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Abstract. This work relates to bivariate Meyer-König and Zeller operators, M_n, $n \in \mathbb{N}$ which are not a tensor product setting. We show the monotonicity of the sequence of operators for n under convexity, moreover we study the property of monotonicity in the sense of Li [9]. Finally, we provide an rth order generalization $M_n^{[r]}$ of M_n and also study approximation of $M_n^{[r]}$.

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1. INTRODUCTION

The Cheney and Sharma modification of the well-known univariate Meyer-König and Zeller operators (MKZ) are defined as

$$M_n^* (f,x) = \sum_{k=0}^{\infty} m_{n,k} (x) f \left(\frac{k}{n+k} \right), \; n \in \mathbb{N} \tag{1.1}$$

for $f \in C[0,1)$ and $x \in [0,1)$, $n \in \mathbb{N}$, where $m_{n,k} (x) = (1-x)^{n+1} \binom{n+k}{k} x^k$ [3].

The monotonic convergence of $\{M_n^* (f;x)\}_{n=1}^{\infty}$ under convexity was investigated in [2] by Cheney and Sharma by means of analytical approach, also studied in [8] by Khan, by means of probabilistic approach. Monotonic convergence of several approximation operators are studied deeply in [6] by Khan, Della-Vecchia and Fassih with probabilistic point of view. Some works related to MKZ operators can be viewed in [1–8, 13] and [10, 11].

We shall use the following standard notation. Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$, $\mathbf{k} = (k_1, k_2) \in \mathbb{N}_0^2$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathbf{x}^{\mathbf{k}} = x_1^{k_1} . x_2^{k_2}$, $\mathbf{k}! = k_1!.k_2!$, $|\mathbf{k}| = k_1 + k_2$, $|\mathbf{x}| =$

$x_1 + x_2$, $\mathbf{e}_i$ denotes the unit vector in $\mathbb{R}^2$. Furthermore $\binom{n}{\mathbf{k}} = \frac{n!}{\mathbf{k}!\,(n-|\mathbf{k}|)!}$, and $\sum_{\mathbf{k}=0}^{\infty} = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}$.

Let $S_2 \subset \mathbb{R}^2$ be the open simplex defined by

$$S_2 = \left\{\mathbf{x} \in \mathbb{R}^2;\ \mathbf{x}_i \geq 0,\ i = 1,2,\ |\mathbf{x}| < 1\right\}.$$

For a function $f \in C(S_2)$, bivariate MKZ operators are defined as

$$M_n(f,\mathbf{x}) = (1-|\mathbf{x}|)^{n+1}\sum_{\mathbf{k}=0}^{\infty} f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|}\right)\binom{n+|\mathbf{k}|}{\mathbf{k}}\mathbf{x}^{\mathbf{k}},\ n \in \mathbb{N}, \tag{1.2}$$

where $C(S_2)$ denotes the space of continuous real valued functions defined on S_2. It is obvious that M_n, $n \in \mathbb{N}$, are not a tensor product extension of the univariate MKZ operators M_n^* given by (1.2).

Now we give the following definitions which we shall use.

Definition 1. A continuous function f is said to be convex in $D \subset \mathbb{R}^m$, if

$$f\left(\sum_{i=1}^{r}\alpha_i \mathbf{x}_i\right) \geq \sum_{i=1}^{r}\alpha_i f(\mathbf{x}_i)$$

for every $x_1, x_2, ..., x_r \in D$ and for every non-negative numbers $\alpha_1, \alpha_2, ..., \alpha_r$ such that $\alpha_1 + \alpha_2 + ... + \alpha_r = 1$.

Definition 2. A continuous function f from $D \subseteq \mathbb{R}^2$ into $\mathbb{R}$ is said to be Lipschitz continuous of order μ, $\mu \in (0,1]$, if there exists a constant $A > 0$ such that for every (x_1,x_2), $(y_1,y_2) \in D$, f satisfies

$$|f(x_1,x_2) - f(y_1,y_2)| \leq A\sum_{i=1}^{2}|x_i - y_i|^{\mu},$$

the set of Lipschitz continuous functions is denoted by $Lip_A(\mu, D)$.

In this work, we firstly show the monotonicity of the sequence of bivariate MKZ operators defined by (1.2) under convexity. Secondly we give a kind of monotonnicity similar to the property given by Li in [9]. Namely, we show that if $f(\mathbf{x})$ is a non-negative function and $x_i^{-1} f(\mathbf{x})$ $(i = 1,2)$ is non-increasing for x_i on $(0,1)$, then for each $n \geq 1$, $x_i^{-1} M_n(f;\mathbf{x})$ is also non-increasing for x_i on $(0,1)$. Moreover we build an $r-th$ order generalization $M_n^{[r]}$ of M_n analogues to Kirov and Popova's construction in [8] and investigate its approximation property.

2. Monotonicity for the Sequence of Bivariate Meyer-König and Zeller Operators

In this section, we study the monotonic convergence of the sequence of bivariate MKZ operator under convexity. Note that monotonic convergence of univariate MKZ operator when f is convex, was first obtained by Cheney and Sharma in [3]. We note here that monotonic convergence of the multivariate Baskakov operator is studied in [2].

Theorem 1. *If f is convex, then $M_n(f;\mathbf{x})$ is strictly monotonically non-decreasing in n, unless f is the linear function (in which case $M_n(f;\mathbf{x}) = M_{n+1}(f;\mathbf{x})$ for all $n \in \mathbb{N}$).*

Proof. We have

$$\begin{aligned}
&M_n(f;\mathbf{x}) - M_{n+1}(f;\mathbf{x}) = (1-|\mathbf{x}|)^{n+1} \times \\
&\times \left\{ \sum_{k_2=1}^{\infty} \left[f\left(0, \frac{k_2}{n+k_2}\right) \binom{n+k_2}{k_2} - f\left(0, \frac{k_2}{n+1+k_2}\right) \binom{n+1+k_2}{k_2} \right] \mathbf{x}^{\mathbf{k}} \right. \\
&+ \sum_{k_1=1}^{\infty} \left[f\left(\frac{k_1}{n+k_1}, 0\right) \binom{n+k_1}{k_1} - f\left(\frac{k_1}{n+1+k_1}, 0\right) \binom{n+1+k_1}{k_1} \right] \mathbf{x}^{\mathbf{k}} \\
&+ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} f\left(\frac{\mathbf{k}}{n+1+|\mathbf{k}|}\right) \binom{n+1+|\mathbf{k}|}{\mathbf{k}} x_1^{k_1+1} x_2^{k_2} \\
&\left. + \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} f\left(\frac{\mathbf{k}}{n+1+|\mathbf{k}|}\right) \binom{n+1+|\mathbf{k}|}{\mathbf{k}} x_1^{k_1} x_2^{k_2+1} \right\}.
\end{aligned}$$

Therefore the last equation reduces to the following:

$$\begin{aligned}
&M_n(f;\mathbf{x}) - M_{n+1}(f;\mathbf{x}) = (1-|\mathbf{x}|)^{n+1} \times \\
&\times \left\{ \sum_{k_2=1}^{\infty} \left[f\left(0, \frac{k_2}{n+k_2}\right) \binom{n+k_2}{k_2} - f\left(0, \frac{k_2}{n+1+k_2}\right) \binom{n+1+k_2}{k_2} \right. \right. \\
&\left. + f\left(0, \frac{k_2-1}{n+k_2}\right) \binom{n+k_2}{k_2-1} \right] x_2^{k_2} + \sum_{k_1=1}^{\infty} \left[f\left(\frac{k_1}{n+k_1}, 0\right) \binom{n+k_1}{k_1} \right. \\
&\left. - f\left(\frac{k_1}{n+1+k_1}, 0\right) \binom{n+1+k_1}{k_1} + f\left(\frac{k_1-1}{n+k_1}, 0\right) \binom{n+k_1}{k_1-1} \right] x_1^{k_1} \\
&+ \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \left[f\left(\frac{k_1-1}{n+|\mathbf{k}|}, \frac{k_2}{n+|\mathbf{k}|}\right) \binom{n+|\mathbf{k}|}{\mathbf{k}-e_1} \right.
\end{aligned}$$

$$+f\left(\frac{k_1}{n+|\mathbf{k}|},\frac{k_2-1}{n+|\mathbf{k}|}\right)\binom{n+|\mathbf{k}|}{\mathbf{k}-e_2}$$
$$\left.\left.+f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|}\right)\binom{n+|\mathbf{k}|}{\mathbf{k}}-f\left(\frac{\mathbf{k}}{n+1+|\mathbf{k}|}\right)\binom{n+1+|\mathbf{k}|}{\mathbf{k}}\right]\mathbf{x}^{\mathbf{k}}\right\}.$$

Let I_1, I_2 and I_3 denote the followings

$$I_1=f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|}\right)\binom{n+|\mathbf{k}|}{\mathbf{k}}-f\left(\frac{\mathbf{k}}{n+1+|\mathbf{k}|}\right)\binom{n+1+|\mathbf{k}|}{\mathbf{k}}$$
$$+f\left(\frac{\mathbf{k}-\mathbf{e}_1}{n+|\mathbf{k}|}\right)\binom{n+|\mathbf{k}|}{\mathbf{k}-\mathbf{e}_1}+f\left(\frac{\mathbf{k}-\mathbf{e}_2}{n+|\mathbf{k}|}\right)\binom{n+|\mathbf{k}|}{\mathbf{k}-\mathbf{e}_2},$$

$$I_2=f\left(0,\frac{k_2}{n+k_2}\right)\binom{n+k_2}{k_2}-f\left(0,\frac{k_2}{n+1+k_2}\right)\binom{n+1+k_2}{k_2}$$
$$+f\left(0,\frac{k_2-1}{n+k_2}\right)\binom{n+k_2}{k_2-1},$$

$$I_3f\left(\frac{k_1}{n+k_1},0\right)\binom{n+k_1}{k_1}-f\left(\frac{k_1}{n+1+k_1},0\right)\binom{n+1+k_1}{k_1}$$
$$+f\left(\frac{k_1-1}{n+k_1},0\right)\binom{n+k_1}{k_1-1}.$$

For I_1, we take

$$\alpha_1=\frac{\binom{n+|\mathbf{k}|}{\mathbf{k}}}{\binom{n+1+|\mathbf{k}|}{\mathbf{k}}},\ \alpha_2=\frac{\binom{n+|\mathbf{k}|}{\mathbf{k}-\mathbf{e}_1}}{\binom{n+1+|\mathbf{k}|}{\mathbf{k}}},\ \alpha_3=\frac{\binom{n+|\mathbf{k}|}{\mathbf{k}-\mathbf{e}_2}}{\binom{n+1+|\mathbf{k}|}{\mathbf{k}}}.$$

Clearly α_1, α_2 and α_3 are non-negative numbers, and $\alpha_1+\alpha_2+\alpha_3=1$. On the other hand if we set

$$\mathbf{x}_1=\frac{\mathbf{k}}{n+|\mathbf{k}|},\ \mathbf{x}_2=\frac{\mathbf{k}-\mathbf{e}_1}{n+|\mathbf{k}|},\ \mathbf{x}_3=\frac{\mathbf{k}-\mathbf{e}_2}{n+|\mathbf{k}|},$$

then it follows that

$$\alpha_1\mathbf{x}_1+\alpha_2\mathbf{x}_2+\alpha_3\mathbf{x}_3=\frac{\mathbf{k}}{n+1+|\mathbf{k}|}.$$

Hence, from the convexity of f we obtain that $I_1 \leq 0$. For I_2, we set

$$\alpha_1 = \frac{\binom{n+k_2}{k_2}}{\binom{n+1+k_2}{k_2}}, \ \ \alpha_2 = \frac{\binom{n+k_2}{k_2-1}}{\binom{n+1+k_2}{k_2}}$$

and

$$\mathbf{y}_1 = \left(0, \frac{k_2}{n+k_2}\right), \ \ \mathbf{y}_2 = \left(0, \frac{k_2-1}{n+k_2}\right),$$

then, clearly, $\alpha_1, \ \alpha_2 \geq 0$, and $\alpha_1 + \alpha_2 = 1$. Morever we have

$$\alpha_1 \mathbf{y}_1 + \alpha_2 \mathbf{y}_2 = \left(0, \frac{k_2}{n+k_2}\right).$$

Thus, from the convexity of f we obtain that $I_2 \leq 0$. Similarly we deduce that $I_3 \leq 0$. So we have proved $M_n(f;\mathbf{x}) \leq M_{n+1}(f;\mathbf{x})$ for all $n \in \mathbb{N}$. Using the similar arguments given in [2], the last part of the proof is given as follows: The equality of $M_n(f;\mathbf{x})$ and $M_{n+1}(f;\mathbf{x})$ can be satisfied only if $I_1 = 0$ for all $k_1, k_2 \in \mathbb{N}$ and $I_2 = I_3 = 0$ for all $k_1, k_2 \in \mathbb{N}_0$. But, if f is convex, then $I_1 = 0$ implies that its graph is a plane in triangle. □

In the following, we show that the bivariate MKZ operators can retain a certain monotony, that is

Theorem 2. *Let f be defined on S_2. If $f(\mathbf{x})$ is a non-negative function such that $\frac{f(\mathbf{x})}{x_i}$ $(i = 1,2)$ is non-increasing for x_i on $(0,1)$, then for each $n \geq 1$, $\frac{M_n(f;\mathbf{x})}{x_i}$, is also non-increasing for x_i on $(0,1)$.*

Proof. Straightforward computation gives that for $n \geq 1$ we have, for example with respect to x_1,

$$\frac{\partial}{\partial x_1}\left(\frac{M_n(f;\mathbf{x})}{x_1}\right) =$$

$$= \frac{\partial}{\partial x_1}\left\{\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(1-x_1-x_2)^{n+1}}{x_1} f\left(\frac{k_1}{n+k_1+k_2}, \frac{k_2}{n+k_1+k_2}\right)\binom{n+k_1+k_2}{k_1+k_2} x_1^{k_1} x_2^{k_2}\right\}$$

$$= \frac{\partial}{\partial x_1}\left\{\sum_{k_2=0}^{\infty}\frac{(1-x_1-x_2)^{n+1}}{x_1} f\left(0, \frac{k_2}{n+k_2}\right)\binom{n+k_2}{k_2} x_2^{k_2} + \sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty}\right.$$

$$\left. \times (1-x_1-x_2)^{n+1} f\left(\frac{k_1}{n+k_1+k_2}, \frac{k_2}{n+k_1+k_2}\right)\binom{n+k_1+k_2}{k_1+k_2} x_1^{k_1-1} x_2^{k_2}\right\}$$

$$= \sum_{k_2=0}^{\infty}\frac{\partial}{\partial x_1}\left(\frac{(1-x_1-x_2)^{n+1}}{x_1}\right) f\left(0, \frac{k_2}{n+k_2}\right)\binom{n+k_2}{k_2} x_2^{k_2} + \sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty}$$

$$\times f\left(\frac{k_1}{n+k_1+k_2},\frac{k_2}{n+k_1+k_2}\right)\binom{n+k_1+k_2}{k_1+k_2}\frac{\partial}{\partial x_1}\left[(1-x_1-x_2)^{n+1}x_1^{k_1-1}x_2^{k_2}\right]$$

$$=\sum_{k_2=0}^{\infty}\left(\frac{-(n+1)(1-x_1-x_2)^n x_1-(1-x_1-x_2)^{n+1}}{x_1^2}\right)f\left(0,\frac{k_2}{n+k_2}\right)\binom{n+k_2}{k_2}x_2^{k_2}$$

$$+\sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty}f\left(\frac{k_1}{n+k_1+k_2},\frac{k_2}{n+k_1+k_2}\right)\binom{n+k_1+k_2}{k_1+k_2}x_2^{k_2}$$

$$\times\left[-(n+1)(1-x_1-x_2)^n x_1^{k_1-1}+(1-x_1-x_2)^{n+1}(k_1-1)x_1^{k_1-2}\right]$$

$$=\sum_{k_2=0}^{\infty}\frac{(1-x_1-x_2)^{n+1}}{x_1^2}\left(\frac{-(n+1)x_1}{1-x_1-x_2}-1\right)f\left(0,\frac{k_2}{n+k_2}\right)\binom{n+k_2}{k_2}x_2^{k_2}$$

$$+\sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty}f\left(\frac{k_1}{n+k_1+k_2},\frac{k_2}{n+k_1+k_2}\right)\binom{n+k_1+k_2}{k_1+k_2}(1-x_1-x_2)^n x_2^{k_2}$$

$$\times\left[-nx_1^{k_1-1}-x_1^{k_1-1}+(k_1-1)x_1^{k_1-2}-(k_1-1)x_1^{k_1-1}-(k_1-1)x_1^{k_1-2}x_2\right]$$

$$\times x_1^{k_1-2}(1-x_2)(1-x_1-x_2)^n x_2^{k_2}$$

$$=\sum_{k_2=0}^{\infty}\frac{(1-x_1-x_2)^{n+1}}{x_1^2}\left(-\frac{(1+nx_1-x_2)}{1-x_1-x_2}\right)f\left(0,\frac{k_2}{n+k_2}\right)\binom{n+k_2}{k_2}x_2^{k_2}$$

$$+\sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty}f\left(\frac{k_1}{n+k_1+k_2},\frac{k_2}{n+k_1+k_2}\right)\binom{n+k_1+k_2}{k_1+k_2}(1-x_1-x_2)^n x_2^{k_2}$$

$$\times\left[-(n+k_1)x_1^{k_1-1}+(k_1-1)(1-x_2)x_1^{k_1-2}\right]$$

$$=\sum_{k_2=0}^{\infty}\frac{(1-x_1-x_2)^n}{x_1^2}(-(1+nx_1-x_2))f\left(0,\frac{k_2}{n+k_2}\right)\binom{n+k_2}{k_2}x_2^{k_2}$$

$$+\sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty}f\left(\frac{k_1}{n+k_1+k_2},\frac{k_2}{n+k_1+k_2}\right)\binom{n+k_1+k_2}{k_1+k_2}x_1^{k_1-1}(-n-k_1)x_2^{k_2}$$

$$\times(1-x_1-x_2)^n+\sum_{k_1=2}^{\infty}\sum_{k_2=0}^{\infty}f\left(\frac{k_1}{n+k_1+k_2},\frac{k_2}{n+k_1+k_2}\right)\binom{n+k_1+k_2}{k_1+k_2}$$

$$=\sum_{k_2=0}^{\infty}\frac{-(1+nx_1-x_2)}{x_1^2}f\left(0,\frac{k_2}{n+k_2}\right)\binom{n+k_2}{k_2}(1-x_1-x_2)^n x_2^{k_2}$$

$$-\sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty}f\left(\frac{k_1}{n+k_1+k_2},\frac{k_2}{n+k_1+k_2}\right)\binom{n+k_1+k_2}{k_1+k_2}x_1^{k_1-1}(n+k_1)x_2^{k_2}$$

$$\times(1-x_1-x_2)^n+\sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty} f\left(\frac{k_1+1}{n+k_1+k_2},\frac{k_2}{n+k_1+k_2}\right)\binom{n+k_1+1+k_2}{k_1+1+k_2}$$
$$\times(1-x_2)\,x_1^{k_1-1}k_1\,(1-x_1-x_2)^n\,x_2^{k_2}$$
$$\leq-\sum_{k_2=0}^{\infty}\frac{(1+nx_1-x_2)}{x_1^2} f\left(0,\frac{k_2}{n+k_2}\right)\binom{n+k_2}{k_2}(1-x_1-x_2)^n\,x_2^{k_2}$$
$$+\sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty} f\left(\frac{k_1+1}{n+k_1+k_2+1},\frac{k_2}{n+k_1+k_2+1}\right)\binom{n+k_1+k_2}{k_1+k_2}\frac{n+k_1+k_2+1}{k_1+k_2+1}\frac{k_1}{x_1}$$
$$\times(1-x_2)\,(1-x_1-x_2)^n\,x_1^{k_1}x_2^{k_2}-\sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty} f\left(\frac{k_1}{n+k_1+k_2},\frac{k_2}{n+k_1+k_2}\right)\binom{n+k_1+k_2}{k_1+k_2}$$
$$\times\frac{(n+k_1)}{k_1}\frac{k_1}{x_1}\,(1-x_1-x_2)^n\,x_1^{k_1}x_2^{k_2}$$

Since for $k_1,k_2\geq 0$

$$\frac{n+k_1+k_2+1}{k_1+k_2+1}<\frac{n+k_1+k_2+1}{k_1+1},\qquad -\frac{n+k_1+k_2}{k_1}<-\frac{n+k_1}{k_1}$$

and

$$1+nx_1-x_2\geq 0,\ 1-x_2\leq 1,\quad for\quad x_1,x_2\in S_2,\ n\in\mathbb{N},$$

so we have for $n\geq 1$

$$\frac{\partial}{\partial x_1}\left(\frac{M_n\,(f;\mathbf{x})}{x_1}\right)\leq$$
$$\leq-\sum_{k_2=0}^{\infty}\frac{(1+nx_1-x_2)}{x_1^2} f\left(0,\frac{k_2}{n+k_2}\right)\binom{n+k_2}{k_2}(1-x_1-x_2)^n\,x_2^{k_2}$$
$$+\sum_{k_1=1}^{\infty}\sum_{k_2=0}^{\infty}\left\{f\left(\frac{k_1+1}{n+k_1+k_2+1},\frac{k_2}{n+k_1+k_2+1}\right)\frac{n+k_1+k_2+1}{k_1+k_2+1}\right.$$
$$\left.-f\left(\frac{k_1}{n+k_1+k_2},\frac{k_2}{n+k_1+k_2}\right)\frac{(n+k_1+k_2)}{k_1}\right\}$$
$$\times\binom{n+k_1+k_2}{k_1+k_2}\frac{k_1}{x_1}\,(1-x_1-x_2)^n\,x_1^{k_1}x_2^{k_2},$$

is non-positive, since f is a non-negative function, *such that* $\frac{f\,(\mathbf{x})}{x_i}$ $(i=1,2)$ *is non-increasing for* x_i on $(0,1)$. Similar calculations can be obtained for x_2. □

3. A GENERALIZATION

This section provides an $r-$th order generalization of the bivariate MKZ operators in the sense of Kirov and Popova's construction [8].

Let $C^r(S_2)$, $r \in \mathbb{N}_0$, denote the space of all functions f defined on S_2 and having all continuous partial derivatives up to order r. By $M_n^{[r]}$, we denote the following generalization of M_n. For $\mathbf{x}, \mathbf{t} \in S_2$,

$$M_n^{[r]}(f;\mathbf{x}) = M_n\left(P_{r,\frac{\mathbf{k}}{n+|\mathbf{k}|}}(\Delta\mathbf{x}, f)\right), \tag{3.1}$$

where

$$\begin{aligned}\Delta\mathbf{x} &:= (\Delta x_1, \Delta x_2) = \mathbf{x} - \frac{\mathbf{k}}{n+|\mathbf{k}|} = \left(x_1 - \frac{k_1}{n+|\mathbf{k}|}, x_2 - \frac{k_2}{n+|\mathbf{k}|}\right), \\ \nabla &= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right), \\ (\Delta\mathbf{x}.\nabla)^r &:= \sum_{i+j=r} \binom{r}{j} (\Delta x_1)^i (\Delta x_2)^j \frac{\partial^r}{\partial x_1^i \partial x_2^j},\end{aligned} \tag{3.2}$$

$\binom{r}{j}$ are binomial coefficients, and

$$\begin{aligned}P_{r,\frac{\mathbf{k}}{n+|\mathbf{k}|}}(\Delta\mathbf{x}, f) &= f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|}\right) + (\Delta\mathbf{x}.\nabla) f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|}\right) \\ &+ \frac{(\Delta\mathbf{x}.\nabla)^2}{2!} f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|}\right) + \dots + \frac{(\Delta\mathbf{x}.\nabla)^r}{r!} f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|}\right),\end{aligned} \tag{3.3}$$

the Taylor polynomial for f at $\dfrac{\mathbf{k}}{n+|\mathbf{k}|} \in S_2$.

Now we state the following pointwise estimate for $M_n^{[r]}$

Theorem 3. *If $f \in C^r(S_2)$ and $\frac{\partial^r f}{\partial x_1^i \partial x_2^j} \in Lip_A(\gamma)$, $i + j = r$, then we have*

$$\left|M_n^{[r]}(f;\mathbf{x}) - f(\mathbf{x})\right| \le \frac{2A}{(r-1)!} \frac{\gamma}{\gamma+r} B(\gamma, r) M_n(g;\mathbf{x}) \tag{3.4}$$

for $x \in S_2$, where $g(\mathbf{s}) = |\mathbf{x}-\mathbf{s}|^{r+\gamma}$, $B(\gamma, r)$ is the familiar beta function, r, $n \in \mathbb{N}_0$, $0 < \gamma \le 1$ and $A > 0$.

Proof. From (3.1) and (3.3) we have

$$f(\mathbf{x}) - M_n^{[r]}(f;\mathbf{x}) = \sum_{\mathbf{k}=0}^{\infty} R_{r,\frac{\mathbf{k}}{n+|\mathbf{k}|}}(\Delta\mathbf{x}, f) \binom{n+|\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1-|\mathbf{x}|)^{n+1}, \tag{3.5}$$

where

$$R_{r,\frac{\mathbf{k}}{n+|\mathbf{k}|}}(\Delta\mathbf{x},f) := f(\mathbf{x}) - \sum_{h=0}^{r} \frac{(\Delta\mathbf{x}.\nabla)^h}{h!} f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|}\right) \binom{n+|\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1-|\mathbf{x}|)^{n+1}. \tag{3.6}$$

(3.6) can be given by

$$R_{r,\frac{\mathbf{k}}{n+|\mathbf{k}|}}(\Delta\mathbf{x},f) = \frac{1}{(r-1)!} \int_0^1 (\Delta\mathbf{x}.\nabla)^r \times \left[f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|} + t\Delta\mathbf{x}\right) - f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|}\right)\right] (1-t)^{r-1} dt. \tag{3.7}$$

Using (3.2),(3.7) results in

$$R_{r,\frac{\mathbf{k}}{n+|\mathbf{k}|}}(\Delta\mathbf{x},f) = \frac{1}{(r-1)!} \int_0^1 \sum_{i+j=r} \binom{r}{j} (\Delta x_1)^i (\Delta x_2)^j \frac{\partial^r}{\partial x_1^i \partial x_2^j} \times \left[f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|} + t\Delta\mathbf{x}\right) - f\left(\frac{\mathbf{k}}{n+|\mathbf{k}|}\right)\right] (1-t)^{r-1} dt. \tag{3.8}$$

Substituting (3.8) into (3.5) and taking into account that $\frac{\partial^r f}{\partial x_1^i \partial x_2^j} \in Lip_A(\gamma)$ we arrive at the following.

$$\begin{aligned} \left| f(\mathbf{x}) - M_n^{[r]}(f;\mathbf{x}) \right| &\le \frac{A}{(r-1)!} \sum_{\mathbf{k}=0}^{\infty} (|\Delta x_1| + |\Delta x_2|)^r \left(|\Delta x_1|^\gamma + |\Delta x_2|^\gamma\right) \\ &\quad \times \binom{n+|\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1-|\mathbf{x}|)^{n+1} \int_0^1 t^\gamma (1-t)^{r-1} dt \\ &\le \frac{2A}{(r-1)!} \sum_{\mathbf{k}=0}^{\infty} (|\Delta x_1| + |\Delta x_2|)^{r+\gamma} \binom{n+|\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1-|\mathbf{x}|)^{n+1} B(\gamma+1,r) \\ &\le \frac{2A}{(r-1)!} \frac{\gamma}{\gamma+r} B(\gamma,r) \sum_{\mathbf{k}=0}^{\infty} \left| \mathbf{x} - \frac{\mathbf{k}}{n+|\mathbf{k}|} \right|^{r+\gamma} \binom{n+|\mathbf{k}|}{\mathbf{k}} \mathbf{x}^{\mathbf{k}} (1-|\mathbf{x}|)^{n+1}, \end{aligned}$$

which proves (3.4). □

For the uniform convergence of $M_n^{[r]}(f)$, let us consider the above mentioned function $g(\mathbf{s}) = |\mathbf{x}-\mathbf{s}|^{r+\gamma}$. Obviously $g(\mathbf{x}) = 0$ and g is continuous on S_2. From multivariate extension of the Bohman-Korovkin theorem (see [12]) we have

$$\|M_n(g)\|_{C(S_2)} \to 0 \text{ as } n \to \infty.$$

Therefore by means of Theorem 3 we arrive at the following result

$$\left\| M_n^{[r]}(f) - f \right\|_{C(S_2)} \to 0 \text{ as } n \to \infty.$$

REFERENCES

[1] A. Altın, O. Doğru, and F. Taşdelen, "The generalization of Meyer-König and Zeller operators by generating functions," *J. Math. Anal. Appl.*, vol. 312, no. 1, pp. 181–194, 2005.

[2] F. Cao, C. Ding, and Z. Xu, "On multivariate Baskakov operator," *J. Math. Anal. Appl.*, vol. 307, no. 1, pp. 274–291, 2005.

[3] E. W. Cheney and A. Sharma, "Bernstein power series," *Can. J. Math.*, vol. 16, pp. 241–252, 1964.

[4] O. Doğru and O. Duman, "Statistical approximation of Meyer-König and Zeller operators based on q-integers," *Publ. Math.*, vol. 68, no. 1-2, pp. 199–214, 2006.

[5] O. Doğru and V. Gupta, "Korovkin-type approximation properties of bivariate q-Meyer-König and Zeller operators," *Calcolo*, vol. 43, no. 1, pp. 51–63, 2006.

[6] M. Khan, B. Della Vecchia, and A. Fassih, "On the monotonicity of positive linear operators," *J. Approx. Theory*, vol. 92, no. 1, pp. 22–37, art. no. at963 113, 1998.

[7] R. A. Khan, "Some probabilistic methods in the theory of approximation operators," *Acta Math. Acad. Sci. Hung.*, vol. 35, pp. 193–203, 1980.

[8] G. Kirov and L. Popova, "A generalization of the linear positive operators," *Math. Balk., New Ser.*, vol. 7, no. 2, pp. 149–162, 1993.

[9] Z. Li, "Bernstein polynomials and modulus of continuity," *J. Approx. Theory*, vol. 102, no. 1, pp. 171–174, art. no. jath.1999.3374, 2000.

[10] A.-J. López-Moreno and F.-J. Muñoz-Delgado, "Asymptotic expansion of multivariate conservative linear operators," *J. Comput. Appl. Math.*, vol. 150, no. 2, pp. 219–251, 2003.

[11] T. Trif, "An elementary proof of the preservation of Lipschitz constants by the Meyer-König and Zeller operators," *JIPAM, J. Inequal. Pure Appl. Math.*, vol. 4, no. 5, p. 3, 2003.

[12] V. I. Volkov, "On the convergence of sequences of linear positive operators in the space of continuous functions of two variables," *Dokl. Akad. Nauk SSSR*, vol. 115, pp. 17–19, 1957.

[13] H. Wang, "Properties of convergence for the q-Meyer-König and Zeller operators," *J. Math. Anal. Appl.*, vol. 335, no. 2, pp. 1360–1373, 2007.

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