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# THE $(P,Q)$ GENERALIZED ANTI-REFLEXIVE EXTREMAL RANK SOLUTIONS TO A SYSTEM OF MATRIX EQUATIONS

QINGFENG XIAO

*Abstract.* Let $n \times n$ complex matrices $P$ and $Q$ be nontrivial generalized reflection matrices, i.e., $P^* = P = P^{-1} \neq I_n$, $Q^* = Q = Q^{-1} \neq I_n$. A complex matrix $A$ with order $n$ is said to be a $(P,Q)$ generalized anti-reflexive matrix, if $PAQ = -A$. We in this paper mainly investigate the $(P,Q)$ generalized anti-reflexive maximal and minimal rank solutions to the system of matrix equation $AX = B$. We present necessary and sufficient conditions for the existence of the maximal and minimal rank solutions, with $(P,Q)$ generalized anti-reflexive, of the system. Expressions of such solutions to this system are also given when the solvability conditions are satisfied. In addition, in correspondence with the minimal rank solution set to the system, the explicit expression of the nearest matrix to a given matrix in the Frobenius norm has been provided.

2000 *Mathematics Subject Classification:* 15A29

*Keywords:* matrix equation, $(P,Q)$ generalized anti-reflexive matrix, maximal rank, minimal rank, optimal approximate solution

## 1. INTRODUCTION

Throughout this paper, let $C_r^{n\times m}$ be the set of all $n \times m$ complex matrices with rank $r$, $UC^{n\times n}$ be the set of all $n \times n$ unitary matrices. Denote by $I_n$ the identity matrix with order $n$. Let $J_n = (e_n e_{n-1} \dots e_1)$, where $e_i$ is the $i^{th}$ column of $I_n$. For a matrix $A$, $A^*$, $A^+$, $\|A\|$ and $r(A)$ represent its conjugate transpose, Moore-Penrose inverse, Frobenius norm and rank, respectively.

**Definition 1.** Let $P, Q \in C^{n\times n}$ be nontrivial generalized reflection matrices, i.e., $P^* = P = P^{-1} \neq I_n$, $Q^* = Q = Q^{-1} \neq I_n$, then matrix $A \in C^{n\times n}$ is said to be the $(P,Q)$ generalized reflexive (or anti-reflexive) matrix, if $PAQ = A$(or $PAQ = -A$).

Obviously, if let $P = Q = J_n$ in Definition 1, then matrix $A$ is the well-known centrosymmetric (or anti-centrosymmetric) matrix, which plays an important role in many areas (see, e.g., [6, 13, 16–19]), and has been widely and extensively studied (see, e.g., [1, 25, 28]). Moreover, let $P = Q$, then matrix $A$ is called generalized centrosymmetric (or anti-centrosymmetric) matrix [12, 22]. As being the extensions

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of the centrosymmetric matrices and generalized centrosymmetric matrices, the generalized reflection matrices and generalized reflexive matrices have many special properties and practical applications, and have also been frequently investigated, see for instance, [3, 7, 8, 26, 27].

In matrix theory and applications, many problems are closely related to the ranks of some matrix expressions with variable entries, so it is necessary to explicitly characterize the possible ranks of the matrix expressions concerned. The study on the possible ranks of matrix equations can be traced back to the late 1970s (see, e.g. [2, 9–11, 23]. Recently, the extremal ranks, i.e. maximal and minimal ranks, of some matrix expressions have found many applications in control theory [4, 5], statistics, and economics (see, e.g. [14, 15, 21]).

In this paper, we consider the $(P,Q)$ generalized anti-reflexive extremal rank solutions of the matrix equation

$$AX = B, \tag{1.1}$$

where $A$ and $B$ are given matrices in $C^{n\times m}$. In 1987, Uhlig [23] gave the maximal and minimal ranks of solutions to system (1.1). By applying the matrix rank method, recently, Tian [20] obtained the minimal rank of solutions to the matrix equation $A = BX + YC$. Xiao et al. [24] in 2009 considered the symmetric minimal rank solution to system (1.1). The $(P,Q)$ generalized reflexive and anti-reflexive matrices with respect to the generalized reflection matrix dual $(P,Q)$ are two classes of important matrices and have engineering and scientific applications. The $(P,Q)$ generalized anti-reflexive maximal and minimal rank solutions of the matrix equation (1.1), however, has not been considered yet. In this paper, we will discuss this problem.

We also consider the matrix nearness problem

$$\min_{X\in S_m} \left\| X - \tilde{X} \right\|_F, \tag{1.2}$$

where $\tilde{X}$ is a given matrix in $C^{n\times m}$ and $S_m$ is the minimal rank solution set of Eq. (1.1).

The matrix nearness problem (1.2) is the so-called optimal approximation problem, which has important application in practice, and has been discussed far and wide (see, e.g., [7, 8, 12, 25, 27] and the references therein).

We organize this paper as follows. In Section 2, we first establish a representation for the generalized reflection matrix dual $(P,Q)$. Then we give necessary and sufficient conditions for the existence of $(P,Q)$ generalized anti-reflexive solution to (1.1). We also give the expressions of such solutions when the solvability conditions are satisfied. In Section 3 we establish formulas of maximal and minimal ranks of $(P,Q)$ generalized anti-reflexive solutions to (1.1), and present the $(P,Q)$ generalized anti-reflexive extremal rank solutions to (1.1). In Section 4 we present the expression of the optimal approximation solution to the set of the minimal rank solution.

## 2. $(P,Q)$ GENERALIZED ANTI-REFLEXIVE SOLUTION TO (1.1)

In this section we first introduce some structure properties of the generalized reflection matrix dual $(P,Q)$ and establish the representations of $(P,Q)$ generalized anti-reflexive matrix. Then we give the necessary and sufficient conditions for the existence of and the expressions for the $(P,Q)$ generalized anti-reflexive solution of Eq. (1.1).

**Lemma 1** ([12]). *Given generalized reflection matrices $P,Q \in C^{n\times n}$. Let*

$$P_1 = \frac{I_n + P}{2} \in C_r^{n\times n}, \quad Q_1 = \frac{I_n + Q}{2} \in C_s^{n\times n}, \quad P_2 = \frac{I_n - P}{2}, \quad Q_2 = \frac{I_n - Q}{2}.$$

*Then $r(P_2) = n-r$, $r(Q_2) = n-s$, and there exist column orthogonal matrices $U_{11} \in C^{n\times r}$, $U_{22} \in C^{n\times(n-r)}$, $V_{11} \in C^{n\times s}$, $V_{22} \in C^{n\times(n-s)}$, such that*

$$P_1 = U_{11}U_{11}^*, \quad P_2 = U_{22}U_{22}^*, \quad Q_1 = V_{11}V_{11}^*, \quad Q_2 = V_{22}V_{22}^*. \tag{2.1}$$

*Remark* 1. Denote $U = [U_{11},U_{22}]$, $V = [V_{11},V_{22}]$, it follows from Lemma 1 that $U,V \in UC^{n\times n}$, and

$$P = U\begin{bmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{bmatrix}U^*, \quad Q = V\begin{bmatrix} I_s & 0 \\ 0 & -I_{n-s} \end{bmatrix}V^*, \tag{2.2}$$

where the symbols "0" stand for null matrices with associated orders (in the sequel, we always mark them like this).

**Lemma 2** ([8]). *Let $A \in C^{n\times n}$ and generalized reflection matrices $P$, $Q$ with the forms of* (2.2)*, then $A$ is the $(P,Q)$ generalized anti-reflexive matrix if and only if*

$$A = U\begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix}V^*, \tag{2.3}$$

*where $M \in C^{r\times(n-s)}$, $N \in C^{(n-r)\times s}$ are arbitrary.*

Given matrices $A_1 \in C^{m\times n}$, $B_1 \in C^{m\times p}$, by making generalized singular value decomposition to $[A_1,B_1]$, we have

$$A_1 = M_1\Sigma_{A_1}U_1, \quad B_1 = M_1\Sigma_{B_1}V_1 \tag{2.4}$$

where $M_1$ is a $m\times m$ nonsingular matrix, $U_1 \in UC^{n\times n}$, $V_1 \in UC^{p\times p}$,

$$\Sigma_{A_1} = \begin{bmatrix} I & 0 & 0 \\ 0 & S_{A_1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{matrix} r_1-s_1 \\ s_1 \\ k_1-r_1 \\ m-k_1 \end{matrix}, \quad \Sigma_{B_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{B_1} & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}\begin{matrix} r_1-s_1 \\ s_1 \\ k_1-r_1 \\ m-k_1 \end{matrix},$$

$k_1 = r[A_1,B_1]$, $s_1 = r(A_1)+r(B_1)-r[A_1,B_1]$, $S_{A_1} = diag(\alpha_1,\ldots,\alpha_{s_1})$, $S_{B_1} = diag(\beta_1,\ldots,\beta_{s_1})$, $r_1 = r(A_1)$, $0 < \alpha_{s_1} \le \cdots \le \alpha_1 < 1$, $0 < \beta_1 \le \cdots \le \beta_{s_1} < 1$, $\alpha_i^2+\beta_i^2 = 1$, $i = 1,\ldots,s_1$.

**Lemma 3.** *Given matrices $A_1 \in C^{m\times n}$, $B_1 \in C^{m\times p}$, the generalized singular value decomposition of the matrix pair $[A_1, B_1]$ is given by (2.4), then matrix equation $A_1X = B_1$ is consistent, if and only if*

$$r[A_1, B_1] = r(A_1), \tag{2.5}$$

*and the expression of its general solution is*

$$X = U_1^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_1}^{-1} S_{B_1} \\ Y_{31} & Y_{32} \end{bmatrix} V_1, \tag{2.6}$$

*where $Y_{31} \in C^{(n-r_1)\times(p-s_1)}$, $Y_{32} \in C^{(n-r_1)\times s_1}$ are arbitrary.*

*Proof.* With (2.4)(2.4) we have

$$r(B_1 - A_1X) = r(M_1\Sigma_{B_1}V_1 - M_1\Sigma_{A_1}U_1X) = r(\Sigma_{B_1} - \Sigma_{A_1}U_1XV_1^*).$$

Let $Y = U_1XV_1^*$ and Partition $Y$ with $Y = (Y_{ij})_{3\times3}$, then

$$\Sigma_{B_1} - \Sigma_{A_1}Y = \begin{bmatrix} -Y_{11} & -Y_{12} & -Y_{13} \\ -S_{A_1}Y_{21} & S_{B_1} - S_{A_1}Y_{22} & -S_{A_1}Y_{23} \\ 0 & 0 & I_{B_1} \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_1 - s_1 \\ s_1 \\ k_1 - r_1 \\ m - k_1 \end{matrix}. \tag{2.7}$$

Noting that $Y_{ij}\,(i = 1,2, j = 1,2,3)$ are arbitrary, then

$$\min r(B_1 - A_1X) = \min r(\Sigma_{B_1} - \Sigma_{A_1}Y) = k_1 - r_1 = r(A_1, B_1) - r(A_1).$$

$A_1X = B_1$ is solvable in $C^{n\times p}$ if and only if $\min r(B_1 - A_1X) = 0$. Then matrix equation $A_1X = B_1$ is consistent, if and only if (2.5) holds. In this case, from (2) and $Y = U_1XV_1^*$, its general solution can be expressed as (2.6). The proof is completed. □

Assume the given generalized reflection matrices $P$, $Q$ with the forms of (2.2). Let

$$AU = [A_2, A_3], \quad BV = [B_2, B_3], \tag{2.8}$$

where $A_2 \in C^{m\times r}$, $A_3 \in C^{m\times(n-r)}$, $B_2 \in C^{m\times s}$, $B_3 \in C^{m\times(n-s)}$, and the generalized singular value decomposition of the matrix pair $[A_2, B_3]$, $[A_3, B_2]$ are, respectively,

$$A_2 = M_2\Sigma_{A_2}U_2, \quad B_3 = M_2\Sigma_{B_3}V_2, \tag{2.9}$$

$$A_3 = M_3\Sigma_{A_3}U_3, \quad B_2 = M_3\Sigma_{B_2}V_3, \tag{2.10}$$

where $U_2 \in UC^{r\times r}$, $V_2 \in UC^{(n-s)\times(n-s)}$, $U_3 \in UC^{(n-r)\times(n-r)}$, $V_3 \in UC^{s\times s}$, nonsingular matrices $M_2, M_3 \in C^{m\times m}$, $k_2 = r[A_2, B_3]$, $r_2 = r(A_2)$, $s_2 = r(A_2) + r(B_3) - r[A_2, B_3]$, and $k_3 = r[A_3, B_2]$, $r_3 = r(A_3)$, $s_3 = r(A_3) + r(B_2) - r[A_3, B_2]$,

$$\Sigma_{A_2} = \begin{bmatrix} I & 0 & 0 \\ 0 & S_{A_2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_2 - s_2 \\ s_2 \\ k_2 - r_2 \\ m - k_2 \end{matrix}, \qquad \Sigma_{B_3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{B_3} & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_2 - s_2 \\ s_2 \\ k_2 - r_2 \\ m - k_2 \end{matrix},$$

$$\Sigma_{A_3} = \begin{bmatrix} I & 0 & 0 \\ 0 & S_{A_3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_3 - s_3 \\ s_3 \\ k_3 - r_3 \\ m - k_3 \end{matrix}, \qquad \Sigma_{B_2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & S_{B_2} & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} r_3 - s_3 \\ s_3 \\ k_3 - r_3 \\ m - k_3 \end{matrix},$$

Then we can establish the existence theorems as follows.

**Theorem 1.** *Let $A, B \in C^{m\times n}$ and generalized reflection matrices $P$, $Q$ of size $n$ be known. Suppose generalized reflection matrices $P$, $Q$ with the forms of (2.2), $AU, BU$ have the partition forms of (2.8), and the generalized singular value decompositions of the matrix pair $[A_2, B_3]$ and $[A_3, B_2]$ are given by (2.9) and (2.10). Then the equation (1.1) has a $(P,Q)$ generalized anti-reflexive solution $X$ if and only if*

$$r[A_2, B_3] = r(A_2), \quad r[A_3, B_2] = r(A_3), \tag{2.11}$$

*and its general solution can be expressed as*

$$X = U \begin{bmatrix} 0 & U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} V_2 \\ U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ W_{31} & W_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} V^*, \tag{2.12}$$

*where $Z_{31} \in C^{(r-r_2)\times(n-s-s_2)}$, $Z_{32} \in C^{(r-r_2)\times s_2}$, $W_{31} \in C^{(n-r-r_3)\times(s-s_3)}$, $W_{32} \in C^{(n-r-r_3)\times s_3}$ are arbitrary.*

*Proof.* Suppose the matrix equation (1.1) has a solution $X$ that is $(P,Q)$ generalized anti-reflexive, then it follows from Lemma 2 that there exist $M \in C^{r\times(n-s)}$, $N \in C^{(n-r)\times s}$ satisfying

$$X = U \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix} V^* \quad and \quad AX = B \tag{2.13}$$

By (2.8), that is

$$[A_2 \ \ A_3] \begin{bmatrix} 0 & M \\ N & 0 \end{bmatrix} = [B_2 \ \ B_3], \tag{2.14}$$

i.e.

$$A_2 M = B_3, \quad A_3 N = B_2. \tag{2.15}$$

Therefore by Lemma 3, (2.11) holds, and

$$M = U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} V_2, \quad N = U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ W_{31} & W_{32} \end{bmatrix} V_3, \tag{2.16}$$

where $Z_{31} \in C^{(r-r_2)\times(n-s-s_2)}$, $Z_{32} \in C^{(r-r_2)\times s_2}$, $W_{31} \in C^{(n-r-r_3)\times(s-s_3)}$, $W_{32} \in C^{(n-r-r_3)\times s_3}$ are arbitrary. Substituting (2.16) into (2.13) yields that the $(P,Q)$ generalized anti-reflexive solution $X$ of the matrix equation (1.1) can be represented by (2.12). The proof is completed. □

## 3. $(P,Q)$ GENERALIZED ANTI-REFLEXIVE EXTREMAL RANK SOLUTIONS TO (1.1)

In this section, we first derive the formulas of the maximal and minimal ranks of $(P,Q)$ generalized anti-reflexive solutions of (1.1), then present the expressions of $(P,Q)$ generalized anti-reflexive maximal and minimal rank solutions to (1.1).

**Theorem 2.** *Suppose that the matrix equation (1.1) has a $(P,Q)$ generalized anti-reflexive solution $X$ and $\Omega$ is the set of all $(P,Q)$ generalized anti-reflexive solutions of (1.1). Then the extreme ranks of $X$ are as follows:*

(1) *The maximal rank of $X$ is*

$$\min\{n-s, r-r(A_2)+r(B_3)\}+\min\{s, n-r-r(A_3)+r(B_2)\}. \tag{3.1}$$

*The general expression of $X$ satisfying (3.1) is*

$$X = U\begin{bmatrix} 0 & U_2^*\begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1}S_{B_3} \\ Z_{31} & Z_{32}\end{bmatrix}V_2 \\ U_3^*\begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ W_{31} & W_{32}\end{bmatrix}V_3 & 0\end{bmatrix}V^*, \tag{3.2}$$

*where $Z_{31} \in C^{(r-r_2)\times(n-s-s_2)}$, $W_{31} \in C^{(n-r-r_3)\times(s-s_3)}$ are chosen such that $r(Z_{31}) = \min(r-r_2, n-s-s_2)$, $r(W_{31}) = \min(n-r-r_3, s-s_3)$, $Z_{32} \in C^{(r-r_2)\times s_2}$, $W_{32} \in C^{(n-r-r_3)\times s_3}$ are arbitrary.*

(2) *The minimal rank of $X$ is*

$$\min_{X\in\Omega} r(X) = r(B_2)+r(B_3). \tag{3.3}$$

*The general expression of $X$ satisfying (3.3) is*

$$X = U\begin{bmatrix} 0 & U_2^*\begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1}S_{B_3} \\ 0 & Z_{32}\end{bmatrix}V_2 \\ U_3^*\begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ 0 & W_{32}\end{bmatrix}V_3 & 0\end{bmatrix}V^*, \tag{3.4}$$

*where $Z_{32} \in C^{(r-r_2)\times s_2}$, $W_{32} \in C^{(n-r-r_3)\times s_3}$ are arbitrary.*

*Proof.* (1) By (2.12),

$$\max_{X\in\Omega} r(X) = \max_{Z_{31}} r\begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1}S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} + \max_{W_{31}} r\begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ W_{31} & W_{32} \end{bmatrix}, \tag{3.5}$$

$$\max_{Z_{31}} r\begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1}S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} = s_2 + \min\{r-r_2, n-s-s_2\} \tag{3.6}$$

$$= \min\{n-s, r-r_2+s_2\} = \min\{n-s, r-r(A_2)+r(B_3)\},$$

and

$$\max_{W_{31}} r\begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ W_{31} & W_{32} \end{bmatrix} = s_3 + \min\{n-r-r_3, s-s_3\} \tag{3.7}$$

$$= \min\{s, n-r-r_3+s_3\} = \min\{s, n-r-r(A_3)+r(B_2)\}.$$

Taking (3.6) and (3.7) into (3.5) yields (3.1).

According to the general expression of the solution in Theorem 1, it is easy to verify the rest of part in (1).

(2) By (2.12),

$$\min_{X\in\Omega} r(X) = \min_{Z_{31}} r\begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1}S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} + \min_{W_{31}} r\begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ W_{31} & W_{32} \end{bmatrix}, \tag{3.8}$$

$$\min_{Z_{31}} r\begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1}S_{B_3} \\ Z_{31} & Z_{32} \end{bmatrix} = s_2 = r(B_3) \tag{3.9}$$

and

$$\min_{W_{31}} r\begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ W_{31} & W_{32} \end{bmatrix} = s_3 = r(B_2). \tag{3.10}$$

Taking (3.9) and (3.10) into (3.8) yields (3.3).

According to the general expression of the solution in Theorem 1, it is easy to verify the rest of part in (2). The proof is completed. □

## 4. THE EXPRESSION OF THE OPTIMAL APPROXIMATION SOLUTION TO THE SET OF THE MINIMAL RANK SOLUTION

Let $\Omega = \{X : AX = B, PXQ = -X\}$ be the solution set and $S_m = \{X \in \Omega : r(X) = \min\{r(X') : X' \in \Omega\}\}$ the set of minimal rank solutions. From (3.4), When the solution set $S_m$ is nonempty, it is easy to verify that $S_m$ is a closed convex set, therefore there exists a unique solution $\hat{X}$ to the matrix nearness Problem (1.2).

**Theorem 3.** *Given matrix $\tilde{X}$, and the other given notations and conditions are the same as in Theorem 1. Let*

$$U^*\tilde{X}V = \begin{bmatrix} \tilde{X}_{11} & \tilde{X}_{12} \\ \tilde{X}_{21} & \tilde{X}_{22} \end{bmatrix}, \quad \tilde{X}_{12} \in C^{r\times(n-s)}, \quad \tilde{X}_{21} \in C^{(n-r)\times s}, \tag{4.1}$$

*and we denote*

$$U_2\tilde{X}_{12}V_2^* = \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{21} & \tilde{Z}_{22} \\ \tilde{Z}_{31} & \tilde{Z}_{32} \end{bmatrix}, \quad U_3\tilde{X}_{21}V_3^* = \begin{bmatrix} \tilde{W}_{11} & \tilde{W}_{12} \\ \tilde{W}_{21} & \tilde{W}_{22} \\ \tilde{W}_{31} & \tilde{W}_{32} \end{bmatrix}. \tag{4.2}$$

*If $S_m$ is nonempty, then Problem (1.2) has a unique $\hat{X}$ which can be represented as*

$$\hat{X} = U \begin{bmatrix} 0 & U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1}S_{B_3} \\ 0 & \tilde{Z}_{32} \end{bmatrix} V_2 \\ U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ 0 & \tilde{W}_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} V^*, \tag{4.3}$$

*where $\tilde{Z}_{32}, \tilde{W}_{32}$ are the same as in (4.2).*

*Proof.* When $S_m$ is nonempty, it is easy to verify from (3.4) that $S_m$ is a closed convex set. Since $C^{n\times n}$ is a uniformly convex Banach space under Frobenius norm, there exists a unique solution for Problem (1.2). By Theorem 2, for any $X \in S_m$, $X$ can be expressed as

$$X = U \begin{bmatrix} 0 & U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1}S_{B_3} \\ 0 & Z_{32} \end{bmatrix} V_2 \\ U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1}S_{B_2} \\ 0 & W_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} V^*, \tag{4.4}$$

where $Z_{32} \in C^{(r-r_2)\times s_2}$, $W_{32} \in C^{(n-r-r_3)\times s_3}$ are arbitrary.

Using the invariance of the Frobenius norm under unitary transformations, we have

$$\|X-\tilde{X}\|^2 = \left\| \begin{bmatrix} 0 & U_2^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_2}^{-1} S_{B_3} \\ 0 & Z_{32} \end{bmatrix} V_2 \\ U_3^* \begin{bmatrix} 0 & 0 \\ 0 & S_{A_3}^{-1} S_{B_2} \\ 0 & W_{32} \end{bmatrix} V_3 & 0 \end{bmatrix} - U^* \tilde{X} V \right\|^2$$

$$= \left\|Z_{32}-\tilde{Z}_{32}\right\|^2 + \left\|W_{32}-\tilde{W}_{32}\right\|^2 + \left\|S_{A_2}^{-1}S_{B_3}-\tilde{Z}_{22}\right\|^2 + \left\|S_{A_3}^{-1}S_{B_2}-\tilde{W}_{22}\right\|^2 + \left\|\tilde{X}_{11}\right\|^2 + \left\|\tilde{X}_{22}\right\|^2 + \left\|\tilde{Z}_{11}\right\|^2 + \left\|\tilde{Z}_{12}\right\|^2 + \left\|\tilde{Z}_{21}\right\|^2 + \left\|\tilde{Z}_{31}\right\|^2 \left\|\tilde{W}_{11}\right\|^2 + \left\|\tilde{W}_{12}\right\|^2 + \left\|\tilde{W}_{21}\right\|^2 + \left\|\tilde{W}_{31}\right\|^2.$$

Therefore, $\|X-\tilde{X}\|$ reaches its minimum if and only if

$$Z_{32} = \tilde{Z}_{32}, \quad W_{32} = \tilde{W}_{32}. \tag{4.5}$$

Substituting (4.5) into (4.4) yields (4.3). The proof is completed. □

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