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FORCED FERMI-PASTA-ULAM LATTICE MAPS

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Abstract. We prove existence and uniqueness results for periodic and quasiperiodic travelling waves of Fermi-Pasta-Ulam (FPU) type lattice maps. We also study quasiperiodic solutions of difference equations on Banach algebras as a natural generalizations of the FPU stationary lattice equations.

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1. INTRODUCTION

Dynamical systems on infinite lattices play an important role in applications to natural sciences. There are several well-known types of lattice equations such as sine-Gordon, Frenkel-Kontorova model, Klein-Gordon, Toda, discrete Schrödinger, Ablowitz-Ladik nonlinear Schrödinger, Fermi-Pasta-Ulam and discrete Nagumo equations. We refer the reader to the huge literature [1–4, 7, 8, 10, 12, 15, 16, 18, 21, 22]. Some of these lattice models are derived by spatial discretization of corresponding partial differential equations. In any case, they are defined as an infinite number of coupled nonlinear oscillators where the theory of ordinary differential equations in infinite dimensions can be applied [9, 17]. Of course, the dynamics of such systems is extremely rich. There are three important types of solutions: stationary ones, spatially localized ones, the so called breathers and the travelling waves.

This paper is devoted to the study of discrete FPU type lattices. The FPU model consists of a chain of particles connected by nonlinear springs

$$\ddot{u}_n = \phi(u_{n+1} - u_n) + \phi(u_{n-1} - u_n), \quad n \in \mathbb{Z} \tag{1.1}$$

for a smooth and odd function ϕ. Most of results are connected with Hamiltonian case presented by (1.1). But motivated by [13, 14], we consider in [11] a 1D (possible)

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damped FPU lattice forced by a travelling wave field:

$$\begin{aligned}\ddot{u}_n = \alpha\,(u_{n+1}+u_{n-1}-2u_n)+\beta(u_{n+1}-u_n)^3+\beta(u_{n-1}-u_n)^3 \\ -\gamma\dot{u}_n+f\cos(\omega t+pn)\,,\end{aligned} \tag{1.2}$$

where $\alpha > 0$, $\beta > 0$, $\gamma \geq 0$, $\omega > 0$, $p \neq 0$, $f \neq 0$ are parameters. Analytical and numerical methods are used. This paper is a continuation of [11] in the sense that we also discretize (1.2) in time. Moreover, we consider general analytical nonlinearities (see (2.1) and (2.3)). First in Section 2, we study forced periodic and quasiperiodic travelling waves of such FPU lattice maps. We also mention a few words on their stationary solutions. Then in Section 3, we extend our method to nonlocal interactions. In Section 4, we end this paper with the investigation of general difference equations on Banach algebras, which are generalizations of stationary lattice equations. Proofs of our results are based on a majorant method [19]. Two illustrative examples are given as well.

2. 1D FORCED FPU LATTICE MAPS WITH LOCAL INTERACTIONS

In this section, we consider a discrete version of the following 1-dimensional damped FPU lattice forced by a travelling wave field (see [13, 14]):

$$\begin{aligned}\ddot{u}_n = \alpha(u_{n+1}-2u_n+u_{n-1})+\varphi_1(u_{n+1}-u_n)+\varphi_2(u_{n-1}-u_n) \\ -\gamma\dot{u}_n+f\cos(\omega t+pn),\end{aligned} \tag{2.1}$$

where $\alpha > 0$, $\gamma \geq 0$, $\omega > 0$, $p \neq 0$, $f \neq 0$ are parameters and φ_1,φ_2 are odd analytic functions with radius of convergence ρ_1,ρ_2, respectively, such that

$$\mathrm{D}\varphi_{1,2}(0)=0. \tag{2.2}$$

Equation (2.1) of the form (1.2) is studied in [11]. We substitute the differentiation by the symmetric difference, i.e.

$$\dot{u}_n(t)\to u_n(t+1/2)-u_n(t-1/2), \qquad \ddot{u}_n(t)\to u_n(t+1)-2u_n(t)+u_n(t-1).$$

So we study the travelling waves of the system

$$\begin{aligned}u_n(t+1)-2u_n(t)+u_n(t-1) = \alpha(u_{n+1}(t)-2u_n(t)+u_{n-1}(t)) \\ +\varphi_1(u_{n+1}(t)-u_n(t))+\varphi_2(u_{n-1}(t)-u_n(t)) \\ -\gamma(u_n(t+1/2)-u_n(t-1/2))+f\cos(\omega t+pn).\end{aligned} \tag{2.3}$$

Putting $u_n(t)=U(\omega t+pn)$ for $U(z+\pi)=-U(z)$ in (2.3), we get

$$\begin{aligned}U(z+\omega)-2U(z)+U(z-\omega) = \alpha(U(z+p)-2U(z)+U(z-p)) \\ +\varphi_1(U(z+p)-U(z))+\varphi_2(U(z-p)-U(z)) \\ -\gamma(U(z+\omega/2)-U(z-\omega/2))+f\cos z\end{aligned} \tag{2.4}$$

with $z = \omega t + pn$. We take Banach spaces

$$W := \left\{ U \in C(\mathbb{R}, \mathbb{R}) \mid U(z) = \sum_{k\in\mathbb{Z}} d_k \,\mathrm{e}^{k\imath z}, \sum_{k\in\mathbb{Z}} |d_k| < \infty \right\},$$

$$X := \left\{ U \in C(\mathbb{R}, \mathbb{R}) \mid U(z) = \sum_{k\in\mathbb{Z}} c_k \,\mathrm{e}^{(2k+1)\imath z}, \sum_{k\in\mathbb{Z}} |c_k| < \infty \right\}$$

with the norms

$$\|U\| := \sum_{k\in\mathbb{Z}} |d_k|, \qquad \|U\| := \sum_{k\in\mathbb{Z}} |c_k|,$$

respectively. Obviously, $X \subset W$.

The following lemma is clear.

Lemma 1. *If $U_1, U_2 \in W$ then $U_1U_2 \in W$ and $\|U_1U_2\| \le \|U_1\|\|U_2\|$. For each $k \in \mathbb{N}$, if $U_1, U_2, \dots, U_{2k+1} \in X$ then $U_1U_2 \dots U_{2k+1} \in X$.*

By setting

$$\begin{aligned} \mathcal{K}U(z) &:= U(z+\omega) - 2U(z) + U(z-\omega) - \alpha(U(z+p) - 2U(z) + U(z-p)) \\ &\quad + \gamma(U(z+\omega/2) - U(z-\omega/2)), \\ \mathcal{F}(U, f)(z) &:= \varphi_1(U(z+p) - U(z)) + \varphi_2(U(z-p) - U(z)) + f\cos z, \end{aligned}$$

equation (2.4) has the form

$$\mathcal{K}U = \mathcal{F}(U, f).$$

Denote $\rho_0 := \min\{\rho_1, \rho_2\}$ and $B(\rho_0) := \{U \in X \mid \|U\| < \rho_0\}$.

We have the next result.

Lemma 2. *Function $\mathcal{F} : B(\rho_0/2) \times \mathbb{R} \to X$ fulfils*

$$\|\mathcal{F}(U, f)\| \le \sum_{k=3}^{\infty} \frac{|\mathrm{D}^k\varphi_1(0)| + |\mathrm{D}^k\varphi_2(0)|}{k!} 2^k \|U\|^k + |f|, \tag{2.5}$$

$$\begin{aligned} &\|\mathcal{F}(U_1, f) - \mathcal{F}(U_2, f)\| \\ &\le \|U_1 - U_2\| \sum_{k=3}^{\infty} \frac{|\mathrm{D}^k\varphi_1(0)| + |\mathrm{D}^k\varphi_2(0)|}{k!} 2^k \sum_{i=0}^{k-1} \|U_1\|^i \|U_2\|^{k-i-1}, \end{aligned} \tag{2.6}$$

$$\|\mathcal{F}(U, f_1) - \mathcal{F}(U, f_2)\| \le |f_1 - f_2| \tag{2.7}$$

for any $U, U_1, U_2 \in B(\rho_0/2) \subset X$ and $f, f_1, f_2 \in \mathbb{R}$.

Proof. Since $\cos z = \frac{\mathrm{e}^{\imath z} + \mathrm{e}^{-\imath z}}{2}$, we get $\|\cos z\| = 1$ and (2.7) easily follows. Next we derive

$$U(z \pm p) - U(z) = \sum_{k\in\mathbb{Z}} c_k \left(\mathrm{e}^{\pm(2k+1)\imath p} - 1 \right) \mathrm{e}^{(2k+1)\imath z}$$

for any $U \in X$ with $U(z) = \sum_{k\in\mathbb{Z}} c_k \,\mathrm{e}^{(2k+1)\imath z}$. So we obtain

$$\|U(z \pm p) - U(z)\| = \sum_{k\in\mathbb{Z}} |c_k| \left|\mathrm{e}^{\pm(2k+1)\imath p} - 1\right| \le 2\|U\| \tag{2.8}$$

for any $U \in X$. For $U \in B(\rho_0)$ we estimate the Taylor series

$$\|\varphi_i(U)\| \le \sum_{k=3}^{\infty} \frac{|\mathrm{D}^k \varphi_i(0)|}{k!} \|U\|^k < \infty$$

for $i = 1,2$. Note that since φ_1, φ_2 are odd, their even Taylor coefficients are zero. Then combining these estimates with (2.8) and Lemma 1, we arrive at (2.5).

Next arguing like for (2.8), we obtain

$$\|U_1(z \pm p) - U_1(z) - (U_2(z \pm p) - U_2(z))\| \le 2\|U_1 - U_2\| \tag{2.9}$$

for any $U_1, U_2 \in X$. Then we apply Lemma 1, estimates (2.8), (2.9) and the formula

$$a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + \cdots + b^{k-1}) \tag{2.10}$$

for any $a, b \in \mathbb{C}$, odd $k \in \mathbb{N}$ to obtain property (2.6). The proof is finished. □

Next if $U \in X$ with $U(z) = \sum_{k\in\mathbb{Z}} c_k \,\mathrm{e}^{(2k+1)\imath z}$, then

$$\mathcal{K}U(z) = \sum_{k\in\mathbb{Z}} \left(4\alpha \sin^2 \frac{2k+1}{2} p - 4\sin^2 \frac{2k+1}{2}\omega + 2\imath\gamma \sin \frac{2k+1}{2}\omega\right) c_k \,\mathrm{e}^{(2k+1)\imath z} \tag{2.11}$$

and so $\mathcal{K} \in L(X)$ with

$$\|\mathcal{K}\|_{L(X)} \le 4\alpha + 4 + 2\gamma.$$

If

$$\Theta := \inf_{k\in\mathbb{Z}} \sqrt{\left(4\alpha \sin^2 \frac{2k+1}{2} p - 4\sin^2 \frac{2k+1}{2}\omega\right)^2 + 4\gamma^2 \sin^2 \frac{2k+1}{2}\omega} > 0 \tag{2.12}$$

for a constant Θ depending on $\alpha, p, \omega, \gamma$, then $\mathcal{K}^{-1} \in L(X)$ is such that

$$\|\mathcal{K}^{-1}\|_{L(X)} \le \frac{1}{\Theta}. \tag{2.13}$$

Now we can prove the following existence result on (2.4) when all parameters except f are fixed.

Theorem 1. *Assume* (2.12) *along with*

$$|f| < |f_l| \tag{2.14}$$

for f_l satisfying

$$A(r) := \Theta r - \sum_{k=3}^{\infty} \frac{|\mathrm{D}^k \varphi_1(0)| + |\mathrm{D}^k \varphi_2(0)|}{k!} 2^k r^k = |f_l|, \tag{2.15}$$

$$\mathrm{D}A(r) = 0 \tag{2.16}$$

for some $r \in (0, \rho_0/2)$. *Then* (2.4) *has a unique solution* $U(f) \in \overline{B(\rho_f)}$ *in a closed ball where* $\rho_f < \rho_0/2$ *is a smallest positive root of* $A(r) = |f|$. *Moreover,* $U(f)$ *can be approximated by an iteration process. Finally, it holds*

$$\|U(f_1) - U(f_2)\| \le \frac{|f_1 - f_2|}{\Theta} \times \left(1 - \frac{1}{\Theta}\sum_{k=3}^{\infty} \frac{|\mathrm{D}^k\varphi_1(0)| + |\mathrm{D}^k\varphi_2(0)|}{(k-1)!} 2^k \rho_{\max\{|f_1|,|f_2|\}}^{k-1}\right)^{-1} \tag{2.17}$$

for any $f_1, f_2 \in \mathbb{R}$ *satisfying* (2.14).

Proof. We rewrite (2.4) as a parametrized fixed point problem

$$U = \mathcal{R}(U, f) := \mathcal{K}^{-1}\mathcal{F}(U, f)$$

in $B(\rho_0/2) \subset X$. We already know that $\mathcal{R} : B(\rho_0/2) \times \mathbb{R} \to X$ is continuous and by (2.5), (2.13) such that

$$\|\mathcal{R}(U, f)\| \le \frac{1}{\Theta}\left(\sum_{k=3}^{\infty} \frac{|\mathrm{D}^k\varphi_1(0)| + |\mathrm{D}^k\varphi_2(0)|}{k!} 2^k \|U\|^k + |f|\right).$$

Next, if there is $0 < \rho_f < \rho_0/2$ such that

$$A(\rho_f) = |f|, \tag{2.18}$$

then $\mathcal{R}(\cdot, f)$ maps $\overline{B(\rho_f)}$ into itself. So it remains to study (2.18). In order to find the largest f_l for which (2.18) has a solution $\rho_{f_l} > 0$, we need to solve $A(r) = |f_l|$ together with (2.16) for $r \in (0, \rho_0/2)$. This implies (2.14). Note that

$$\pm \mathrm{D}A(r) = \pm\Theta \mp \sum_{k=3}^{\infty} \frac{|\mathrm{D}^k\varphi_1(0)| + |\mathrm{D}^k\varphi_2(0)|}{(k-1)!} 2^k r^{k-1} > \mathrm{D}A(\rho_{f_l}) = 0$$

for $r \in (0, \rho_0/2)$, $\pm r < \pm\rho_{f_l}$, and $\lim_{r\to(\rho_0/2)^-} DA(r) = -\infty$ (see Sections 7.21, 7.22 and 7.31 of [20]). Hence ρ_{f_l} is uniquely determined by (2.16). Moreover, continuity of $A(r)$ with $A(0) = 0$, $A(\rho_{f_l}) = |f_l|$ yield that $0 < \rho_f < \rho_{f_l}$ whenever $0 < |f| < |f_l|$, i.e. $\mathrm{D}A(\rho_f) > 0$. So assuming (2.12), (2.14) and by (2.2), we know that (2.18) has a positive solution $\rho_f < \rho_0/2$. We take the smallest one. So $\mathcal{R}(\cdot, f)$ maps $\overline{B(\rho_f)}$ into itself and, moreover, by (2.6), (2.13)

$$\|\mathcal{R}(U_1, f) - \mathcal{R}(U_2, f)\| \le \frac{\|U_1 - U_2\|}{\Theta}\sum_{k=3}^{\infty} \frac{|\mathrm{D}^k\varphi_1(0)| + |\mathrm{D}^k\varphi_2(0)|}{(k-1)!} 2^k \rho_f^{k-1}$$

for any $U_1, U_2 \in \overline{B(\rho_f)}$. Hence, $\mathcal{R}(\cdot, f)$ is a contraction on $\overline{B(\rho_f)}$ with a contraction constant

$$\frac{1}{\Theta}\sum_{k=3}^{\infty} \frac{|\mathrm{D}^k\varphi_1(0)| + |\mathrm{D}^k\varphi_2(0)|}{(k-1)!} 2^k \rho_f^{k-1} = \frac{\Theta - \mathrm{D}A(\rho_f)}{\Theta} < \frac{\Theta - \mathrm{D}A(\rho_{f_l})}{\Theta} = 1.$$

The proof of the existence and uniqueness is finished by the Banach fixed point theorem [5]. Next, let $f_1, f_2 \in \mathbb{R}$ satisfy (2.14), then $U(f_i) \in \overline{B(\rho_{f_i})} \subset \overline{B(\rho_{f_3})}$ for $i = 1,2$ and $f_3 := \max\{|f_1|,|f_2|\}$. Note f_3 satisfies (2.14). By (2.6), (2.7) and (2.13), we derive

$$\begin{aligned}\|U(f_1) - U(f_2)\| &= \|\mathcal{R}(U(f_1), f_1) - \mathcal{R}(U(f_2), f_2)\| \\ &\leq \|\mathcal{R}(U(f_1), f_1) - \mathcal{R}(U(f_2), f_1)\| + \|\mathcal{R}(U(f_2), f_1) - \mathcal{R}(U(f_2), f_2)\| \\ &\leq \frac{\|U(f_1) - U(f_2)\|}{\Theta} \sum_{k=3}^{\infty} \frac{|\mathrm{D}^k \varphi_1(0)| + |\mathrm{D}^k \varphi_2(0)|}{(k-1)!} 2^k \rho_{f_3}^{k-1} + \frac{|f_1 - f_2|}{\Theta}\end{aligned}$$

which implies (2.17). □

Remark 1.

1. If $\gamma > 0$ and

$$\omega \in Q := \left\{ \frac{2l_1 + 1}{l_2}\pi \mid l_1, l_2 \in \mathbb{N} \right\},$$

then (2.12) holds and for $\omega = \frac{2l_1+1}{l_2}\pi$

$$\Theta \geq 2\gamma \inf_{k \in \mathbb{Z}} \left| \sin\left(\frac{2k+1}{2} \frac{2l_1+1}{l_2}\pi \right) \right| \geq 2\gamma \left| \sin \frac{\pi}{2l_2} \right| \geq \frac{2\gamma}{l_2}$$

since $\sin x \geq 2x/\pi$ for $x \in [0, \pi/2]$. So we can replace Θ with $2\gamma/l_2$ in the above considerations.

2. Clearly, if we change $p \leftrightarrow -p$, Θ remains the same and Theorem 1 holds as well. Thus we can assume $p > 0$. Next, each element $\frac{2l_1+1}{l_2}\pi$ of Q can be chosen such that $2l_1 + 1, l_2$ are relatively prime (their only common divisor is 1). In the next steps we always consider this form.

If $p = \frac{2k_1+1}{k_2}\pi \in Q$, then we know (cf. [6]) that there are integers a, b such that $a(2k_1 + 1) = 1 + 2bk_2$. Obviously, a is odd. So there exists $k \in \mathbb{Z}$ such that $2k + 1 = a(1 + 2k_2)$. Consequently,

$$(2k+1)(2k_1+1) = 2bk_2(1+2k_2) + 2k_2 + 1 \equiv 1 \pmod{2k_2},$$

i.e. for such k it holds

$$\sin^2 \frac{2k+1}{2} p = \sin^2 \frac{\pi}{2k_2}.$$

In addition, on setting $l_i := i + k + 2ki$ for $i = 1, \ldots, k_2 - 1$ we get

$$(2l_i + 1)(2k_1 + 1) = (2i+1)(2k+1)(2k_1+1) \equiv 2i + 1 \pmod{2k_2}$$

for each $i = 1, \ldots, k_2 - 1$. Therefore,

$$\left\{ \sin^2 \frac{2k+1}{2} p \mid k \in \mathbb{Z} \right\} = \left\{ \sin^2 \frac{k\pi}{2k_2} \mid k = 1, 3, \ldots, 2k_2 - 1 \right\}$$

for fixed $p = \frac{2k_1+1}{k_2}\pi \in Q$. Same discussion for $\omega \in Q$ yields:

If $p = \frac{2k_1+1}{k_2}\pi, \omega = \frac{2l_1+1}{l_2}\pi \in Q$ are such that

$$\left\{\alpha \sin^2 \frac{k\pi}{2k_2} \mid k = 1,3,\ldots,2k_2-1\right\} \cap \left\{\sin^2 \frac{l\pi}{2l_2} \mid l = 1,3,\ldots,2l_2-1\right\} = \varnothing,$$

then

$$\Theta \geq 2 \min_{\substack{k=1,3,\ldots,2k_2-1 \\ l=1,3,\ldots,2l_2-1}} \left|\alpha \sin^2 \frac{k\pi}{2k_2} - \sin^2 \frac{l\pi}{2l_2}\right| =: \Theta_1 > 0.$$

Thus for such p and ω we can replace Θ with Θ_1 in the above considerations.

3. One can investigate a discretization of equation (2.1) with more than one forcing which are of different periods, i.e.

$$\begin{aligned} \ddot{u}_n &= \alpha(u_{n+1} - 2u_n + u_{n-1}) + \varphi_1(u_{n+1} - u_n) + \varphi_2(u_{n-1} - u_n) \\ &\quad -\gamma\dot{u}_n + f_1 \cos(\omega_1 t + p_1 n) + f_2 \cos(\omega_2 t + p_2 n) \end{aligned}$$

by assuming $U(z_1 + \pi, z_2 + \pi) = -U(z_1, z_2)$ with $u_n(t) = U(\omega_1 t + p_1 n, \omega_2 t + p_2 n)$, and introducing Banach spaces

$$W := \left\{U \in C(\mathbb{R}^2, \mathbb{R}) \mid U(z) = \sum_{k,l\in\mathbb{Z}} d_{kl}\, \mathrm{e}^{\imath(kz_1+lz_2)}, \sum_{k,l\in\mathbb{Z}} |d_{kl}| < \infty\right\},$$

$$X := \left\{U \in C(\mathbb{R}^2, \mathbb{R}) \mid U(z) = \sum_{k+l\in 2\mathbb{Z}+1} c_{kl}\, \mathrm{e}^{\imath(kz_1+lz_2)}, \sum_{k,l\in 2\mathbb{Z}+1} |c_{kl}| < \infty\right\}$$

with the norms

$$\|U\| := \sum_{k,l\in\mathbb{Z}} |d_{kl}|, \qquad \|U\| := \sum_{k,l\in 2\mathbb{Z}+1} |c_{kl}|,$$

respectively.

The other possibility is to consider forcing by a composed travelling wave field, i.e.

$$\begin{aligned} \ddot{u}_n &= \alpha(u_{n+1} - 2u_n + u_{n-1}) + \varphi_1(u_{n+1} - u_n) + \varphi_2(u_{n-1} - u_n) \\ &\quad -\gamma\dot{u}_n + f \cos(\omega_1 t + p_1 n)\cos(\omega_2 t + p_2 n) \end{aligned}$$

with $\omega_1, \omega_2 \in \mathbb{R}$, $\frac{\omega_1}{\omega_2} \notin \mathbb{Q}$, by assuming $U(z_1 + \pi, z_2) = -U(z_1, z_2) = U(z_1, z_2 + \pi)$ where $u_n(t) = U(\omega_1 t + p_1 n, \omega_2 t + p_2 n)$. In this case, one takes Banach spaces

$$W := \left\{U \in C(\mathbb{R}^2, \mathbb{R}) \mid U(z) = \sum_{k,l\in\mathbb{Z}} d_{kl}\, \mathrm{e}^{\imath(kz_1+lz_2)}, \sum_{k,l\in\mathbb{Z}} |d_{kl}| < \infty\right\},$$

$$X := \left\{U \in C(\mathbb{R}^2, \mathbb{R}) \mid U(z) = \sum_{k,l\in\mathbb{Z}} c_{kl}\, \mathrm{e}^{(2k+1)\imath z_1+(2l+1)\imath z_2}, \sum_{k,l\in\mathbb{Z}} |c_{kl}| < \infty\right\}$$

with the norms

$$\|U\| := \sum_{k,l\in\mathbb{Z}} |d_{kl}|, \qquad \|U\| := \sum_{k,l\in\mathbb{Z}} |c_{kl}|,$$

respectively.

4. If u_n in (2.1) is independent of t, i.e. $u_n(t) = u_n$ for all $t \in \mathbb{R}$, equation (2.3) becomes a nonlinear difference equation

$$0 = \alpha(u_{n+1} - 2u_n + u_{n-1}) + \varphi_1(u_{n+1} - u_n) + \varphi_2(u_{n-1} - u_n) + f \cos pn \tag{2.19}$$

and by adding another forcing with different period, one can study the existence of quasiperiodic solution $\{u_n\}_{n\in\mathbb{Z}}$ of the infinite system of equations

$$\begin{aligned} 0 = \alpha(u_{n+1} - 2u_n + u_{n-1}) + \varphi_1(u_{n+1} - u_n) + \varphi_2(u_{n-1} - u_n) \\ + f_1 \cos p_1 n + f_2 \cos p_2 n, \quad n \in \mathbb{Z} \end{aligned} \tag{2.20}$$

with $\frac{p_1}{p_2} \notin \mathbb{Q}$.

Example 1. Consider equation

$$\begin{aligned} U(z+\omega) - 2U(z) + U(z+\omega) = \alpha(U(z+p) + U(z-p) - 2U(z)) \\ +\beta(U(z+p) - U(z))^3 + \beta(U(z-p) - U(z))^3 \\ -\gamma(U(z+\omega/2) - U(z-\omega/2)) + f \cos z \end{aligned} \tag{2.21}$$

with parameters $\alpha > 0$, $\beta > 0$, $\gamma \geq 0$, $\omega > 0$, $p \neq 0$, $f \neq 0$.

Corollary 1. *Assuming (2.12) and*

$$|f| < \sqrt{\frac{\Theta^3}{108\beta}}, \tag{2.22}$$

equation (2.21) has a unique solution $U(f) \in \overline{B(\rho_f)} \subset X$ where

$$\rho_f = \sqrt{\frac{\Theta}{12\beta}} \sin\left(\frac{1}{3} \arcsin\left(|f| \sqrt{\frac{108\beta}{\Theta^3}}\right)\right). \tag{2.23}$$

Moreover, $U(f)$ can be approximated by an iteration process. Finally, it holds

$$\|U(f_1) - U(f_2)\| \leq \frac{|f_1 - f_2|}{\Theta\left(1 - 4\sin^2\left(\frac{1}{3}\arcsin\left(\max\{|f_1|, |f_2|\}\sqrt{\frac{108\beta}{\Theta^3}}\right)\right)\right)}$$

for any $f_1, f_2 \in \mathbb{R}$ satisfying (2.22).

Proof. In this case $\rho_0 = \infty$ and the corollary follows directly from Theorem 1. The smallest positive root ρ_f of equation

$$A(\rho_f) = \Theta\rho_f - 16\beta\rho_f^3 = |f|$$

is know to have a form of (2.23) [6]. □

3. 1D FORCED FPU LATTICE MAPS WITH NONLOCAL INTERACTIONS

In this section, we consider a discretization of 1-dimensional damped FPU lattice forced by a travelling wave field with nonlocal interactions:

$$\ddot{u}_n = \sum_{j\in\mathbb{Z}} \alpha_j \left(u_{n+j} - u_n\right) + \sum_{j\in\mathbb{Z}} \varphi_j(u_{n+j} - u_n) - \gamma \dot{u}_n + f\cos(\omega t + pn), \tag{3.1}$$

where $\alpha_j = \alpha_{-j} > 0$, $\alpha_0 = 0$ and

$$\sum_{j\in\mathbb{N}} \alpha_j < \infty,$$

$\gamma \geq 0,\ \omega > 0,\ p \neq 0,\ f \neq 0$ are parameters and φ_j is odd analytic function with radius of convergence ρ_j for $j \in \mathbb{Z}\backslash\{0\}$, $\varphi_0 \equiv 0$ such that

$$\rho_0 := \inf_{j\in\mathbb{Z}\backslash\{0\}} \rho_j > 0, \quad \mathrm{D}\varphi_j(0) = 0,\ j \in \mathbb{Z}\backslash\{0\}$$

and

$$\sum_{j\in\mathbb{Z}\backslash\{0\}} \sum_{k=3}^{\infty} \frac{|\mathrm{D}^k \varphi_j(0)|}{k!} r^k < \infty$$

for all $r \in (0, \rho_0/2)$.

As in the previous section, we apply symmetric difference and study the equation

$$\begin{aligned} u_n(t+1) - 2u_n(t) + u_n(t-1) = &\sum_{j\in\mathbb{N}} \alpha_j (u_{n+j}(t) - 2u_n(t) + u_{n-j}(t)) \\ &+ \sum_{j\in\mathbb{N}} \varphi_j(u_{n+j}(t) - u_n(t)) + \varphi_{-j}(u_{n-j}(t) - u_n(t)) \\ &- \gamma(u_n(t+1/2) - u_n(t-1/2)) + f\cos(\omega t + pn). \end{aligned} \tag{3.2}$$

Putting $u_n(t) = U(\omega t + pn)$, $U(z+\pi) = -U(z)$ for $U \in B(\rho_0/2)$ in (3.2), we get

$$\begin{aligned} U(z+\omega) - 2U(z) + U(z-\omega) = &\sum_{j\in\mathbb{N}} \alpha_j (U(z+pj) - 2U(z) + U(z-pj)) \\ &+ \sum_{j\in\mathbb{N}} \varphi_j(U(z+pj) - U(z)) + \varphi_{-j}(U(z-pj) - U(z)) \\ &- \gamma(U(z+\omega/2) - U(z-\omega/2)) + f\cos z \end{aligned} \tag{3.3}$$

with $z = \omega t + pn$. By setting

$$\begin{aligned}\mathcal{J}U(z) &:= U(z+\omega) - 2U(z) + U(z-\omega)\\ &- \sum_{j\in\mathbb{N}} \alpha_j (U(z+pj) - 2U(z) + U(z-pj)) + \gamma (U(z+\omega/2) - U(z-\omega/2)),\\ \mathcal{G}(U,f)(z) &:= \sum_{j\in\mathbb{N}} \varphi_j (U(z+pj) - U(z)) + \varphi_{-j}(U(z-pj) - U(z)) + f\cos z,\end{aligned}$$

equation (3.3) has the form

$$\mathcal{J}U = \mathcal{G}(U,f).$$

Following the proof of Lemma 2, we have the following result.

Lemma 3. *Function* $\mathcal{G} : B(\rho_0/2) \times \mathbb{R} \to X$ *satisfies*

$$\|\mathcal{G}(U,f)\| \le \sum_{k=3}^{\infty} \frac{\sum_{j\in\mathbb{N}} |D^k \varphi_j(0)| + |D^k \varphi_{-j}(0)|}{k!} 2^k \|U\|^k + |f|,$$

$$\begin{aligned}&\|\mathcal{G}(U_1,f) - \mathcal{G}(U_2,f)\|\\ &\le \|U_1 - U_2\| \sum_{k=3}^{\infty} \frac{\sum_{j\in\mathbb{N}} |D^k \varphi_j(0)| + |D^k \varphi_{-j}(0)|}{k!} 2^k \sum_{i=0}^{k-1} \|U_1\|^i \|U_2\|^{k-i-1},\\ &\|\mathcal{G}(U,f_1) - \mathcal{G}(U,f_2)\| \le |f_1 - f_2|\end{aligned}$$

for any $U, U_1, U_2 \in B(\rho_0/2)$ *and* $f, f_1, f_2 \in \mathbb{R}$.

Next if $U \in X$ with $U(z) = \sum_{k\in\mathbb{Z}} c_k \mathrm{e}^{(2k+1)\imath z}$, then

$$\begin{aligned}\mathcal{J}U(z) = \sum_{k\in\mathbb{Z}} &\left(4\sum_{j\in\mathbb{N}} \alpha_j \sin^2 \frac{2k+1}{2} pj - 4\sin^2 \frac{2k+1}{2}\omega + 2\imath\gamma \sin \frac{2k+1}{2}\omega \right)\\ &\times c_k \mathrm{e}^{(2k+1)\imath z}.\end{aligned}$$

Hence $\mathcal{J} \in L(X)$ with

$$\|\mathcal{J}\|_{L(X)} \le 4\sum_{j\in\mathbb{N}} \alpha_j + 4 + 2\gamma.$$

Moreover, assuming

$$\Gamma := \inf_{k\in\mathbb{Z}} \sqrt{\left(4\sum_{j\in\mathbb{N}} \alpha_j \sin^2 \frac{2k+1}{2} pj - 4\sin^2 \frac{2k+1}{2}\omega \right)^2 + 4\gamma^2 \sin^2 \frac{2k+1}{2}\omega} > 0, \tag{3.4}$$

$\mathcal{J}^{-1} : X \to X$ is such that

$$\|\mathcal{J}^{-1}\|_{L(X)} \le \frac{1}{\Gamma}.$$

Summarising, we arrive at the following result.

Theorem 2. *Suppose* (3.4). *By replacing*

$$|\mathrm{D}^k\varphi_1(0)| \leftrightarrow \sum_{j\in\mathbb{N}} |\mathrm{D}^k\varphi_j(0)|, \qquad |\mathrm{D}^k\varphi_2(0)| \leftrightarrow \sum_{j\in\mathbb{N}} |\mathrm{D}^k\varphi_{-j}(0)|, \qquad \Theta \leftrightarrow \Gamma$$

for each $k = 1,2,\dots,$ *the statements of Theorem 1 are valid for* (3.3).

Remark 1.1. remains true with Γ instead of Θ.

4. Quasiperiodic solutions of difference equations

Motivated by stationary FPU lattice equations (2.19) and (2.20), in this section, we study the existence of quasiperiodic solutions of a general nonlinear difference equation with a quasiperiodic perturbation. We take coefficients from Banach complex algebra V and introduce Banach space

$$W := \left\{ c = \{c_{kp}\}_{k,p\in\mathbb{Z}} \mid c_{kp} \in V,\ \forall k,p \in \mathbb{Z},\ \sum_{k,p\in\mathbb{Z}} \|c_{kp}\| < \infty \right\}$$

with the norm

$$\|c\| := \sum_{k,p\in\mathbb{Z}} \|c_{kp}\|.$$

We shall seek the solutions in

$$S_{\omega_1\omega_2} := \left\{ U = \{U_n\}_{n\in\mathbb{Z}} \mid U_n = \sum_{k,p\in\mathbb{Z}} c_{kp}\, \mathrm{e}^{(\omega_1 k+\omega_2 p)\imath n},\ c \in W,\ \forall n \in \mathbb{Z} \right\}.$$

For each $U \in S_{\omega_1\omega_2}$ there may be many different $c \in W$ such that

$$U_n = \sum_{k,p\in\mathbb{Z}} c_{kp}\, \mathrm{e}^{(\omega_1 k+\omega_2 p)\imath n}.$$

Therefore, we denote $U = U(c) = \{U_n\}_{n\in\mathbb{Z}} = \{U_n(c)\}_{n\in\mathbb{Z}}$ to emphasize which coefficients are considered. Moreover, we write $\|U(c)\| = \|c\|$ $(= \|U_n(c)\|)$. On the other side, if $\|U\| = a$, then there exist coefficients $c \in W$ such that $U = U(c)$ and $\|c\| = a$. For the set $S_{\omega_1\omega_2}$ we have the next statement which can be easily verified.

Lemma 4. *Let* $X(x), Y(y) \in S_{\omega_1\omega_2}$. *Then the following holds true:*

(1) $\|X(x)\| < \infty$,
(2) *for all* $\alpha,\beta \in \mathbb{C}$: $Z = \alpha X(x) + \beta Y(y) \in S_{\omega_1\omega_2}$ *and there exist coefficients* $z \in W$ *such that* $Z = Z(z)$ *and* $z_{kp} = \alpha x_{kp} + \beta y_{kp}$ *for each* $k,p \in \mathbb{Z}$,
(3) $Z = \{Z_n\}_{n\in\mathbb{Z}} = \{X_nY_n\}_{n\in\mathbb{Z}} \in S_{\omega_1\omega_2}$ *and there exists* $z \in W$ *such that* $Z = Z(z)$ *and* $\|z\| \le \|x\|\|y\|$,
(4) *if* $j \in \mathbb{Z}$ *is arbitrary and fixed, then* $Z = \{Z_n\}_{n\in\mathbb{Z}} = \{X_{n+j}\}_{n\in\mathbb{Z}} \in S_{\omega_1\omega_2}$.

Let $\omega_1, \omega_2 \in \mathbb{R}$ be such that $\frac{\omega_1}{\omega_2} \notin \mathbb{Q}$ and $j \in \mathbb{N}$ be fixed. Consider the following nonlinear difference equation

$$\mathcal{L}(\sigma)X = Y + F(X), \qquad (\mathcal{L}(\sigma)X_n = Y_n + F_n(X_n), \forall n \in \mathbb{Z}) \tag{4.1}$$

where $X = \{X_n\}_{n\in\mathbb{Z}}, Y = \{Y_n\}_{n\in\mathbb{Z}} \in S_{\omega_1\omega_2}$, $\mathcal{L}(\sigma) : S_{\omega_1\omega_2} \to S_{\omega_1\omega_2}$ is a linear difference operator given by

$$\mathcal{L}(\sigma) = \alpha_0\sigma^j + \alpha_1\sigma^{j-1} + \cdots + \alpha_{j-1}\sigma + \alpha_j$$

with shift operator σ ($\sigma^m(X_n) = X_{n+m}$) and constants $\alpha_0, \dots, \alpha_j \in \mathbb{C}$, i.e.

$$\mathcal{L}(\sigma)X_n = \alpha_0 X_{n+j} + \alpha_1 X_{n+j-1} + \cdots + \alpha_{j-1}X_{n+1} + \alpha_j X_n,$$

and $F : S_{\omega_1\omega_2} \to S_{\omega_1\omega_2}$ is a given nonlinear function such that

$$F_n(z) = \sum_{j=0}^{\infty}(a_j + b_j\,\mathrm{e}^{\imath\omega_1 n} + c_j\,\mathrm{e}^{\imath\omega_2 n})z^j$$

with coefficients $a_j, b_j, c_j \in \mathbb{C}$ for each $j = 0, 1, 2, \dots$, which is analytic with radius of convergence ρ_n for each $n \in \mathbb{Z}$ such that $\inf_{n\in\mathbb{Z}} \rho_n > 0$. Let the radius of convergence $\widetilde{\rho}$ of function

$$v(r) := \sum_{j=0}^{\infty}(|a_j| + |b_j| + |c_j|)r^j$$

be positive. Obviously, $\widetilde{\rho} \le \inf_{n\in\mathbb{Z}} \rho_n$. Denoting $B(\rho) := \{X \in S_{\omega_1\omega_2} \mid \|X\| < \rho\}$ and $\mathcal{F}(X,Y) := Y + F(X)$ or equivalently $\mathcal{F}_n(X_n, Y_n) := Y_n + F_n(X_n)$ for each $n \in \mathbb{Z}$, equation (4.1) has the form

$$\mathcal{L}(\sigma)X = \mathcal{F}(X,Y), \qquad (\mathcal{L}(\sigma)X_n = \mathcal{F}_n(X_n, Y_n), \forall n \in \mathbb{Z}) \tag{4.2}$$

and we have the following result.

Lemma 5. *Function $\mathcal{F} : B(\widetilde{\rho}) \times B(\widetilde{\rho}) \to S_{\omega_1\omega_2}$ fulfils*

$$\|\mathcal{F}_n(X_n, Y_n)\| \le \|Y_n\| + \sum_{j=0}^{\infty}\big(|a_j| + |b_j| + |c_j|\big)\|X_n\|^j, \tag{4.3}$$

$$\|\mathcal{F}_n(X_n, Y_n) - \mathcal{F}_n(Z_n, Y_n)\| \le \|X_n - Z_n\| \sum_{j=1}^{\infty}\big(|a_j| + |b_j| + |c_j|\big)\sum_{l=0}^{j-1}\|X_n\|^l\|Z_n\|^{j-l-1}, \tag{4.4}$$

$$\|\mathcal{F}_n(X_n, Y_n) - \mathcal{F}_n(X_n, Z_n)\| \le \|Y_n - Z_n\| \tag{4.5}$$

for each $n \in \mathbb{Z}$ and $X, Y, Z \in B(\widetilde{\rho}) \subset S_{\omega_1\omega_2}$.

Proof. The first and the last statements are trivial. The second one uses formula (2.10). □

Next, if $X(x) \in S_{\omega_1\omega_2}$, i.e. $X_n = \sum_{k,p\in\mathbb{Z}} x_{kp}\,\mathrm{e}^{(\omega_1 k+\omega_2 p)\imath n}$, then

$$\mathcal{L}(\sigma)X_n =$$

$$\sum_{k,p\in\mathbb{Z}} x_{kp}\left(\alpha_0\,\mathrm{e}^{(\omega_1 k+\omega_2 p)\imath j}+\cdots+\alpha_{j-1}\,\mathrm{e}^{(\omega_1 k+\omega_2 p)\imath}+\alpha_j\right)\mathrm{e}^{(\omega_1 k+\omega_2 p)\imath n}$$

$$= \sum_{k,p\in\mathbb{Z}} \mathcal{L}\left(\mathrm{e}^{(\omega_1 k+\omega_2 p)\imath}\right) x_{kp}\,\mathrm{e}^{(\omega_1 k+\omega_2 p)\imath n}$$

and so

$$\|\mathcal{L}(\sigma)\|_{L(S_{\omega_1\omega_2})} \le \sum_{l=0}^{j} |\alpha_l|.$$

If

$$\Theta := \inf_{\alpha\in[0,2\pi]} |\mathcal{L}(\mathrm{e}^{\imath\alpha})| > 0, \tag{4.6}$$

then $\mathcal{L}(\sigma)^{-1} \in L(S_{\omega_1\omega_2})$ is such that

$$\|\mathcal{L}(\sigma)^{-1}\|_{L(S_{\omega_1\omega_2})} \le \frac{1}{\Theta}. \tag{4.7}$$

Now we state the sufficient condition for the existence of a quasiperiodic solution of equation (4.1) with given Y and function F. Note that it does not imply the uniqueness of the solution due to non-uniqueness of the coefficients y.

Theorem 3. *Let $Y = Y(y) \in S_{\omega_1\omega_2}$ be fixed and condition (4.6) be fulfilled. If there exists $0 < \widetilde{r} < \widetilde{\rho}$ such that*

$$\frac{\|Y\| + v(\widetilde{r})}{\Theta} \le \widetilde{r}, \qquad \frac{\mathrm{D}v(\widetilde{r})}{\Theta} < 1, \tag{4.8}$$

then there exists at least one solution $X \in \overline{B(\widetilde{r})} \subset S_{\omega_1\omega_2}$ of equation (4.1), i.e. there is $x \in W$ satisfying $\|x\| \le \widetilde{r}$ and $X = X(x)$. This solution can be approximated by an iteration process.

Proof. We formulate equation (4.2) as a parametrized fixed point problem

$$X = \mathcal{R}(X,Y) := \mathcal{L}(\sigma)^{-1}\mathcal{F}(X,Y)$$

in $B(\widetilde{\rho}) \subset S_{\omega_1\omega_2}$. By (4.7) and (4.3) we know that

$$\|\mathcal{R}(X,Y)\| \le \frac{\|Y\| + v(r)}{\Theta}$$

if $\|X\| \le \widetilde{r}$. Hence, if the first condition in (4.8) is satisfied, function $\mathcal{R}(\cdot,Y)$ maps $\overline{B(\widetilde{r})}$ into itself. The second condition in (4.8) establishes the contractivity of $\mathcal{R}(\cdot,Y)$, since by (4.4)

$$\|\mathcal{R}(X,Y) - \mathcal{R}(Z,Y)\| \le \frac{\|X-Z\|}{\Theta}\sum_{j=1}^{\infty}\left(|a_j|+|b_j|+|c_j|\right)\sum_{l=0}^{j-1}\|X\|^l\|Z\|^{j-l-1}$$

$$\leq \frac{\mathrm{D}v(r)}{\Theta}\|X - Z\|$$

for all $X, Z \in \overline{B(\widetilde{r})}$. The rest follows from Banach fixed point theorem. □

Example 2. Consider equation

$$X_{n+2} + aX_{n+1} + X_n = d_1 \cos\sqrt{2}n + d_2 \sin 3n + bX_n^3 \tag{4.9}$$

for each $n \in \mathbb{Z}$ with parameters $a \in \mathbb{R}$, $|a| > 2$, $b, d_1, d_2 \in \mathbb{C}$.

In this case, we have $\omega_1 = \sqrt{2}$, $\omega_2 = 3$, $\mathcal{L}(\sigma) = \sigma^2 + a\sigma + \sigma^0$,

$$W := l_1 = \left\{ c = \{c_{kp}\}_{k,p\in\mathbb{Z}} \mid c_{kp} \in \mathbb{C}, \forall k, p \in \mathbb{Z}, \sum_{k,p\in\mathbb{Z}} |c_{kp}| < \infty \right\},$$

$$Y_n = d_1 \cos\sqrt{2}n + d_2 \sin 3n = \sum_{k,p\in\mathbb{Z}} y_{kp}\, \mathrm{e}^{(\sqrt{2}k+3p)\imath n}$$

with $y_{\pm 1,0} = d_1/2$, $y_{0,1} = -y_{0,-1} = d_2/2$ and $y_{kp} = 0$ for each $(k, p) \in \mathbb{Z}^2 \setminus \{(\pm 1, 0), (0, \pm 1)\}$, and $F_n(X_n) = bX_n^3$. Hence $v(r) = |b|r^3$ for any $r \geq 0$, $\|Y\| = |d_1| + |d_2|$ and we seek the solution X in space

$$S_{\omega_1\omega_2} := \left\{ U = \{U_n\}_{n\in\mathbb{Z}} \mid U_n = \sum_{k,p\in\mathbb{Z}} c_{kp}\, \mathrm{e}^{(\sqrt{2}k+3p)\imath n}, c \in W, \forall n \in \mathbb{Z} \right\}.$$

Corollary 2. *If*

$$|d_1| + |d_2| < \sqrt{\frac{4(|a|-2)^3}{27|b|}}, \tag{4.10}$$

then equation (4.9) has at least one quasiperiodic solution $X \in \overline{B(\widetilde{r})} \subset S_{\omega_1\omega_2}$ in the closed ball with radius

$$\widetilde{r} = \sqrt{\frac{4(|a|-2)}{3|b|}} \sin\left(\frac{1}{3}\arcsin\left((|d_1| + |d_2|)\sqrt{\frac{27|b|}{4(|a|-2)^3}}\right)\right). \tag{4.11}$$

Proof. For $\mathcal{L}$ we have

$$|\mathcal{L}(\mathrm{e}^{\imath\alpha})| = |\mathrm{e}^{2\imath\alpha} + a\,\mathrm{e}^{\imath\alpha} + 1|$$
$$= |(a + 2\cos\alpha)\cos\alpha + \imath(a + 2\cos\alpha)\sin\alpha| = |a + 2\cos\alpha|$$

and so $\Theta = |a| - 2 > 0$. Next, $v(r) = |b|r^3$ and conditions (4.8) are equivalent to

$$\frac{|d_1| + |d_2| + |b|\widetilde{r}^3}{|a| - 2} \leq \widetilde{r}, \qquad \frac{3|b|\widetilde{r}^2}{|a| - 2} < 1.$$

Moreover, the smaller $\widetilde{r}$ satisfying these conditions, the more precise X. Thus from the first one it follows that $\widetilde{r}$ has to be a positive real root of function $A(r) :=$

$|d_1|+|d_2|-r(|a|-2)+|b|r^3$. The second one implies, it is the smallest one and nondegenerate. Assumption (4.10) is a necessary condition for $A(r)$ to have two distinct positive real roots. Finally, from [6] we know that $\widetilde{r}$ is given by (4.11). □

Remark 2.

1. The theory derived in this section can be extended to arbitrary number $q>2$ of angle speeds $\omega_1,\dots,\omega_q$ assuming $\frac{\omega_i}{\omega_j}\notin\mathbb{Q}$ for each $i,j=1,\dots,q$, $i\neq j$.

2. Example 2 is a very special application of the theory of Section 4. More generally, in (4.9) one can take $d_1,d_2\in L(\mathbb{C}^N,\mathbb{C}^N)$, i.e. $N\times N$ complex-valued matrices. Then coefficient space V is a Banach complex algebra of $N\times N$ matrices.

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