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Zhenbang Li and Changchun Liu

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ON THE BLOW-UP OF SOLUTIONS FOR THE UNSTABLE SIXTH ORDER PARABOLIC EQUATION

ZHENBANG LI AND CHANGCHUN LIU

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Abstract. We study the universal blow-up of sixth-order parabolic thin film equation with the initial boundary conditions. We prove that the problem in finite time blow-up will happen, if the initial datum $u_0 \in C^{6+\alpha}(\overline{\Omega})$ with $-\int_\Omega \left(H(u_0)+\frac{1}{2}|\Delta u_0|^2\right)dx \geq 0$. And then, we get some nondegeneracy results on blow-up for this problem.

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1. INTRODUCTION

In this paper, we consider the following initial boundary problem of sixth-order equation

$$\begin{cases} u_t - \Delta(\Delta^2 u - |u|^{p-1}u) = 0, & \text{in } \Omega\times(0,T), \\ u = \Delta u = \Delta^2 u = 0, & \text{on } \partial\Omega\times[0,T), \\ u = u_0, & \text{in } \Omega\times\{0\}, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N$ is a bounded smooth domain, $p > 1$.

During the past years, only a few works have been devoted to the sixth-order parabolic equation [1, 4, 5, 7].

Recently, Evans, Galaktionov and King [4, 5] considered the sixth-order thin film equation containing an unstable (backward parabolic) second-order term

$$\frac{\partial u}{\partial t} = \operatorname{div}\left[|u|^n \nabla \Delta^2 u\right] - \Delta(|u|^{p-1}u), n > 0, p > 1.$$

By a formal matched expansion technique, they show that, for the first critical exponent $p = p_0 = n + 1 + \frac{4}{N}$ for $n \in (0, \frac{5}{4})$, where N is the space dimension, the free-boundary problem with zero-height, zero-contact-angle, zero-moment, and zero-flux conditions at the interface admits a countable set of continuous branches of radially symmetric self-similar blow-up solutions $u_k(x,t) = (T-t)^{-\frac{N}{nN+6}} f_k(y)$, $y = \frac{x}{(T-t)^{\frac{1}{nN+6}}}$, where $T > 0$ is the blow-up time.

In fact, when $n = 0$, the equation (1.1) is obtained. In this paper we study the universal blow-up and some nondegeneracy results on blow-up of the equation (1.1). Our method about universal finite time blow-up is similar to that of Elliott and Zheng [3] which treats the blow-up problem for Cahn-Hilliard equation. We can show that if the initial datum $u_0 \in C^{6+\alpha}(\overline{\Omega})$ with $-\int_\Omega \left(H(u_0) + \frac{1}{2}|\Delta u_0|^2\right) dx \geq 0$, then the solution to the above problem (1.1) should blow up in finite time.

We also establish some nondegeneracy results on the blow-up of the problem. We mainly follow the purpose of Giga and Kohn [6] and Cheng and Zheng [2]. More accurately, there is a constant $\varepsilon > 0$, depending on n, p and the constant in the estimates of the fundamental solution to $u_t - \Delta^3 u = 0$ (see (3.1) below), such that if u is a solution of the equation

$$u_t - \Delta(\Delta^2 u - |u|^{p-1}u) = 0, \qquad \text{on} \quad Q_r = B_r(a) \times [t_1 - r^6, t_1),$$

where $1 < p < 3, a \in \mathbb{R}^n, t_1 \in \mathbb{R}$ and $0 < r \leq 1$, and if

$$|u(x,t)| \leq \varepsilon(t_1 - t)^{-\frac{2}{3(p-1)}} \qquad \text{for} \quad \text{all} \quad (x,t) \in Q_r, \tag{1.2}$$

then u does not blow up at (a, t_1).

The following sections include our main results. In Section 2, we establish universal finite time blow-up. Section 3 is devoted to the nondegeneracy results on the blow-up.

2. Universal finite time blow-up

Theorem 1. *Assume $u_0 \in C^{6+\alpha}(\overline{\Omega})$ with $-\int_\Omega \left(H(u_0) + \frac{1}{2}|\Delta u_0|^2\right) dx \geq 0$. Then the solution of the problem (1.1) must blow up at a finite time, namely, for some $T > 0$*

$$\lim_{t \to T} \|u(t)\| = +\infty,$$

where $H(u) = -\frac{|u|^{p+1}}{p+1}$.

Proof. Let

$$F(t) = \int_\Omega \left(H(u) + \frac{1}{2}|\Delta u|^2\right) dx,$$

then

$$\begin{aligned} \frac{dF(t)}{dt} &= \int_\Omega \left(-|u|^{p-1}u\varphi(u)u_t + \frac{1}{2}\Delta u \Delta u_t\right) dx \\ &= \int_\Omega \left(-|u|^{p-1}u + \frac{1}{2}\Delta^2 u\right) u_t dx \\ &= -\int_\Omega |\nabla \left(-|u|^{p-1}u + \frac{1}{2}\Delta^2 u\right)|^2 dx \leq 0. \end{aligned}$$

So

$$2\int_\Omega H(u)dx - 2F(0) \leq -\|\Delta u\|^2, \tag{2.1}$$

where

$$F(0)=\int_{\Omega}\left(H(u_0)+\frac{1}{2}|\Delta u_0|^2\right)dx.$$

Let ϕ be the unique solution to

$$\begin{cases}\Delta\phi=u, & \text{in } \Omega,\\ \nabla\phi=0, & \text{on } \partial\Omega.\end{cases}$$

It is easy to get that

$$\|\nabla\phi\|^2\le C\|\Delta\phi\|_2^2\le C\|u\|^2. \tag{2.2}$$

Now multiplying (1.1) by ϕ and integrating with respect x, we obtain

$$\begin{aligned}\frac{d}{dt}\|\nabla\phi\|^2&=-2\int_{\Omega}\varphi(u)udx-2\|\Delta u\|^2dx\\ &\ge 4\int_{\Omega}H(u)dx-4F(0)-2\int_{\Omega}\varphi(u)udx\\ &=\int_{\Omega}(2-\frac{4}{p+1})|u|^{p+1}dx-4F(0)\\ &\ge\frac{2(p-1)}{p+1}\left(\int_{\Omega}u^2dx\right)^{\frac{p+1}{2}}-4F(0).\end{aligned} \tag{2.3}$$

Combining (2.2), (2.3) and $-F(0)\ge 0$, we have

$$\frac{d}{dt}\|\nabla\phi\|^2\ge\frac{2C(p-1)}{p+1}\|\nabla\phi\|^{p+1}. \tag{2.4}$$

Let $y(t)=\|\nabla\phi\|_2^2$ with $t\in[0,T)$, then

$$y'(t)\ge\gamma\,(y(t))^{\frac{p+1}{2}}, \tag{2.5}$$

where $\gamma=\frac{2C(p-1)}{p+1}$. A direct integration of (2.5) then yields

$$y^{\frac{p-1}{2}}(t)\ge\frac{1}{y^{\frac{1-p}{2}}(0)-\frac{p-1}{2}\gamma t}.$$

It turns out that the solution of the problem (1.1) will blow up in finite time. The proof of this theorem is completed. □

3. Nondegeneracy results on the blow-up

Let $\Gamma(x,t)$ be the fundamental solution to $u_t-\Delta^3u=0$. According to [8], we have the follow inequalities:

$$|D_t^{\mu}D_x^{\nu}\Gamma(x,t)|\le Ct^{-\frac{1}{6}(n+6\mu+\nu)}\exp\left\{-\omega\frac{|x|^{\frac{6}{5}}}{t^{\frac{1}{5}}}\right\},\qquad t>0, \tag{3.1}$$

where $C>0$, $\omega>0$ are constants, and μ, ν are nonnegative integers.

Our purpose in this section is to have some nondegeneracy results on the blow-up. We state that the solution $u(x,t)$ to blows up at (a,t_1) if it is not locally bounded nearby, i.e., if there is a sequence $\{(x_k,\tau_k)\} \subset \Omega \times [0,t_1)$ with $(x_k,\tau_k) \to (a,t_1)$ as $k \to \infty$ such that $|u(x_k,\tau_k)| \to \infty$.

Theorem 2. *There is a constant $\varepsilon > 0$, depending on n, p and the constant in (3.1), such that if u is a solution of the equation*

$$u_t - \Delta(\Delta^2 u - |u|^{p-1}u) = 0, \quad on \quad Q_r = B_r(a) \times [t_1 - r^6, t_1),$$

where $1 < p < 3, a \in \mathbb{R}^n, t_1 \in \mathbb{R}$ and $0 < r \leq 1$, and if

$$|u(x,t)| \leq \varepsilon (t_1 - t)^{-\frac{2}{3(p-1)}} \qquad for \quad all \quad (x,t) \in Q_r, \tag{3.2}$$

then u does not blow up at (a,t_1).

Next, we introduce the two lemma which will be used in the article and whose proofs can be found in [2] and [6].

Lemma 1. *For $0 < a < 1$, $\theta > 0$, and $0 < h < 1$, the integral*

$$I(h) = \int_h^1 (s-h)^{-a} s^{-\theta} ds,$$

satisfies

$$(1) \quad I(h) \leq \left(\frac{1}{1-a} + \frac{1}{a+\theta-1} \right) \quad if \quad a + \theta > 1,$$

$$(2) \quad I(h) \leq \frac{1}{1-a} + |\log h| \quad if \quad a + \theta = 1,$$

$$(3) \quad I(h) \leq \frac{1}{1-a-\theta} \quad if \quad a + \theta < 1.$$

Lemma 2. *If $y(t)$, $r(t)$ and $q(t)$ are continuous functions defined on $[t_0,t_1]$, such that $y(t) \leq y_0 + \int_{t_0}^t y(s)r(s)ds + \int_{t_0}^t q(s)ds$, $t_0 \leq t \leq t_1$, and $r(t) \geq 0$ on $[t_0,t_1]$, then*

$$y(t) \leq exp\left\{ \int_{t_0}^t r(\tau)d\tau \right\} \left[y_0 + \int_{t_0}^t q(\tau) exp\left\{ -\int_{t_0}^t r(\sigma)d\sigma \right\} d\tau \right].$$

Then, we began to prove the main Theorem 2.

Proof. Without loss of generality, we may assume $a = 0$ and $t_1 = 0$. By scaling, it is sufficient to consider the case $r = 1$. In the fact, if u satisfies the assumptions of the theorem with $r < 1$, then $u_r(x,t) = r^{\frac{4}{p-1}} u(rx, r^6 t)$ satisfies them with $r = 1$ (using the same ε), and clearly u_r blow up at $(0,0)$ if u does.

Let ϕ be a smooth function supported on $B_1(0)$ such that $\phi \equiv 1$ on $B_{\frac{1}{2}}(0)$ and $0 \leq \phi \leq 1$. Consider $\omega = \phi u$; then $\omega_t - \Delta^3 \omega == g$ where

$$\begin{aligned} g =& -2\nabla\Delta^2 u\nabla\phi - \Delta^2 u\Delta\phi \\ & -\Delta(u\Delta^2\phi + 4\nabla\Delta u\nabla\phi + 6\Delta u\Delta\phi + 4\nabla u\nabla\Delta\phi) - \phi\Delta(|u|^{p-1}u) \end{aligned}$$

The semigroup representation formula for ω gives that

$$\omega(t) = e^{(t+1)\Delta^3}\omega(-1) + \int_{-1}^{t} e^{(t-s)\Delta^3} g(s)ds \qquad \text{for} \quad -1 \le t < 0, \tag{3.3}$$

where $e^{t\Delta^3}$ is the semigroup associated with the equation $u_t - \Delta^3 u = 0$ in $\mathbb{R}^n$, i.e.,

$$(e^{t\Delta^3}h)(x) = \int_{\mathbb{R}^n} \Gamma(x-y,t)h(y)dy.$$

Notice that $\int_{\mathbb{R}^n} \Gamma(x-y,t)dy = 1$. It follows that

$$\|e^{t\Delta^3}h\| \le \|h\|_\infty. \tag{3.4}$$

The (3.1) implies that

$$\begin{aligned} |(e^{t\Delta^3}D_i h)(x)| &= |\int_{\mathbb{R}^n} \Gamma(x-y,t)D_i h(y)dy| \\ &= |\int_{\mathbb{R}^n} \frac{\partial}{\partial x_i}\Gamma(x-y,t)h(y)dy| \le Ct^{-\frac{1}{6}}\|h\|_\infty, \quad \forall i = 1,2,\dots,n, \end{aligned}$$

So, we get that

$$\begin{aligned} &\|e^{t\Delta^3}D_i h\|_\infty \le Ct^{-\frac{1}{6}}\|h\|_\infty, \quad \|e^{t\Delta^3}D_{ij} h\|_\infty \le Ct^{-\frac{1}{3}}\|h\|_\infty, \\ &\|e^{t\Delta^3}D_{ijk} h\|_\infty \le Ct^{-\frac{1}{2}}\|h\|_\infty, \quad \|e^{t\Delta^3}D_{ijkm} h\|_\infty \le Ct^{-\frac{2}{3}}\|h\|_\infty, \\ &\|e^{t\Delta^3}D_{ijkmq} h\|_\infty \le Ct^{-\frac{5}{6}}\|h\|_\infty, \end{aligned} \tag{3.5}$$

where $i,j,k,m,q \in \{1,2,\cdots,n\}$.

Now let $g = g_1 + g_2$, where $g_2 = -\phi\Delta(|u|^{p-1}u)$. As above, we estimate

$$\begin{aligned} &\left|\int_{-1}^{t} e^{(t-s)\Delta^3} g_2(s)ds\right| \\ \le& \int_{-1}^{t}\left|\int_{\mathbb{R}^n} \Delta(\phi\Gamma(x-y,t-s))(|u|^{p-1}u)(y,s)dy\right|ds \\ \le& \int_{-1}^{t}\left|\int_{\mathbb{R}^n} \Delta\Gamma(x-y,t-s)\phi|u|^{p-1}u(y)dy\right|ds \\ &+ \int_{-1}^{t}\left|\int_{\mathbb{R}^n} (\Gamma(x-y,t-s)\Delta\phi + 2\nabla\Gamma(x-y,t-s)\cdot\nabla\phi)|u|^{p-1}u(y)dy\right|ds \\ &\le C\int_{-1}^{t}(t-s)^{-\frac{1}{3}}\|\phi u^p\|_\infty(s)ds + C\int_{-1}^{t}\|\Delta\phi u^p\|_\infty(s)ds \end{aligned}$$

$$
\begin{aligned}
&+C\int_{-1}^{t}(t-s)^{-\frac{1}{6}}\|\nabla\phi u^{p}\|_{\infty}(s)ds\\
&\le C\int_{-1}^{t}(t-s)^{-\frac{1}{3}}\|u\|_{\infty}^{p-1}\|\omega\|_{\infty}(s)ds+C\int_{-1}^{t}\|u^{p}\|_{\infty}(s)ds\\
&+C\int_{-1}^{t}(t-s)^{-\frac{1}{6}}\|u\|_{\infty}^{p}(s)ds\\
&\le C\varepsilon^{p-1}\int_{-1}^{t}(t-s)^{-\frac{1}{3}}(-s)^{-\frac{2}{3}}\|\omega\|_{\infty}(s)ds+C\varepsilon^{p}\int_{-1}^{t}(-s)^{-\frac{2p}{3(p-1)}}ds\\
&+C\varepsilon^{p}\int_{-1}^{t}(t-s)^{-\frac{1}{6}}(-s)^{-\frac{2p}{3(p-1)}}ds, \qquad (3.6)
\end{aligned}
$$

due to our assumption.

On the other hand, it is found similarly that

$$
\begin{aligned}
&\left|\int_{-1}^{t}e^{(t-s)\Delta^{3}}g_{1}(s)ds\right|\\
&=\Big|\int_{-1}^{t}\int_{\mathbb{R}^{n}}\Gamma(x-y,t-s)(-2\nabla\Delta^{2}u\nabla\phi-\Delta^{2}u\Delta\phi\\
&-\Delta(u\Delta^{2}\phi+4\nabla\Delta u\nabla\phi+6\Delta u\Delta\phi+4\nabla u\nabla\Delta\phi))dyds\Big|\\
&\le C\int_{-1}^{t}(t-s)^{-\frac{5}{6}}\|u\|_{\infty}(s)ds\le C\varepsilon\int_{-1}^{t}(t-s)^{-\frac{5}{6}}(-s)^{-\frac{2}{3(p-1)}}ds. \qquad (3.7)
\end{aligned}
$$

By (3.2)-(3.4), (3.6) and (3.7), we get that for $-1\le t<0$,

$$
\begin{aligned}
\|\omega(t)\|_{\infty}&\le\varepsilon+\varepsilon^{p-1}\int_{-1}^{t}(t-s)^{-\frac{1}{3}}(-s)^{-\frac{2}{3}}\|\omega\|_{\infty}(s)ds\\
&+C\varepsilon^{p}\int_{-1}^{t}(t-s)^{-\frac{1}{6}}(-s)^{-\frac{2p}{3(p-1)}}ds+C\varepsilon\int_{-1}^{t}(t-s)^{-\frac{5}{6}}(-s)^{-\frac{2}{3(p-1)}}ds\\
&\le\varepsilon+C\varepsilon^{p-1}\int_{-1}^{t}(t-s)^{-\frac{1}{3}}(-s)^{-\frac{2}{3}}\|\omega\|_{\infty}(s)ds+C\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}}, \qquad (3.8)
\end{aligned}
$$

due to $1<p<3$ and Lemma (1).

Let $y(t)=\|\omega(t)\|_{\infty}$; therefore

$$
y(t)\le\varepsilon+C\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}}+C\varepsilon^{p-1}\int_{-1}^{t}(t-s)^{-\frac{1}{3}}(-s)^{-\frac{2}{3}}y(s)ds. \qquad (3.9)
$$

Define $f(t)=\chi_{[-1,0]}(t)y(t)$, $\forall t<0$. We introduce a special maximal function on $(-\infty,0)$:

$$
(Mf)(t)=\sup_{r>0}\frac{1}{r}\int_{t-r}^{t}|f(s)|ds, \qquad \forall t\in(-\infty,0).
$$

Now $\forall r > 0$,

$$\int_{-1}^{t}(t-s)^{-\frac{1}{3}}(-s)^{-\frac{2}{3}}y(s)ds = \int_{-\infty}^{t}(t-s)^{-\frac{1}{3}}(-s)^{-\frac{2}{3}}f(s)ds$$
$$= \int_{t-r}^{t}(t-s)^{-\frac{1}{3}}(-s)^{-\frac{2}{3}}f(s)ds + \int_{-\infty}^{t-r}(t-s)^{-\frac{1}{3}}(-s)^{-\frac{2}{3}}f(s)ds$$
$$= I_1 + I_2.$$

We compute these two integrals, respectively.

$$\begin{aligned} I_1 &\le (-t)^{-\frac{2}{3}}\int_{t-r}^{t}(t-s)^{-\frac{1}{3}}f(s)ds \\ &= (-t)^{-\frac{2}{3}}\sum_{k=0}^{\infty}\int_{t-\frac{r}{2^k}}^{t-\frac{r}{2^{k+1}}}(t-s)^{-\frac{1}{3}}f(s)ds \\ &\le (-t)^{-\frac{2}{3}}\sum_{k=0}^{\infty}\left(\frac{r}{2^{k+1}}\right)^{-\frac{1}{3}}\int_{t-\frac{r}{2^k}}^{t-\frac{r}{2^{k+1}}}f(s)ds \\ &\le (-t)^{-\frac{2}{3}}\sum_{k=0}^{\infty}\left(\frac{1}{2^{k+1}}\right)^{\frac{2}{3}}r^{\frac{2}{3}}(Mf)(t) \\ &= Cr^{\frac{2}{3}}(-t)^{-\frac{2}{3}}(Mf)(t), \end{aligned}$$

and

$$I_2 \le r^{-\frac{1}{3}}\int_{-\infty}^{t-r}(-s)^{-\frac{2}{3}}f(s)ds \le r^{-\frac{1}{3}}\int_{-\infty}^{t}(-s)^{-\frac{2}{3}}f(s)ds = r^{-\frac{1}{3}}\int_{-1}^{t}(-s)^{-\frac{2}{3}}f(s)ds.$$

Then,

$$f(t) \le \varepsilon + C\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}} + C\varepsilon^{p-1}\left[r^{\frac{2}{3}}(-t)^{-\frac{2}{3}}(Mf)(t) + r^{-\frac{1}{3}}\int_{-1}^{t}(-s)^{-\frac{2}{3}}f(s)ds\right],$$

for all $r > 0$ and $t \in (-\infty, 0)$.

Let

$$r = \frac{\int_{-1}^{t}(-s)^{-\frac{2}{3}}f(s)ds}{(-t)^{-\frac{2}{3}}(Mf)(t)},$$

so we have

$$\begin{aligned} f(t) &\le \varepsilon + C\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}} + C\varepsilon^{p-1}\left((-t)^{-\frac{1}{3}}\int_{-1}^{t}(-s)^{-\frac{2}{3}}f(s)ds\right)^{\frac{2}{3}}((Mf)(t))^{\frac{1}{3}} \\ &\le \varepsilon + C\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}} + C\varepsilon^{p-1}(-t)^{-\frac{1}{3}}\int_{-1}^{t}(-s)^{-\frac{2}{3}}f(s)ds \\ &\quad + C\varepsilon^{p-1}(Mf)(t). \end{aligned} \tag{3.10}$$

If we define

$$g(t) = (-t)^{-\frac{1}{3}} \int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s)ds,$$

then

$$g'(t) = (-t)^{-1}\left[\frac{1}{3}(-t)^{-\frac{1}{3}} \int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s) + f(t)\right] \geq 0.$$

Hence $g(t)$ is increasing in $(-\infty, 0)$.

Then we get

$$\begin{aligned}\max_{-1\leq\tau\leq t} f(\tau) \leq &\varepsilon + C\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}} \\ &+ C\varepsilon^{p-1} g(t) + C\varepsilon^{p-1} \max_{-1\leq\tau\leq t}(Mf)(\tau), \quad \forall t \in [-1,0), \quad (3.11)\end{aligned}$$

where we have used $\frac{1}{6} - \frac{2}{3(p-1)} < 0$ since $1 < p < 3$.

Clearly, $\max_{-1\leq\tau\leq t}(Mf)(\tau) \leq \max_{-1\leq\tau\leq t} f(\tau)$ by our definition of the maximal function. Therefore (3.11) implies that for any $-1 \leq t < 0$,

$$\begin{aligned}&\max_{-1\leq\tau\leq t} f(\tau) \leq \\ &\frac{1}{1-C\varepsilon^{p-1}}\left[\varepsilon + C\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}} + C\varepsilon^{p-1}(-t)^{-\frac{1}{3}} \int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s)ds\right],\end{aligned}$$

provided that $C\varepsilon^{p-1} < 1$. Especially,

$$\begin{aligned}f(t) \leq &\frac{1}{1-C\varepsilon^{p-1}}\left[\varepsilon + C\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}} + C\varepsilon^{p-1}(-t)^{-\frac{1}{3}} \int_{-1}^{t} (-s)^{-\frac{2}{3}} f(s)ds\right] \\ &\forall t \in [-1,0).\end{aligned}$$

Then for $\varepsilon > 0$ small enough, we obtain

$$\begin{aligned}(-t)^{\frac{1}{3}} f(t) \leq 2&\left[\varepsilon + C\varepsilon(-t)^{\frac{1}{2}-\frac{2}{3(p-1)}} + C\varepsilon^{p-1} \int_{-1}^{t} (-s)^{-1}(-s)^{\frac{1}{3}} f(s)ds\right] \\ &\forall t \in [-1,0).\end{aligned}$$

Define $h(t) = (-t)^{\frac{1}{3}} f(t)$; then

$$h(t) \leq 2\varepsilon + 2C\varepsilon(-t)^{\frac{1}{2}-\frac{2}{3(p-1)}} + 2C\varepsilon^{p-1} \int_{-1}^{t} (-s)^{-1} h(s)ds, \quad (3.12)$$

Applying Lemma(2), we have

$$\begin{aligned}h(t) &\leq (-t)^{-2C\varepsilon^{p-1}}\left[2\varepsilon + C(p,\varepsilon)\varepsilon(-t)^{\frac{1}{2}-\frac{2}{3(p-1)}+2C\varepsilon^{p-1}}\right] \\ &\leq 2\varepsilon(-t)^{-2C\varepsilon^{p-1}} + C(p,\varepsilon)\varepsilon(-t)^{\frac{1}{2}-\frac{2}{3(p-1)}}, \quad \forall t \in [-1,0).\end{aligned}$$

Then $f(t) \leq 2\varepsilon(-t)^{-\frac{1}{3}-2C\varepsilon^{p-1}} + C(p,\varepsilon)\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}}$, $\forall t \in [-1,0)$, or

$$y(t) \leq 2\varepsilon(-t)^{-\frac{1}{3}-2C\varepsilon^{p-1}} + C(p,\varepsilon)\varepsilon(-t)^{\frac{1}{6}-\frac{2}{3(p-1)}}, \quad \forall t \in [-1,0). \quad (3.13)$$

Choose $\varepsilon > 0$ small enough that $\frac{1}{3} + 2C\varepsilon^{p-1} < \frac{2}{3(p-1)}$ which is possible since $1 < p < 3$. Define $\alpha = \max\{\frac{1}{3} + 2C\varepsilon^{p-1}, \frac{2}{3(p-1)} - \frac{1}{6}\} \leq \frac{2}{3(p-1)}$, it is easy to find that $\alpha > \frac{1}{3}$; then (3.13) implies $y(t) \leq C(p,\varepsilon)\varepsilon(-t)^{-\alpha}$, $\forall t \in [-1,0)$. Hence

$$|u(x,t)| \leq C(p,\varepsilon)\varepsilon(-t)^{-\alpha}, \quad \forall (x,t) \in B_{\frac{1}{2}}(0) \times [-1,0). \tag{3.14}$$

Now let $\tilde{\phi}$ be a function supported on $B_{\frac{1}{2}}(o)$ with $\tilde{\phi} \equiv 1$ on $B_{\frac{1}{4}}(0)$ and $0 \leq \tilde{\phi} \leq 1$, and define $\tilde{\omega} = \tilde{\phi} u$; then we go back to (3.6)-(3.8) and we have that

$$\begin{aligned}
\|\tilde{\omega}(t)\|_\infty &\leq \varepsilon + C\int_{-1}^{t} (t-s)^{-\frac{1}{3}} \|u\|_\infty^{p-1} \|\tilde{\omega}\|_\infty ds + C\int_{-1}^{t} \|u\|_\infty^{p} ds \\
&\quad + C\int_{-1}^{t} (t-s)^{-\frac{1}{6}} \|u\|_\infty^{p} ds + C\varepsilon \int_{-1}^{t} (t-s)^{-\frac{5}{6}} \|u\|_\infty ds \\
&\leq \varepsilon + C\varepsilon^{p-1} \int_{-1}^{t} (t-s)^{-\frac{1}{3}} (-s)^{-\alpha(p-1)} (-s)^{-\alpha} ds + C\varepsilon^{p} \int_{-1}^{t} (-s)^{-\alpha p} ds \\
&\quad + C\varepsilon^{p} \int_{-1}^{t} (t-s)^{-\frac{1}{6}} (-s)^{-\alpha p} ds + C\varepsilon \int_{-1}^{t} (t-s)^{-\frac{5}{6}} (-s)^{-\alpha} ds \\
&\leq \varepsilon + C\varepsilon^{p-1} \int_{-1}^{t} (t-s)^{-\frac{1}{3}} (-s)^{-\alpha p} ds + C\varepsilon^{p} \int_{-1}^{t} (-s)^{-\alpha p} ds \\
&\quad + C\varepsilon^{p} \int_{-1}^{t} (t-s)^{-\frac{1}{6}} (-s)^{-\alpha p} ds + C\varepsilon \int_{-1}^{t} (t-s)^{-\frac{5}{6}} (-s)^{-\alpha} ds
\end{aligned} \tag{3.15}$$

due to (3.14).

Since $\frac{1}{3} < \alpha < \frac{2}{3(p-1)}$, we get

$$\frac{5}{6} - \alpha p > \frac{2}{3} - \alpha p > \frac{1}{6} - \alpha.$$

Hence by Lemma(1), we obtain

$$\|\tilde{\omega}(t)\|_\infty \leq \varepsilon + C\varepsilon^{p-1} + C\varepsilon^{p-1}(-t)^{\frac{1}{6}-\alpha} \leq (2 + C\varepsilon^{p-1})(-t)^{\frac{1}{6}-\alpha}, \quad \forall t \in [-1,0),$$

Which means, for small $\varepsilon > 0$,

$$|u(x,t)| \leq (2 + C\varepsilon^{p-1})(-t)^{\frac{1}{6}-\alpha}, \quad \forall (x,t) \in B_r(0) \times [-1,0). \tag{3.16}$$

Iterating the argument finitely many times we can get that there is a number $0 < r_0 < \frac{1}{4}$ such that

$$|u(x,t)| \leq K(-t)^{-\frac{1}{6p}}, \quad \forall (x,t) \in B_{r_0}(0) \times [-1,0), \tag{3.17}$$

where K is constant.

Next, we choose another cut-off function $\hat{\phi}$ supported on B_{r_0} such that $\hat{\phi} \equiv 1$ on $B_{\frac{r_0}{2}}$ and define $\hat{\omega} = \hat{\phi} u$. Going back to (3.15) and applying Lemma(1), we have

$$\begin{aligned}
\|\hat{\omega}(t)\|_\infty &\le \varepsilon + C\int_{-1}^{t} (t-s)^{-\frac{1}{3}} \|u\|_\infty^{p-1} \|\hat{\omega}\|_\infty ds + C\int_{-1}^{t} \|u\|_\infty^{p} ds \\
&\quad + C\int_{-1}^{t} (t-s)^{-\frac{1}{6}} \|u\|_\infty^{p} ds + C\varepsilon \int_{-1}^{t} (t-s)^{-\frac{5}{6}} \|u\|_\infty ds \\
&\le \varepsilon + CK^{p-1} \int_{-1}^{t} (t-s)^{-\frac{1}{3}} (-s)^{-\frac{1}{6}} ds + CK^{p} \int_{-1}^{t} (-s)^{-\frac{1}{6}} ds \\
&\quad + CK^{p} \int_{-1}^{t} (t-s)^{-\frac{1}{6}} (-s)^{-\frac{1}{6p}} ds + CK \int_{-1}^{t} (t-s)^{-\frac{5}{6}} (-s)^{-\frac{1}{6p}} ds \\
&\le \varepsilon + CK^{p-1},
\end{aligned} \tag{3.18}$$

which means that $|u(x,t)| \le C$ in $B_{\frac{r_0}{2}} \times [-1,0)$. This completes the proof of the theorem.

□

Using the same argument, we can easily draw the following conclusion.

Theorem 3. *Suppose $p \ge 3$, then for any $\delta \in (0, \frac{2}{3(p-1)})$, there is a constant $\varepsilon > 0$, depending on n, p and the constant in (3.1), such that if u is a solution of the equation*

$$u_t - \Delta(\Delta^2 u - |u|^{p-1} u) = 0, \qquad on \quad Q_r = B_r(a) \times [t_1 - r^6, t_1)$$

where $a \in \mathbb{R}^n, t_1 \in \mathbb{R}$ and $0 < r \le 1$, and if

$$|u(x,t)| \le \varepsilon (t_1 - t)^{-\frac{2}{3(p-1)}} \qquad for \quad all \quad (x,t) \in Q_r,$$

then u does not blow up at (a, t_1).

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Authors' addresses

Zhenbang Li
Department of Mathematics, Jilin University, Changchun 130012, China
E-mail address: jamesbom23@yahoo.com.cn

Changchun Liu
Department of Mathematics, Jilin University, Changchun 130012, China
E-mail address: liucc@jlu.edu.cn