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REMARKS ON A CONJECTURE ABOUT RANDIĆ INDEX AND GRAPH RADIUS

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Abstract. Let G be a nontrivial connected graph. The radius $r(G)$ of G is the minimum eccentricity among eccentricities of all vertices in G. The Randić index of G is defined as $R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}}$, and the Harmonic index is defined as $H(G) = \sum_{uv \in E(G)} \frac{2}{d_G(u)+d_G(v)}$, where $d_G(x)$ is the degree of the vertex x in G. In 1988, Fajtlowicz conjectured that for any connected graph G, $R(G) \geq r(G) - 1$. This conjecture remains still open so far. More recently, Deng et al. proved that this conjecture is true for connected graphs with cyclomatic number no more than 4 by means of Harmonic index. In this short paper, we use a class of composite graphs to construct infinite classes of connected graphs, with cyclomatic number greater than 4, for which the above conjecture holds. In particular, for any given positive odd number $k \geq 7$, we construct a connected graph with cyclomatic number k such that the above conjecture holds for this graph.

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1. INTRODUCTION

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. For a graph G, we let $d_G(v)$ be the degree of a vertex v in G and $d_G(u, v)$ be the distance between two vertices u and v in G. Other notation and terminology not defined here will conform to those in [3].

Topological indices are numerical parameters of a graph which characterize the topological structure of the graph. Topological indices are usually graph invariants associated with a graph. As one of the most well-known and successful topological indices, *Randić index*, defined as $R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}}$, was introduced by Randić in [15]. Randić index gained much popularity during the past decades. Since its appearance, tremendous attention has been focused on the upper and lower bounds of this index. Bollobás and Erdös [2] proved that the Randić index of a graph of order

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n without isolated vertices is bounded from below by $\sqrt{n-1}$; they left as an open problem to determine the minimum value of the Randić index for a graph G with given minimum degree $\delta(G)$. Delorme et al. [5] gave a partial solution to this open problem for the case of $\delta(G)=2$. Furthermore, they completely solved this open problem for the case when G is a triangle-free graph with given minimum degree $\delta(G)$. Balister et al. [1] determined the maximal Randić index of a tree with a given number of vertices and leaves. About reviews of mathematical properties of the Randić index, the interested readers are referred to [12, 13]. Fajtlowicz [10] and Caporossi and Hansen [4] conjectured that the Randić index can be bounded from below in terms of the graph radius. In [8], Deng et al. gave a partial solution to this conjecture by studying the relationship between the Harmonic index and graph radius.

The *Harmonic index* of a connected graph G is defined by Fajtlowicz [9] as $H(G)=\sum\limits_{uv\in E(G)} \frac{2}{d_G(u)+d_G(v)}$. Favaron et al. [11] considered the relationship between the Harmonic index and graph eigenvalues. Zhong [17] determined the minimum and maximum values of the Harmonic index for simple connected graphs and trees, and characterized the corresponding extremal graphs. Deng et al. [6] considered the relationship between the Harmonic index $H(G)$ and the chromatic number $\chi(G)$ and proved that $\chi(G)\le 2H(G)$. This result strengthens a conjecture about the Randić index and the chromatic number which is based on the system AutoGraphiX. Deng et al. [7] gave a best possible lower bound for the Harmonic index of a graph and a triangle-free graph with minimum degree no less than two and characterized the corresponding extremal graphs, respectively.

In [10], Fajtlowicz proposed a conjecture concerning the relationship between the Randić index and graph radius, which reads as follows.

Conjecture 1 ([10]). *For any connected G, $R(G)\ge r(G)-1$.*

Caporossi and Hansen [4] partially proved this conjecture by showing that $R(T)\ge r(T)+\sqrt{2}-\frac{3}{2}$ for any tree T. Liu and Gutman [14], and You and Liu [16] proved that the conjecture is true for unicyclic, bicyclic and tricyclic graphs.

In [8], Deng et al. proved the following result.

Theorem 1 ([8]). *Let G be a graph with cyclomatic number $k\ge 1$. Then $H(G)\ge r(G)-\frac{31}{105}(k-1)$. In particular, $H(G)>r(G)-1$ for a graph with cyclomatic number no more than 4.*

Remark 1. For any connected graph G, since

$$R(G)=\sum_{uv\in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}} \ge \sum_{uv\in E(G)} \frac{2}{d_G(u)+d_G(v)} = H(G)$$

and by Theorem 1, $H(G) > r(G)-1$ for a connected graph G with cyclomatic number no more than 4, Conjecture 1 holds for those connected graphs G with cyclomatic number no more than 4.

In this short paper, we use a class of composite graphs to construct infinite classes of connected graphs, with cyclomatic number greater than 4, for which Conjecture 1 holds. In particular, for any given positive odd number $k \geq 7$, we can construct a connected graph with cyclomatic number being k such that the above conjecture holds for this graph. Consequently, our results extend those of Deng et al. in [6].

2. Main results

We first introduce a class of composite graphs, with which we are able to construct our desired graphs for which Conjecture 1 holds. The *double graph* G^* of a given graph G is constructed by making two copies of (including the initial edge set of each) and adding edges u_1v_2 and u_2v_1 for every edge uv of G.

For each vertex u in G, we call the corresponding vertices u_1 and u_2, in G^*, the *clone vertices* of u. If an n-vertex connected graph G has a vertex of degree $n-1$, then G is said to be *well-connected*.

Concerning the Harmonic index and radius of double graphs, we have the following result.

Theorem 2. *Let G be a nontrivial connected graph of order n, and let G^* be its double graph. We have*

(a) *If G is not well-connected, then $r(G^*) = r(G)$, and if G is well-connected, then $r(G^*) = r(G)+1$;*
(b) $H(G^*) = 2H(G)$.

Proof. It can be easily seen that if G is connected, then G^* is also connected. For the sake of convenience, we label all vertices of G as $v_1, \ldots, v_n$. Suppose that x_i and y_i are the corresponding clone vertices, in G^*, of v_i for each $i = 1, \ldots, n$. Given a vertex v_i in G. According to the definition of double graph, for any vertex v_j, different from v_i, in G, we have

$$d_{G^*}(x_i, x_j) = d_{G^*}(x_i, y_j) = d_{G^*}(y_i, x_j) = d_{G^*}(y_i, y_j) = d_G(v_i, v_j). \tag{2.1}$$

Moreover, we have $d_{G^*}(x_i) = d_{G^*}(y_i) = 2d_G(v_i)$ for $i = 1, \ldots, n$. Furthermore, we have

$$d_{G^*}(x_i, y_i) = 2 \tag{2.2}$$

for $i = 1, \ldots, n$, since there exists at least one vertex, say x_k (or y_k), such that both x_i and y_i are adjacent to x_k (or y_k).

If G is not well-connected, we have $ec_G(x_i) \geq 2$ for $i = 1, \ldots, n$, that is, $r(G) \geq 2$. Combining this fact and Eq.s (2.1) and (2.2), we have $ec_{G^*}(x_i) = ec_G(v_i) = ec_{G^*}(y_i)$ for any $i = 1, \ldots, n$, i.e., $r(G) = r(G^*)$. If G is well-connected, then G has a vertex, say v_i, of degree $n-1$. Thus, we have $ec_G(v_i) = 1$ and $r(G) = 1$. By

Eq. (2.2), we have $r(G^*) = ec_{G^*}(x_i) = d_{G^*}(x_i, y_i) = 2 = r(G)+1$. This proves (a).

Now, we prove (b). Let $h_G(uv) = \frac{2}{d_G(u)+d_G(v)}$. Then $H(G) = \sum_{uv\in E(G)} h_G(uv)$. For $1 \le i, j \le n$ and $i \ne j$,

$$h_{G^*}(x_i x_j) = \frac{2}{d_{G^*}(x_i)+d_{G^*}(x_j)} = \frac{1}{d_G(v_i)+d_G(v_j)} = \frac{1}{2} h_G(v_i v_j). \tag{2.3}$$

By symmetry, we have

$$h_{G^*}(x_i x_j) = h_{G^*}(x_i y_j) = h_{G^*}(y_i x_j) = h_{G^*}(y_i y_j) \tag{2.4}$$

for each $i, j = 1, \ldots, n$ and $i \ne j$. By means of Eq.s (2.3) and (2.4),

$$\begin{aligned} H(G^*) &= 4 \sum_{x_i x_j \in E(G^*)} h_{G^*}(x_i x_j) \\ &= 4 \sum_{v_i v_j \in E(G)} \frac{1}{2} h_G(v_i v_j) \\ &= 2H(G), \end{aligned}$$

which proves (b). □

We use Theorem 2 to prove the following consequence.

Theorem 3. *There exist infinite class of connected graphs, with cyclomatic number greater than 4, for which Conjecture 1 holds. In particular, for any given positive odd number $k \ge 7$, there exists a connected graph with cyclomatic number being k such that Conjecture 1 holds for this graph.*

Proof. We first take a connected graph G of order $n \ge 4$ and size m such that it is not well-connected and its cyclomatic number is less than or equal to 4. By Theorem 1, $H(G) > r(G)-1$.

Now, we construct the double graph G^* of G. Since G is not well-connected, then by Theorem 2, $H(G^*) = 2H(G) > 2(r(G)-1) = 2r(G^*)-2 > r(G^*)-1$, as $r(G^*) \ge 2$. Thus, $R(G^*) \ge H(G^*) > r(G^*)-1$. Moreover, we can find that the cyclomatic number of G^* is $|E(G^*)|-|V(G^*)|+1 = 4m-2n+1 \ge 4(n-1)-2n+1 = 2n-3 \ge 5$ for $n \ge 4$.

For a graph G, we now define its k*-th iterated double graph* G^{k*} as

$$G^{1*} = G^* \text{ and } G^{k*} = (G^{(k-1)*})^* \text{ for } k \ge 1$$

and $G^{0*} = G$ for consistence.

Note that G^* is also not well-connected. Repeatedly using Theorem 2, we can prove that

$$R(G^{k*}) \ge H(G^{k*}) = 2^k H(G) > 2^k(r(G)-1) = 2^k(r(G^{k*})-1) > r(G^{k*})-1.$$

Similarly, the cyclomatic number of G^{k*} is $|E(G^{k*})|-|V(G^{k*})|+1 = 4^k m - 2^k n + 1 \geq 4^k(n-1) - 2^k n + 1 = 2^k(2^k-1)n - 4^k + 1 = 2^k[2^k(n-1)-n]+1 \geq 2n-3 \geq 5$ for $k \geq 1$ and $n \geq 4$.

Now, for a given positive odd number $k = 2k^{'}+1 \geq 7$, we construct a connected graph with cyclomatic number being k. In fact, we first take a graph G with $k^{'}$ vertices and $k^{'}$ edges. Let G^* be the double graph of G. Then the cyclomatic number of G^* is equal to $|E(G^*)|-|V(G^*)|+1 = 2k^{'}+1 = k$. By the previous proof, we know that $R(G^*) \geq H(G^*) > r(G^*)-1$. This completes the proof. □

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