $\mathfrak{U} \subseteq \mathcal{F}$, we have that $F(E) \leq Z_{\mathcal{F}}(G)$ by Lemma 5. In view of the solvability of $E$ and Lemma 6, $F^*(E) = F(E) \leq Z_{\mathcal{F}}(G)$. By Lemma 7, $E \leq Z_{\mathcal{F}}(G)$. Since $G/Z_{\mathcal{F}}(G) \cong (G/E)/(Z_{\mathcal{F}}(G)/E) \in \mathcal{F}$, we have $G \in \mathcal{F}$. □

**Theorem 6.** *Let $\mathcal{F}$ be a saturated formation containing $\mathfrak{U}$. If there is a normal subgroup $E$ of a group $G$ such that $G/E \in \mathcal{F}$ and every cyclic subgroup of $E$ with* *prime order or order 4 having no supersoluble supplement in $G$ is $\tau$-supplemented in $G$, then $G \in \mathcal{F}$.*

*Proof.* Since $E \leq Z_{\mathcal{U}}(G)$ by Theorem 3 and $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$ by Lemma 5, we have $E \leq Z_{\mathcal{F}}(G)$. Hence $G/Z_{\mathcal{F}}(G) \cong (G/E)/(Z_{\mathcal{F}}(G)/E) \in \mathcal{F}$. It follows that $G \in \mathcal{F}$. □

**Theorem 7.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let $G$ be a group. If there is a normal subgroup $E$ such that $G/E \in \mathcal{F}$ and every cyclic subgroup of $F^*(E)$ with prime order or order 4 having no supersoluble supplement in $G$ is $\tau$-supplemented in $G$, then $G \in \mathcal{F}$.*

*Proof.* By Theorem 3, $F^*(E) \leq Z_{\mathcal{U}}(G)$ and since $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$ by Lemma 5, we have $F^*(E) \leq Z_{\mathcal{F}}(G)$. In view of Lemma 7, $E \leq Z_{\mathcal{F}}(G)$. Hence $G \in \mathcal{F}$ since $G/E \in \mathcal{F}$. □

**Theorem 8.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a solvable normal subgroup $E$ such that $G/E \in \mathcal{F}$. If every cyclic subgroup of $F(E)$ with prime order or order 4 having no supersoluble supplement in $G$ is $\tau$-supplemented in $G$, then $G \in \mathcal{F}$.*

*Proof.* Since $E$ is solvable, $F^*(E) = F(E)$ by Lemma 6. Applying Theorem 7, we have $G \in \mathcal{F}$. □

**Corollary 3** ([22, Theorem 3.1])**.** *Assume that $G$ is solvable and every maximal subgroup of the Sylow subgroups of $F(G)$ is normal in $G$. Then $G$ is supersolvable.*

**Corollary 4** ([4, Corollary 4.4])**.** *Suppose that $G$ is a solvable group with a normal subgroup $E$ such that $G/E$ is supersolvable. If all maximal subgroups of any Sylow subgroup of $F(E)$ are $S$-quasinormal in $G$, then $G$ is supersolvable.*

A subgroup $H$ of a group $G$ is $c$-normal in $G$ if there is a normal subgroup $K$ of $G$ such that $G = HK$ and $H \cap K \leq H_G$, where $H_G$ is the normal core of $H$ in $G$.

**Corollary 5** ([15, Theorem 2])**.** *Let $G$ be a group and $E$ a solvable normal subgroup of $G$ such that $G/E$ is supersolvable. If every maximal subgroup of each Sylow subgroup of $F(E)$ is $c$-normal in $G$, then $G$ is supersolvable.*

**Corollary 6** ([34, Theorem 2])**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups. Suppose that $G$ is a group with a solvable normal subgroup $E$ such that $G/E \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(E)$ are $s$-semipermutable in $G$, then $G \in \mathcal{F}$.*

**Corollary 7** ([16, Theorem 1.2])**.** *Suppose that $G$ is a solvable group with a normal subgroup $E$ such that $G/E$ is supersolvable. If every maximal subgroup of each Sylow subgroup of $F(E)$ is complement in $G$, then $G$ is supersolvable.*

**Corollary 8** ([30, Theorem 1]). *Let $\mathcal{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that $G$ is a group with a solvable normal subgroup $E$ such that $G/E \in \mathcal{F}$. If every maximal subgroup of each Sylow subgroup of $F(E)$ is $c$-normal in $G$, then $G \in \mathcal{F}$.*

**Corollary 9** ([1, Theorem 1.4]). *Let $\mathcal{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that $G$ is a solvable group with a normal subgroup $E$ such that $G/E \in \mathcal{F}$. If every maximal subgroup of each Sylow subgroup of $F(E)$ is $S$-quasinormal in $G$, then $G \in \mathcal{F}$.*

**Corollary 10** ([28, Theorem 4.5]). *Let $\mathcal{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that $G$ is a group with a solvable normal subgroup $E$ such that $G/E \in \mathcal{F}$. If every maximal subgroup of each Sylow subgroup of $F(E)$ is $c$-supplemented in $G$, then $G \in \mathcal{F}$.*

**Corollary 11** ([9, Theorem 1.6]). *Let $\mathcal{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that $G$ is a group with a solvable normal subgroup $E$ such that $G/E \in \mathcal{F}$. If every maximal subgroup of each Sylow subgroup of $F(E)$ is complemented in $G$, then $G \in \mathcal{F}$.*

A subgroup $H$ is called $Q$-supplemented in a group $G$, if there exists a subgroup $K$ of $G$ such that $G = HK$ and $H \cap K$ is contained in $H_{QG}$, where $H_{QG}$ is the maximal quasinormal subgroup of $G$ contained in $H$.

**Corollary 12** ([21, Theorem 3.6]). *Let $\mathcal{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that $G$ is a group with a solvable normal subgroup $E$ such that $G/E \in \mathcal{F}$. If every maximal subgroup of each Sylow subgroup of $F(E)$ is $Q$-supplemented in $G$, then $G \in \mathcal{F}$.*

**Corollary 13** ([34, Theorem 3]). *Let $\mathcal{F}$ be a saturated formation containing $\mathfrak{U}$. If there is a normal subgroup $E$ of $G$ such that $G/E \in \mathcal{F}$ and every cyclic subgroup of $E$ with prime order or order* 4 *is $s$-semipermutable in $G$, then $G \in \mathcal{F}$.*

**Corollary 14** ([29, Theorem 4.2]). *If every cyclic subgroup of a group $G$ with prime order or order* 4 *is $c$-normal in $G$, then $G$ is supersolvable.*

**Corollary 15** ([35, Theorem 3.1]). *Let $G$ be a group and $E$ a normal subgroup of a group $G$ such that $G/E$ is supersolvable. If every minimal subgroup of $E$ is $c$-supplemented in $G$ and if every cyclic subgroup of $E$ with order* 4 *is $c$-normal in $G$, then $G$ is supersolvable.*

**Corollary 16** ([7, Theorem 4.1]). *If every cyclic subgroup of $G^{\mathfrak{U}}$ with prime order or order* 4 *is $c$-supplemented in $G$, then $G$ is supersolvable.*

**Corollary 17** ([23, Theorem 3.9]). *Let $\mathcal{F}$ be a saturated formation containing $\mathfrak{U}$. Then $G \in \mathcal{F}$ if and only if there is a normal subgroup $E$ of $G$ such that $G/E \in \mathcal{F}$ and the subgroups prime order or order* 4 *of $E$ with are $c$-normal in $G$.*

**Corollary 18** ([2, Theorem 1]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If there is a normal subgroup $E$ of $G$ such that $G/E \in \mathcal{F}$ and every cyclic subgroup of $E$ with prime order or order 4 is $S$-quasinormal in $G$, then $G \in \mathcal{F}$.*

**Corollary 19** ([6, Theorem 3.4]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If every cyclic subgroup of $G^{\mathcal{F}}$ with prime order or order 4 is $c$-normal in $G$, then $G \in \mathcal{F}$.*

**Corollary 20** ([31, Theorem 3.2]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let $G$ be a group. If there is a normal subgroup $E$ such that $G/E \in \mathcal{F}$ and the subgroups of $F^*(E)$ with prime order or order 4 are $c$-normal in $G$, then $G \in \mathcal{F}$.*

**Corollary 21** ([18, Theorem 3.3]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let $G$ be a group. If there is a normal subgroup $E$ such that $G/E \in \mathcal{F}$ and the subgroups of $F^*(E)$ with prime order or order 4 are $S$-quasinormal in $G$, then $G \in \mathcal{F}$.*

**Corollary 22** ([32, Theorem 1.2]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let $G$ be a group. If there is a normal subgroup $E$ such that $G/E \in \mathcal{F}$ and the subgroups of $F^*(E)$ with prime order or order 4 are $c$-supplemented in $G$, then $G \in \mathcal{F}$.*

**Corollary 23** ([3, Corollary 1]). *Suppose that $G$ is a group with a normal solvable subgroup $E$ such that $G/E$ is supersolvable. If every subgroup of $F(E)$ of prime order or order 4 is $S$-quasinormal in $G$, then $G$ is supersolvable.*

**Corollary 24** ([15, Theorem 3]). *Let $G$ be a group and $E$ a solvable normal subgroup of $G$ such that $G/E$ is supersolvable. If all minimal subgroups and all cyclic subgroups of $F(E)$ with order 4 are $c$-normal in $G$, then $G$ is supersolvable.*

**Corollary 25** ([28, Theorem 4.1]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a solvable normal subgroup $E$ such that $G/E \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups of $F(E)$ with order 4 is $c$-supplemented in $G$, then $G \in \mathcal{F}$.*

**Corollary 26** ([30, Theorem 2]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a solvable normal subgroup $E$ such that $G/E \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups of $F(E)$ with order 4 is $c$-normal in $G$, then $G \in \mathcal{F}$.*

**Corollary 27** ([17, Theorem 3]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. A group $G \in \mathcal{F}$ if and only if there is a solvable normal subgroup $E$ of $G$ such that $G/E \in \mathcal{F}$ and the subgroups of $F(E)$ with prime order or order 4 is $c$-normal in $G$.*

**Corollary 28** ([3, Theorem ]). *A group $G \in \mathcal{F}$ if and only if there is a solvable normal subgroup $E$ of $G$ such that $G/E \in \mathcal{F}$ and the subgroups of $F(E)$ with prime order or order 4 is $S$-quasinormal in $G$.*

**Corollary 29** ([34, Theorem 4]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a solvable normal subgroup $E$ such that $G/E \in \mathcal{F}$. If all minimal subgroups and all cyclic subgroups of $F(E)$ with order 4 is s-semipermutable in $G$, then $G \in \mathcal{F}$.*

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