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INEQUALITIES FOR MAPPINGS WHOSE SECOND DERIVATIVES ARE QUASI-CONVEX OR h -CONVEX FUNCTIONS

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Abstract. In this paper, we establish some new integral inequalities for quasi-convex and h -convex functions by using a kernel and fairly elementary analysis.

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1. INTRODUCTION

One of the most famous inequality for convex functions is so called Hermite-Hadamard's inequality as follows: Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$

be a convex function defined on the interval I of real numbers and $a, b \in I$, with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}.$$

For recent results, refinements, counterparts, generalizations and new *Hermite-Hadamard* type inequalities see [9], [2],[4] and [11].

Definition 1 (See [7]). A function $f : [a, b] \rightarrow \mathbb{R}$ is said quasi-convex on $[a, b]$ if

$$f(\lambda x + (1-\lambda)y) \leq \sup\{f(x), f(y)\}, \quad (QC)$$

holds for all $x, y \in [a, b]$ and $\lambda \in [0, 1]$.

Definition 2 (see [10]). Let $h : J \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative function. We say that $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is an h -convex function or that f belongs to the class $SX(h, I)$, if f is nonnegative and for all $x, y \in I$ and $\alpha \in [0, 1]$ we have

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha) f(x) + h(1-\alpha) f(y).$$

In recent years, several authors have been studied on integral inequalities. One of the well known of these inequalities is Simpson's inequality as following:

Theorem 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a four times continuously differentiable mapping on (a, b) and $\|f^{(4)}\|_{\infty} = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$. Then, the following inequality holds:

$$\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_{\infty} (b-a)^4.$$

For recent refinements, counterparts, generalizations and new Simpson's type inequalities, see the papers [8], [1], [3], [5].

In [6], Özdemir *et al.* proved the following lemma:

Lemma 1. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L_1[a, b]$, where $a, b \in I$ with $a < b$ and $r \in \mathbb{R}^+$ then the following equality holds:

$$\begin{aligned} & \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)} \int_a^b f(x) dx \\ &= (b-a)^2 \int_0^1 k(t) f''(tb + (1-t)a) dt \end{aligned} \quad (1.1)$$

where

$$k(t) = \begin{cases} \frac{t}{r} \left(\frac{1}{r+1} - t \right), & t \in \left[0, \frac{1}{2} \right) \\ (1-t) \left(\frac{t}{r} - \frac{1}{r+1} \right), & t \in \left[\frac{1}{2}, 1 \right] \end{cases}.$$

In this article, by using functions whose second derivatives' absolute values are quasi-convex or h -convex, we obtained new inequalities related to the left hand side of Simpson inequality. Also new inequalities are proved.

2. RESULTS FOR QUASI-CONVEX FUNCTIONS

Theorem 2. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L_1[a, b]$ and $|f''|^q$ is quasi-convex, where $a, b \in I$ with $a < b$, $r \geq 1$ and $q \geq 1$, then one has the following inequality;

$$\begin{aligned} & \left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)} \int_a^b f(x) dx \right| \\ & \leq \frac{r^3 - 3r + 6}{12r(r+1)^3} (b-a)^2 \left(\sup \left\{ |f''(a)|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}}. \end{aligned} \quad (2.1)$$

Proof. Suppose that $q = 1$, from Lemma 1 and by using quasi-convexity of $|f''|$, we obtain

$$\begin{aligned} & \left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r} \int_0^1 f(tb + (1-t)a) dt \right| \\ & \leq (b-a)^2 \left\{ \int_0^{\frac{1}{2}} \left| \frac{t}{r} \left(\frac{1}{r+1} - t \right) \right| |f''(tb + (1-t)a)| dt \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 \left| (1-t) \left(\frac{t}{r} - \frac{1}{r+1} \right) \right| |f''(tb + (1-t)a)| dt \right\} \\ & \leq (b-a)^2 \left\{ \int_0^{\frac{1}{2}} \left| \frac{t}{r} \left(\frac{1}{r+1} - t \right) \right| \sup\{|f''(a)|, |f''(b)|\} dt \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 \left| (1-t) \left(\frac{t}{r} - \frac{1}{r+1} \right) \right| \sup\{|f''(a)|, |f''(b)|\} dt \right\} \\ & = \frac{r^3 - 3r + 6}{12r(r+1)^3} (b-a)^2 \sup\{|f''(a)|, |f''(b)|\}. \end{aligned}$$

Now suppose that $q > 1$, from Lemma 1 we have

$$\begin{aligned} & \left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r} \int_0^1 f(tb + (1-t)a) dt \right| \\ & \leq (b-a)^2 \int_0^1 |k(t)| |f''(tb + (1-t)a)| dt \end{aligned}$$

By using Power mean inequality we get

$$\begin{aligned} & \left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r} \int_0^1 f(tb + (1-t)a) dt \right| \\ & \leq (b-a)^2 \left(\int_0^1 |k(t)| dt \right)^{1-\frac{1}{q}} \left(\int_0^1 |k(t)| |f''(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f''|^q$ is quasi-convex, we have

$$\begin{aligned} & \left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r} \int_0^1 f(tb + (1-t)a) dt \right| \\ & \leq \frac{r^3 - 3r + 6}{12r(r+1)^3} (b-a)^2 \left(\sup \left\{ |f''(a)|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}}. \end{aligned}$$

This completes the proof. \square

Corollary 1. *Under the assumptions of Theorem 2, if we choose $r = 1$ in (2.1), we have the following inequality:*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{24} \left(\sup \left\{ |f''(a)|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}}. \end{aligned}$$

Corollary 2. *If we take $q = 1$ and $r = 1$ in (2.1), we obtain an inequality which includes the left-hand side of the Corollary 3 in [6]:*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{24} \sup \left\{ |f''(a)|, |f''(b)| \right\} \\ & \leq \frac{(b-a)^2}{24} (|f''(a)| + |f''(b)|) \end{aligned}$$

Corollary 3. *If we choose $r = 2$ in (2.1), we obtain*

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{81} \left(\sup \left\{ |f''(a)|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}} \\ & \leq \frac{(b-a)^2}{81} \left(|f''(a)|^q + |f''(b)|^q \right)^{\frac{1}{q}} \end{aligned}$$

Let $a_1 = |f''(a)|^q$, $b_1 = |f''(b)|^q$. Here $0 < \frac{1}{q} < 1$ for $q > 1$, using the fact that

$$\sum_{k=1}^n (a_k + b_k)^s \leq \sum_{k=1}^n (a_k)^s + \sum_{k=1}^n (b_k)^s \text{ for } 0 \leq s \leq 1, a_1, a_2, \dots, a_n \geq 0,$$

$b_1, b_2, \dots, b_n \geq 0$, we get

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{81} (|f''(a)| + |f''(b)|). \end{aligned}$$

Theorem 3. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L_1[a, b]$. If $|f''|^p$ is quasi-convex on $[a, b]$, where $a, b \in I$ with $a < b$, $r \geq 1$ and $p > 1$, then the following inequality hold

$$\begin{aligned} & \left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)} \int_a^b f(x) dx \right| \quad (2.2) \\ & \leq \frac{(b-a)^2}{4} \left(\sup \{ |f''(a)|^p, |f''(b)|^p \} \right)^{\frac{1}{p}} \\ & \times \left(\frac{1}{2} \right)^{\frac{1}{p}} \left(\frac{1}{r(r+1)^{\frac{q+2}{q}}} \right) \left\{ \frac{(r-1)^{q+1} [2(q+2) - (r-1)(q+1) + 2^{2+q}]}{(q+1)(q+2)} \right\}^{\frac{1}{q}} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By using Lemma 1 and triangle inequality, we have

$$\begin{aligned} & \left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)} \int_a^b f(x) dx \right| \\ & \leq (b-a)^2 \left\{ \left| \int_0^{\frac{1}{2}} \frac{t}{r} \left(\frac{1}{r+1} - t \right) f''(tb + (1-t)a) dt \right| \right. \\ & \quad \left. + \left| \int_{\frac{1}{2}}^1 (1-t) \left(\frac{t}{r} - \frac{1}{r+1} \right) f''(tb + (1-t)a) dt \right| \right\} \\ & = (b-a)^2 \left\{ \left| \int_0^{\frac{1}{2}} f''(tb + (1-t)a) \left(\frac{1}{r+1} - t \right) \frac{t}{r} dt \right| \right. \end{aligned}$$

$$+ \left\{ \int_{\frac{1}{2}}^1 f''(tb + (1-t)a) \left(\frac{t}{r} - \frac{1}{r+1} \right) (1-t) dt \right\}.$$

By using weighted version of Hölder's inequality which is described as

$$\left| \int_I f(s)g(s)h(s)ds \right| \leq \left(\int_I |f(s)|^p h(s)ds \right)^{\frac{1}{p}} \left(\int_I |g(s)|^q h(s)ds \right)^{\frac{1}{q}}$$

for $p > 1$, $p^{-1} + q^{-1} = 1$, h is non-negative on I and provided all the other integrals exist and are finite; we have

$$\begin{aligned} & \left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)} \int_a^b f(x) dx \right| \\ & \leq (b-a)^2 \\ & \times \left\{ \left(\int_0^{\frac{1}{2}} |f''(tb + (1-t)a)|^p \frac{t}{r} dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left| \frac{1}{r+1} - t \right|^q \frac{t}{r} dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_{\frac{1}{2}}^1 |f''(tb + (1-t)a)|^p (1-t) dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left| \frac{t}{r} - \frac{1}{r+1} \right|^q (1-t) dt \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

By using the quasi-convexity of $|f''|^p$, we get

$$\begin{aligned} & \left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)} \int_a^b f(x) dx \right| \\ & \leq (b-a)^2 \left(\sup \{ |f''(a)|^p, |f''(b)|^p \} \right)^{\frac{1}{p}} \\ & \times \left\{ \left(\int_0^{\frac{1}{2}} \frac{t}{r} dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left| \frac{1}{r+1} - t \right|^q \frac{t}{r} dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_{\frac{1}{2}}^1 (1-t) dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left| \frac{t}{r} - \frac{1}{r+1} \right|^q (1-t) dt \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

$$\begin{aligned}
 & + \left(\int_{\frac{1}{2}}^1 (1-t) dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left| \frac{t}{r} - \frac{1}{r+1} \right|^q (1-t) dt \right)^{\frac{1}{q}} \Bigg\} \\
 & = \frac{(b-a)^2}{4} \left(\sup \{ |f''(a)|^p, |f''(b)|^p \} \right)^{\frac{1}{p}} \\
 & \times \left(\frac{1}{2} \right)^{\frac{1}{p}} \left(\frac{1}{r(r+1)^{\frac{q+2}{q}}} \right) \left\{ \frac{(r-1)^{q+1} [2(q+2) - (r-1)(q+1) + 2^{2+q}]}{(q+1)(q+2)} \right\}^{\frac{1}{q}}.
 \end{aligned}$$

So the proof is complete. □

Corollary 4. *Under the assumptions of Theorem 3, if we choose $r = 1$ in the inequality (2.2), we have the following inequalities:*

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{(b-a)^2}{16} \left(\frac{1}{2} \right)^{\frac{1}{q}} \left(\sup \{ |f''(a)|^p, |f''(b)|^p \} \right)^{\frac{1}{p}} \\
 & \leq \frac{(b-a)^2}{16} \left(\sup \{ |f''(a)|^p, |f''(b)|^p \} \right)^{\frac{1}{p}}
 \end{aligned}$$

where $\left(\frac{1}{2}\right)^{\frac{1}{q}} \leq 1, q \in (1, \infty)$.

Corollary 5. *If we choose $r = 2$ in (2.2), we obtain*

$$\begin{aligned}
 & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{(b-a)^2}{24} \left(\frac{1}{2} \right)^{\frac{1}{p}} \left(\frac{1}{3} \right)^{\frac{2}{q}} \left(\frac{(q+3) + 2^{q+2}}{(q+1)(q+2)} \right) \left(\sup \{ |f''(a)|^p, |f''(b)|^p \} \right)^{\frac{1}{p}}.
 \end{aligned}$$

3. RESULTS FOR h -CONVEX FUNCTIONS

Theorem 4. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L_1[a, b]$ and $|f''|$ is h -convex, where $a, b \in I$ with $a < b$ and $r \geq 1$, then one has the following inequality;*

$$\left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)} \int_a^b f(x) dx \right| \tag{3.1}$$

$$\leq (b-a)^2 K(h,r) (|f''(a)| + |f''(b)|)$$

where

$$K(h,r) = \frac{3r^2 - 9r + 8}{480r^2(r+1)^2} + \int_0^1 h^2(t) dt.$$

Proof. From Lemma 1 and properties of absolute value, we have

$$\begin{aligned} & \left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)} \int_a^b f(x) dx \right| \\ & \leq (b-a)^2 \int_0^1 |k(t)| |f''(tb + (1-t)a)| dt \\ & = (b-a)^2 \left\{ \int_0^{\frac{1}{2}} \left| \frac{t}{r} \left(\frac{1}{r+1} - t \right) \right| |f''(tb + (1-t)a)| dt \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 \left| (1-t) \left(\frac{t}{r} - \frac{1}{r+1} \right) \right| |f''(tb + (1-t)a)| dt \right\}. \end{aligned}$$

Since $|f''|$ is h -convex, we get

$$\begin{aligned} & \left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)} \int_a^b f(x) dx \right| \\ & \leq (b-a)^2 \left[\int_0^{\frac{1}{2}} \left| \frac{t}{r} \left(\frac{1}{r+1} - t \right) \right| |h(t)f''(b) + h(1-t)f''(a)| dt \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 \left| (1-t) \left(\frac{t}{r} - \frac{1}{r+1} \right) \right| |h(t)f''(b) + h(1-t)f''(a)| dt \right] \\ & \leq (b-a)^2 \left[\int_0^{\frac{1}{2}} \left| \frac{t}{r} \left(\frac{1}{r+1} - t \right) \right| \{h(t)|f''(b)| + h(1-t)|f''(a)|\} dt \right. \\ & \quad \left. + \int_{\frac{1}{2}}^1 \left| (1-t) \left(\frac{t}{r} - \frac{1}{r+1} \right) \right| \{h(t)|f''(b)| + h(1-t)|f''(a)|\} dt \right] \end{aligned}$$

$$\begin{aligned}
 & + \int_{\frac{1}{2}}^1 \left| (1-t) \left(\frac{t}{r} - \frac{1}{r+1} \right) \right| \{ h(t) |f''(b)| + h(1-t) |f''(a)| \} dt \\
 & = (b-a)^2 [J_1 + J_2].
 \end{aligned} \tag{3.2}$$

So we can write

$$\begin{aligned}
 J_1 & = \int_0^{\frac{1}{2}} \left| \frac{t}{r} \left(\frac{1}{r+1} - t \right) \right| \{ h(t) |f''(b)| + h(1-t) |f''(a)| \} dt \\
 & = |f''(a)| \int_0^{\frac{1}{2}} \left| \frac{t}{r} \left(\frac{1}{r+1} - t \right) \right| h(1-t) dt + |f''(b)| \int_0^{\frac{1}{2}} \left| \frac{t}{r} \left(\frac{1}{r+1} - t \right) \right| h(t) dt.
 \end{aligned}$$

Since $cd \leq \frac{1}{2}(c^2 + d^2)$ for $c, d \in \mathbb{R}$ we have

$$\begin{aligned}
 J_1 & \leq |f''(a)| \left\{ \frac{1}{2} \left[\int_0^{\frac{1}{2}} \left(\frac{t}{r} \left(\frac{1}{r+1} - t \right) \right)^2 dt + \int_0^{\frac{1}{2}} h^2(1-t) dt \right] \right\} \\
 & + |f''(b)| \left\{ \frac{1}{2} \left[\int_0^{\frac{1}{2}} \left(\frac{t}{r} \left(\frac{1}{r+1} - t \right) \right)^2 dt + \int_0^{\frac{1}{2}} h^2(t) dt \right] \right\} \\
 & = |f''(a)| \left\{ \frac{1}{2} \left[\frac{3r^2 - 9r + 8}{480r^2(r+1)^2} + \int_{\frac{1}{2}}^1 h^2(t) dt \right] \right\} \\
 & + |f''(b)| \left\{ \frac{1}{2} \left[\frac{3r^2 - 9r + 8}{480r^2(r+1)^2} + \int_0^{\frac{1}{2}} h^2(t) dt \right] \right\}.
 \end{aligned} \tag{3.3}$$

By similar calculation we get

$$J_2 \leq |f''(a)| \left\{ \frac{1}{2} \left[\frac{3r^2 - 9r + 8}{480r^2(r+1)^2} + \int_0^{\frac{1}{2}} h^2(t) dt \right] \right\} \tag{3.4}$$

$$+ |f''(b)| \left\{ \frac{1}{2} \left[\frac{3r^2 - 9r + 8}{480r^2(r+1)^2} + \int_{\frac{1}{2}}^1 h^2(t) dt \right] \right\}.$$

If we use the inequalities (3.3) and (3.4) in (3.2) we get the desired result. \square

Corollary 6. *In the inequality (3.1), if we choose $h(t) = 1$ and $r = 1$, we get*

$$\left| \frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{37.13}{25.3.5} (b-a)^2 \left(\frac{|f''(a)| + |f''(b)|}{2} \right).$$

Corollary 7. *In Theorem 6, if we choose $h(t) = t$ and $r = 2$ we have*

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ \leq \frac{11.131}{25.33.5} (b-a)^2 \left(\frac{|f''(a)| + |f''(b)|}{2} \right).$$

Theorem 5. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L_1[a, b]$. If $|f''|^p$ is h -convex on $[a, b]$, where $a, b \in I$ with $a < b$, $r \geq 1$ and $p > 1$, then the following inequality hold*

$$\left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)} \int_a^b f(x) dx \right| \quad (3.5) \\ \leq \frac{(b-a)^2}{4r} \left\{ \frac{2^{q+2} + (r-1)^{q+1} (qr + q + r + 3)}{2(r+1)^{q+2} (q+1)(q+2)} \right\}^{\frac{1}{q}} \\ \times \left\{ \left[|f''(b)|^p A(h,t) + |f''(a)|^p B(h,t) \right]^{\frac{1}{p}} \right. \\ \left. + \left[|f''(b)|^p B(h,t) + |f''(a)|^p A(h,t) \right]^{\frac{1}{p}} \right\}$$

where $A(h,t) = \frac{1}{24} + \int_0^{\frac{1}{2}} h^2(t) dt$ and $B(h,t) = \frac{1}{24} + \int_{\frac{1}{2}}^1 h^2(t) dt$.

Proof. By using Lemma 1, triangle inequality and weighted version of Hölder's inequality we have

$$\begin{aligned} & \left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)} \int_a^b f(x) dx \right| \\ & \leq (b-a)^2 \\ & \times \left\{ \left(\int_0^{\frac{1}{2}} |f''(tb + (1-t)a)|^p \frac{t}{r} dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left| \frac{1}{r+1} - t \right|^q \frac{t}{r} dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_{\frac{1}{2}}^1 |f''(tb + (1-t)a)|^p (1-t) dt \right)^{\frac{1}{p}} \left(\int_{\frac{1}{2}}^1 \left| \frac{t}{r} - \frac{1}{r+1} \right|^q (1-t) dt \right)^{\frac{1}{q}} \right\} \end{aligned}$$

By using h -convexity of $|f''|^p$ we have

$$\begin{aligned} & \left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)} \int_a^b f(x) dx \right| \\ & \leq (b-a)^2 \\ & \times \left\{ \left(\int_0^{\frac{1}{2}} [h(t) |f''(b)|^p + h(1-t) |f''(a)|^p] \frac{t}{r} dt \right)^{\frac{1}{p}} \left(\int_0^{\frac{1}{2}} \left| \frac{1}{r+1} - t \right|^q \frac{t}{r} dt \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\int_{\frac{1}{2}}^1 [h(t) |f''(b)|^p + h(1-t) |f''(a)|^p] (1-t) dt \right)^{\frac{1}{p}} \right. \\ & \left. \left(\int_{\frac{1}{2}}^1 \left| \frac{t}{r} - \frac{1}{r+1} \right|^q (1-t) dt \right)^{\frac{1}{q}} \right\} \\ & = (b-a)^2 \end{aligned}$$

$$\begin{aligned} & \times \left\{ \left(\frac{|f''(b)|^p}{r} \int_0^{\frac{1}{2}} th(t) dt + \frac{|f''(a)|^p}{r} \int_0^{\frac{1}{2}} th(1-t) dt \right)^{\frac{1}{p}} \right. \\ & \quad \left(\int_0^{\frac{1}{2}} \left| \frac{1}{r+1} - t \right|^q t dt \right)^{\frac{1}{q}} \\ & \quad + \left(|f''(b)|^p \int_{\frac{1}{2}}^1 (1-t)h(t) dt + |f''(a)|^p \int_{\frac{1}{2}}^1 (1-t)h(1-t) dt \right)^{\frac{1}{p}} \\ & \quad \left. \left(\int_{\frac{1}{2}}^1 \left| \frac{t}{r} - \frac{1}{r+1} \right|^q (1-t) dt \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Since $cd \leq \frac{1}{2}(c^2 + d^2)$ for $c, d \in \mathbb{R}$ we have

$$\begin{aligned} & \left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)} \int_a^b f(x) dx \right| \\ & \leq (b-a)^2 \left\{ \left(\int_0^{\frac{1}{2}} \left| \frac{1}{r+1} - t \right|^q t dt \right)^{\frac{1}{q}} \right. \\ & \quad \times \left[\frac{|f''(b)|^p}{2r} \left(\int_0^{\frac{1}{2}} t^2 dt + \int_0^{\frac{1}{2}} h^2(t) dt \right) \right. \\ & \quad \left. \left. + \frac{|f''(a)|^p}{2r} \left(\int_0^{\frac{1}{2}} t^2 dt + \int_0^{\frac{1}{2}} h^2(1-t) dt \right) \right] \right\}^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}
 & + \left(\int_{\frac{1}{2}}^1 \left| \frac{t}{r} - \frac{1}{r+1} \right|^q (1-t) dt \right)^{\frac{1}{q}} \times \left[\frac{|f''(b)|^p}{2} \left(\int_{\frac{1}{2}}^1 (1-t)^2 dt + \int_{\frac{1}{2}}^1 h^2(t) dt \right) \right. \\
 & \left. + \frac{|f''(a)|^p}{2} \left(\int_{\frac{1}{2}}^1 (1-t)^2 dt + \int_{\frac{1}{2}}^1 h^2(1-t) dt \right) \right]^{\frac{1}{p}} \Bigg\}
 \end{aligned}$$

If we simplify the expression, we have

$$\begin{aligned}
 & \left| \frac{1}{r(r+1)} [f(a) + f(b)] + \frac{2}{r+1} f\left(\frac{a+b}{2}\right) - \frac{2}{r(b-a)} \int_a^b f(x) dx \right| \\
 & \leq \frac{(b-a)^2}{4r} \left\{ \frac{2^{q+2} + (r-1)^{q+1} (qr + q + r + 3)}{2(r+1)^{q+2} (q+1)(q+2)} \right\}^{\frac{1}{q}} \\
 & \times \left\{ \left[|f''(b)|^p \left(\frac{1}{24} + \int_0^{\frac{1}{2}} h^2(t) dt \right) + |f''(a)|^p \left(\frac{1}{24} + \int_{\frac{1}{2}}^1 h^2(t) dt \right) \right]^{\frac{1}{p}} \right. \\
 & \left. + \left[|f''(b)|^p \left(\frac{1}{24} + \int_{\frac{1}{2}}^1 h^2(t) dt \right) + |f''(a)|^p \left(\frac{1}{24} + \int_0^{\frac{1}{2}} h^2(t) dt \right) \right]^{\frac{1}{p}} \right\}
 \end{aligned}$$

then the proof is complete. □

Corollary 8. *If we choose $h(t) = 1, r = 1$ in the inequality (3.5) we get*

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{(b-a)^2}{4} \left\{ \frac{1}{(q+1)(q+2)} \right\}^{\frac{1}{q}} \left\{ \frac{13}{12} (|f''(a)|^p + |f''(b)|^p) \right\}^{\frac{1}{p}}.
 \end{aligned}$$

Corollary 9. *If we choose $h(t) = 1, r = 2$ in (3.5), we get*

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\leq \frac{(b-a)^2}{4} \left\{ \frac{2^{q+2} + (3q+5)}{2 \cdot 3^{q+2} (q+1)(q+2)} \right\}^{\frac{1}{q}} \left\{ \frac{13}{24} (|f''(a)|^p + |f''(b)|^p) \right\}^{\frac{1}{p}}.$$

Corollary 10. *If we choose $h(t) = t$, $r = 2$ in the inequality (3.5), we have*

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{8} \left\{ \frac{2^{q+2} + (3q+5)}{2 \cdot 3^{q+2} (q+1)(q+2)} \right\}^{\frac{1}{q}} \\ & \quad \times \left\{ \left(\frac{|f''(a)|^p}{12} + \frac{|f''(b)|^p}{3} \right)^{\frac{1}{p}} + \left(\frac{|f''(a)|^p}{3} + \frac{|f''(b)|^p}{12} \right)^{\frac{1}{p}} \right\}. \end{aligned}$$

Corollary 11. *In addition to the conditions given above, if we choose $p = q = 2$, we have*

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{48\sqrt{2}} \left\{ \left(\frac{|f''(a)|^2}{12} + \frac{|f''(b)|^2}{3} \right)^{\frac{1}{2}} + \left(\frac{|f''(a)|^2}{3} + \frac{|f''(b)|^2}{12} \right)^{\frac{1}{2}} \right\}. \end{aligned}$$

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