Miskolc Mathematical Notes
Vol. 14 (2013), No 1, pp. 41-47

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2013.599

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HYERS-ULAM STABILITY AND APPLICATIONS IN GAUGE SPACES

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Received 18 September, 2012

Abstract. Using the weakly Picard operator technique, we will present some Ulam- Hyers stability results for operatorial equations and some applications in gauge spaces.

2000 *Mathematics Subject Classification:* 47H10; 54H25

Keywords: Hyers-Ulam stability, weakly Picard operator, ψ-weakly Picard operator, fixed point, integral equation

1. INTRODUCTION

In 1959, G. Marinescu [10] extended the Banach Contraction Principle to locally convex spaces, while I. Colojoară [4] and N. Gheorghiu [7] to gauge spaces and R. J. Knill [9] to uniform spaces. In 1971, Cain and Nashed [3] extended the notion of contraction to Hausdorff locally convex linear spaces. They showed that on sequentially complete subset, the Banach Contraction Principle is still valid. V.G. Angelov [1] introduced the notion of generalized φ-contractive single-valued map in gauge spaces in 1987, meanwhile the concept for multivalued operators was given in 1998 (see V.G. Angelov [2]). In 2000, M. Frigon [6] introduced the notion of generalized contraction in gauge spaces and proved that every generalized contraction on a complete gauge space (sequentially complete gauge space) has a unique fixed point.

Definition 1. Let X be any set. A map $p : X \times X \to \mathbb{R}_+$ is called a pseudometric (or, a gauge) in X whenever

(1) $p(x,y) \geq 0$, for all $x, y \in X$;
(2) If $x = y$, then $p(x,y) = 0$;
(3) $p(x,y) = p(y,x)$, for all $x, y \in X$;
(4) $p(x,z) \leq p(x,y) + p(y,z)$, for every triple of point.

This work was possible with the financial support of the Faculty of Business, Babeş-Bolyai University Cluj-Napoca.

Definition 2. A family $\mathcal{P} = \{p_\alpha\}_{\alpha \in A}$ of pseudometrics on X (or a gauge structure on X), where A is a directed set, is said to be separating if for each pair of points $x, y \in X$, with $x \neq y$, there is a $p_\alpha \in \mathcal{P}$ such that $p_\alpha(x, y) \neq 0$.

A pair $(X, \mathcal{P})$ of a nonempty set X and a separating gauge structure $\mathcal{P}$ on X is called a gauge space.

It is well known (see Dugundji [5], pages 198-204) that any family $\mathcal{P}$ of pseudometrics on a set X induces on X a uniform structure $\mathcal{U}$ and conversely, any uniform structure $\mathcal{U}$ on X is induced by a family of pseudometrics on X. In addition, we have that $\mathcal{U}$ is separating (or Hausdorff) if and only if $\mathcal{P}$ is separating. Thus we may identify the gauge spaces and the Hausdorff uniform spaces.

A sequence $(x_n)_{n \in \mathbb{N}}$ of elements in X is said to be Cauchy if for every $\varepsilon > 0$ and $\alpha \in A$, there is an N with $p_\alpha(x_n, x_{n+p}) \leq \varepsilon$ for all $n \geq N$ and $p \in \mathbb{N}$.

The sequence $(x_n)_{n \in \mathbb{N}}$ is called convergent if there exists an $x_0 \in X$ such that for every $\varepsilon > 0$ and $\alpha \in A$, there is an N with $p_\alpha(x_0, x_n) \leq \varepsilon$ for all $n \geq N$.

Definition 3. A gauge space is called sequentially complete if any Cauchy sequence is convergent.

A subset of X is said to be sequentially closed if it contains the limit of any convergent sequence of its elements.

For further details see J. Dugundji [5], A. Granas, J. Dugundji [8].

Let X be a nonempty set an $f : X \to X$ be an operator. Then $x \in X$ is called fixed point for f if and only if $x = f(x)$. The set $Fix(f) := \{x \in X \mid x = f(x)\}$ is called the fixed point set of f.

Definition 4. Let $(X, \mathcal{P})$ be a gauge space and let $f : (X, \mathcal{P}) \to (X, \mathcal{P})$ be a single-valued operator. By definition, f is weakly Picard (briefly WPO) operator if the sequence of successive approximations $f^n(x)$ converges for all $x \in X$ and the limit (which may depend on X) is a fixed point of f.

If f is WPO, then we consider the operator $f^\infty : (X, (P)) \to (X, (P))$ defined by $f^\infty(x) = \lim\limits_{n \to \infty} f^n(x)$.

Definition 5. Let $(X, \mathcal{P})$ be a gauge space and let $f : (X, \mathcal{P}) \to (X, \mathcal{P})$ be a WPO and $\psi = \{\psi_\alpha\}_{\alpha \in A}$ be a family of mappings such that $\psi_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous in 0 and $\psi_\alpha(0) = 0$. By definition the operator f is ψ_α-WPO if

$$p_\alpha(x, f^\infty(x)) \leq \psi_\alpha(p_\alpha(x, f(x))), \text{ for all } x \in X, \alpha \in A.$$

If there exists $c = \{c_\alpha\}_{\alpha \in A} \in (0, \infty)^A$ such that $\psi_\alpha(t) := c_\alpha \cdot t$, for each $t \in \mathbb{R}_+$ and $\alpha \in A$ then the operator f is c_α-WPO.

For the theory of weakly Picard operators, see [11] for the single-valued case.

The purpose of this paper is to present some results concerning the Hyers-Ulam stability of some operatorial inclusions (such as the fixed point inclusion, the coincdence point equation or inclusion, etc.) in gauge spaces, using the weakly Picard operator technique.

2. Hyers-Ulam stability for fixed point equations

We will present first the concept of Hyers-Ulam stability in the setting of gauge spaces.

Definition 6. Let $(X,\mathcal{P})$ be a gauge space and let $f:(X,\mathcal{P})\to(X,\mathcal{P})$ be a single-valued operator. The fixed point equation

$$x = f(x),\ x\in X \tag{2.1}$$

is called generalized Hyers-Ulam stable if and only if there exists $\psi=\{\psi_\alpha\}_{\alpha\in A}$ a family of mappings, $\psi_\alpha:\mathbb{R}_+\to\mathbb{R}_+$ increasing, continuous in 0 and $\psi_\alpha(0)=0$ such that for each $\varepsilon=\{\varepsilon_\alpha\}_{\alpha\in A}\in(0,\infty)^A$ and for each solution y^* of the inequation

$$p_\alpha(y,f(y))\le\varepsilon_\alpha,\ \alpha\in A, \tag{2.2}$$

there exists a solution x^* of the fixed point equation (2.1) such that

$$p_\alpha(y^*,x^*)\le\psi_\alpha(\varepsilon_\alpha),\ \text{for all } \alpha\in A.$$

If there exists $c=\{c_\alpha\}_{\alpha\in A}\in(0,\infty)^A$ such that $\psi_\alpha(t):=c_\alpha\cdot t$, for each $t\in\mathbb{R}_+$ and $\alpha\in A$ then the fixed point equation (2.1) is said to be Hyers-Ulam stable.

We refer to [12] for the particular case of Hyers-Ulam stability in metric spaces. Our first abstract result is as follows.

Theorem 1. *Let $(X,\mathcal{P})$ be a gauge space and let $f:(X,\mathcal{P})\to(X,\mathcal{P})$ be a ψ_α-WPO. Then, the fixed point equation (2.1) is generalized Hyers-Ulam stable.*

Proof. Let $\varepsilon=\varepsilon_\alpha\in(0,\infty)^A$ and let $y^*\in f^\infty(x,y)$ be an ε-solution of (2.2), i.e., $p_\alpha(y^*,f(y^*))\le\varepsilon_\alpha$, for all $\alpha\in A$. Since f is a ψ_α-WPO, for each $x\in X$ and $\alpha\in A$ we have

$$p_\alpha(x,f^\infty(x)\le\psi_\alpha(p_\alpha(x,f(x))).$$

Then choosing $x^*=f^\infty(y^*)$ we have

$$p_\alpha(y^*,x^*)=p_\alpha(y^*,f^\infty(y^*))\le\psi_\alpha(p_\alpha(y^*,f(y^*)))\le\psi_\alpha(\varepsilon_\alpha).$$

Thus the fixed point equation (2.1) is generalized Hyers-Ulam stable. □

In 1974, Tarafdar [13] expressed the notion of contraction in Hausdorff uniform spaces, using the observation that a uniformity on X determines a family of gauges $\{p_\alpha\}$. A Hyers-Ulam stability result for the case of Tarafdar contraction in gauge spaces is as follows.

Theorem 2. *Let $(X,\mathcal{P})$ be a gauge space and let $f:(X,\mathcal{P})\to(X,\mathcal{P})$ be an a_α-contraction, i.e. for every $\alpha\in A$ there exists $a=\{a_\alpha\}_{\alpha\in A}\in(0,1)^A$ such that*

$$p_\alpha(f(x),f(y))\le a_\alpha\cdot p_\alpha(x,y),\ \text{for all } x,y\in X.$$

Then $F_f=\{x^*\}$ and the fixed point equation (2.1) is Hyers-Ulam stable.

Proof. From Tarafdar [13] we get that f has a unique fixed point $x^*\in X$ and, for each $x\in X$, we have that $f^n(x)\to x^*$. Thus, f is a Picard operator. Moreover, it is a c_α-WPO, with $c_\alpha:=\frac{1}{1-a_\alpha}$. Applying Theorem 1 we obtain the conclusion. □

An extension of the previous result concerns the case of graphic-contractions.

Theorem 3. *Let $(X,\mathcal{P})$ be a sequentially complete gauge space. Let $f:(X,\mathcal{P})\to(X,\mathcal{P})$ be an operator. If f is a graphic a_α-contraction, i.e., for every $\alpha\in A$ there exists $a=\{a_\alpha\}_{\alpha\in A}\in(0,1)^A$ such that*

$$p_\alpha(f^2(x),f(x))\le a_\alpha\cdot p_\alpha(x,f(x)),\ \text{for all } x\in X$$

and f has closed graph, then $F_f\neq\emptyset$ and the equation (2.1) is Hyers-Ulam stable.

Proof. Let $x_0\in X$ and $x_n\in f(x_{n-1})=f^n(x_0), n=1,2,\ldots$ If m and n are positive integers, $m<n$, then for each $\alpha\in A$ we have:

$$\begin{aligned}
p_\alpha(x_m,x_n) &= p_\alpha(f^m(x_0),f^n(x_0))\\
&\le p_\alpha(f^m(x_0),f^{m+1}(x_0))+p_\alpha(f^{m+1}(x_0),f^{m+2}(x_0))+\ldots\\
&\quad + p_\alpha(f^{n-1}(x_0),f^n(x_0))\\
&\le a_\alpha p_\alpha(f^{m-1}(x_0),f^m(x_0))+a_\alpha p_\alpha(f^m(x_0),f^{m+1}(x_0))+\ldots\\
&\quad + a_\alpha p_\alpha(f^{n-2}(x_0),f^{n-1}(x_0))\\
&\le a_\alpha^m p_\alpha(x_0,f(x_0))+a_\alpha^{m+1}p_\alpha(x_0,f(x_0))+\ldots+a_\alpha^{n-1}p_\alpha(x_0,f(x_0))\\
&= p_\alpha(x_0,f(x_0))a_\alpha^m(1+a_\alpha+\ldots+a_\alpha^{n-m+1})\\
&\le p_\alpha(x_0,f(x_0))a_\alpha^m\frac{1-a_\alpha^{n-m}}{1-a_\alpha}.
\end{aligned}$$

Hence the sequence (x_n) is Cauchy, therefore (x_n) converges to a point $x^*\in X$. From the continuity of f we get that x^* is a fixed point for f. So, we have

$$p_\alpha(x_m,x_n)\le p_\alpha(x_0,f(x_0))a_\alpha^m\frac{1-a_\alpha^{n-m}}{1-a_\alpha}.$$

If we choose in the above inequality $m=0$ and let $n\to\infty$ we obtain:

$$p_\alpha(x_0,x^*)\le p_\alpha(x_0,f(x_0))\frac{1}{1-a_\alpha},\ \text{for all } \alpha\in A.$$

Thus f is a c_α-WPO with $c_\alpha:=\frac{1}{1-a_\alpha}$. Therefore the second conclusion follows from Theorem 1. □

3. Applications

We will apply some of the above results to nonlinear integral equations on the real axis.

$$x(t) = \int_0^t K(t,s,x(s))ds + g(t),\ t \in \mathbb{R}_+. \tag{3.1}$$

We give the notion of Hyers-Ulam stability for the integral equation.

Definition 7. The integral equation (3.1) is called Hyers-Ulam stable if and only if there exists $c = \{c_\alpha\}_{\alpha\in A} \in (0,\infty)^A$ such that for each $\varepsilon = \{\varepsilon_\alpha\}_{\alpha\in A} \in (0,\infty)^A$ and for any ε-solution y^* of (1) (i.e., any $y^* \in C([0,\infty],\mathbb{R}^n)$ which satisfies the inequality

$$|y^*(t) - \int_0^t K(t,s,x(s))ds - g(t)| \le \varepsilon_\alpha,\ \text{for each } t \ge 0) \tag{3.2}$$

there exists a solution x^* of the equation (3.1) such that

$$|y^*(t) - x^*(t)| \le c_\alpha \cdot \varepsilon_\alpha,\ \text{for each } t \ge 0.$$

Theorem 4. *Consider equation (3.1). Suppose that:*
i) $K : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}_+ \to \mathbb{R}^n$ are continuous;
ii) there exists $k > 0$ such that

$$|K(t,s,u) - K(t,s,v)| \le k|u - v|,\ \textit{for each } t,s \in \mathbb{R}_+,\ u,v \in \mathbb{R}^n;$$

Then the integral equation (3.1) has a unique solution x^* in $C([0,+\infty),\mathbb{R}^n)$ and equation (3.1) is Hyers-Ulam stable.

Proof. Let $X := C([0,+\infty),\mathbb{R}^n)$ and the family of pseudo-norms

$$\|x\|_n := \max_{t\in[0,n]} |x(t)|e^{-\tau t},\ \text{where } \tau > 0.$$

Define now $d_n(x,y) := \|x - y\|_n$ for $x,y \in X$.

Then $\mathcal{P} := (d_n)_{n\in\mathbb{N}^*}$ is family of gauges on X. Then $(X,\mathcal{P})$ is a complete gauge space.

Define $A : C([0,+\infty),\mathbb{R}^n) \to C([0,+\infty),\mathbb{R}^n)$, by the formula

$$Ax(t) := \int_0^t K(t,s,x(s))ds + g(t),\ t \in \mathbb{R}_+.$$

For each $x,y \in X$ and for $t \in [0,n]$, we have successively:

$$|Ax(t) - Ay(t)| \le \int_0^t |K(t,s,x(s)) - K(t,s,y(s))|ds \le \int_0^t k|x(s) - y(s)|ds$$
$$= k\int_0^t |x(s) - y(s)|e^{-\tau s}e^{\tau s}ds \le k\int_0^t e^{\tau s}(|x(s) - y(s)|e^{-\tau s})ds$$
$$\le k d_n(x,y)\int_0^t e^{\tau s}ds \le \frac{k}{\tau} d_n(x,y)e^{\tau t}.$$

Hence, for $\tau > k$ and denoting $L := \frac{k}{\tau} < 1$ we obtain

$$d_n(Ax, Ay) \leq L d_n(x, y), \text{ for each } x, y \in X.$$

The conclusion follows now from Theorem 2. □

Consider now the following equation

$$x(t) = \int_{-t}^{t} K(t, s, x(s))ds + g(t), \; t \in \mathbb{R}. \tag{3.3}$$

Theorem 5. *Consider the equation (3.3). Suppose that:*
i) $K : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R} \to \mathbb{R}^n$ are continuous;
ii) there exists $k > 0$ such that

$$|K(t, s, u) - K(t, s, v)| \leq k|u - v|, \text{ for each } t, s \in \mathbb{R}, \; u, v \in \mathbb{R}^n;$$

Then the integral equation (3.3) has a unique solution x^* in $C(\mathbb{R}, \mathbb{R}^n)$ and equation (3.3) is Hyers-Ulam stable.

Proof. We consider the gauge space $X := (C(\mathbb{R}, \mathbb{R}^n), \mathcal{P} := (d_n)_{n \in \mathbb{N}})$ where

$$d_n(x, y) = \max_{-n \leq t \leq n} \left(|x(t) - y(t)| \cdot e^{-\tau|t|} \right), \quad \tau > 0,$$

and the operator $B : X \to X$ defined by

$$Bx(t) = \int_{-t}^{t} K(t, s, x(s)) ds + g(t).$$

From condition (ii), for $x, y \in X$, we have

$$|Bx(t) - By(t)| \leq \int_{-t}^{t} k|x(s) - y(s)|e^{-\tau|s|}e^{\tau|s|}ds \leq$$

$$k \int_{-t}^{t} e^{\tau|s|}(|x(s) - y(s)|e^{-\tau|s|})ds \leq k d_n(x, y) \left| \int_{-t}^{t} e^{\tau|s|}ds \right| \leq$$

$$k d_n(x, y) \int_{-|t|}^{|t|} e^{\tau|s|}ds \leq \frac{2k}{\tau} d_n(x, y) e^{\tau|t|}, \quad t \in [-n; n].$$

Thus, for any $\tau \geq 2k$, if we denote $L := \frac{2k}{\tau} < 1$, we obtain

$$d_n(B(x), B(y)) \leq L d_n(x, y)), \quad \text{for all } x, y \in E, \text{ and for } n \in \mathbb{N}.$$

The conclusion follows again by Theorem 2. □

Acknowledgement

For the first author, this work was possible with the financial support of a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0094.

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