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ON AN INEQUALITY OF REDHEFFER

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Abstract. We offer two new proofs of famous Redheffer's inequality, as well establish two converse inequalities for it. Also a hyperbolic analogue is pointed out.

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1. INTRODUCTION

In 1969, Redheffer [8] proposed the following inequality

$$\frac{\pi^2 - x^2}{\pi^2 + x^2} \le \frac{\sin x}{x}, \quad x \in \mathbb{R}, \tag{1.1}$$

which was proved by Williams [12] in the same year. In literature, this inequality is known as *Redheffer's inequality*. By using the infinite product and induction method, Williams verified this inequality also in 1969, [11]. Motivated by his work, many developments such as generalizations, refinements and applications took place, e.g., see [4].

Thereafter some Redheffer's-type inequalities for other trigonometric, hyperbolic and Bessels function were established, e.g., see [2, 3, 6, 10, 13] and the references therein.

Recently a new proof of (1.1) has appeared in [5], where authors are using the Lagrange mean value theorem, combined with induction , which is very complicated for the reader.

The inequality (1.1) is valid for all $x \in \mathbb{R}$. It is immediate that we may assume $x > 0$ and $x \in (0, \pi)$ as for $x > \pi$, we may let $x = \pi + t$ for $t > 0$, then inequality (1.1) becomes

$$\frac{\sin t}{t} < \frac{2\pi^2 + 3\pi t + t^2}{2\pi^2 + 2\pi t + t^2}.$$

This is obvious, as right side is greater than one, and left side less than one. Thus, we may consider $x \in (0, \pi)$.

So far, all the authors have given the proof of (1.1) by using the induction method. From our proof, it is obvious that the induction method is not needed. In this paper we give new interesting proofs of (1.1), which are based on the elementary calculus. The authors think that this proof could be one from the "Book" (See [1]).

2. New proof of inequality and its converse

Lemma 1. *For all $x \in (0,\pi)$ one has*

$$x+\sin(x) > x^2\frac{\cos(x/2)}{\sin(x/2)}.$$

Proof. Let $h_1(x) = x+\sin(x)-x^2\cos(x/2)/\sin(x/2)$. Then

$$h_1'(x) = 1+\cos(x)-\frac{2x\sin(x)-x^2}{2\sin(x/2)^2} = \frac{\sin(x)^2+x^2-2x\sin(x)}{2\sin(x/2)^2} > 0,$$

where we have used $1+\cos(x) = 2\cos(x/2)^2$ and $2\sin(x/2)\cos(x/2) = \sin(x)$. As $h_1(x) > h_1(0) = 0$, the result follows. □

Theorem 1. *The following inequalities*

$$\frac{\pi^2-x^2}{\pi^2+x^2} \le \frac{\sin(x)}{x} < \frac{12-x^2}{12+x^2} \tag{2.1}$$

hold for $x \in (0,\pi]$.

Proof. Let

$$f_1(x) = \frac{x^2(x+\sin(x))}{x-\sin(x)},$$

for $x \in (0,\pi]$. After some elementary computations, one has

$$\frac{(x-\sin(x))^2}{2x}f_1'(x) = x^2(1+\cos(x))-\sin(x)(x+\sin(x)) = g_1(x).$$

As $1+\cos(x) = 2\cos(x/2)^2$, and $\sin(x) = 2\sin(x/2)\cos(x/2)$, we get

$$\begin{aligned} g_1'(x) &= 2\cos\left(\frac{x}{2}\right)\left[x^2\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)(x+\sin(x))\right] \\ &= 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\left[x^2\frac{\cos(x/2)}{\sin(x/2)}-(x+\sin(x))\right] < 0 \end{aligned}$$

by Lemma 1. Thus $f_1'(x) < 0$, and f_1 is strictly decreasing in $x \in (0,\pi]$. We get $f_1(x) > \lim_{x\to\pi} f_1(x) = \pi^2$, which is equivalent to

$$\frac{x+\sin(x)}{x-\sin(x)} > \frac{\pi^2}{x^2},$$

thus the first inequality in (2.1) follows. The second inequality in (2.1) follows similarly from $f_1(x) < \lim_{x\to 0} f_1(x) = 12$. □

The right side of (2.1) improves the known inequality:

$$\frac{\sin(x)}{x} < 1 - \frac{x^2}{\pi^2}, \quad 0 < x < \frac{\pi}{2}.$$

Indeed, this follows by $x^2 + 12 < 2\pi^2$, which is true, as $\pi^2/4 + 12 < 2\pi^2$ becomes $48 < 7\pi^2$.

Proposition 1. *The second inequality in (2.1) refines the relation*

$$\frac{\sin(x)}{x} < \frac{\cos(x)+2}{3}, \quad 0 < x < \frac{\pi}{2},$$

so-called Cusa-Huygens inequality [7, 9].

Proof. It is equivalent to prove that,

$$(12 + x^2)\cos(x) + 5x^2 - 12 = s(x) > 0.$$

One has successively:

$$s'(x) = 2x(5 + \cos(x)) - (12 + x^2)\sin(x) - 12,$$

$$s''(x) = -4x\sin(x) - (10 + x^2)\cos(x) + 10,$$

$$s'''(x) = x^2\sin(x) + 6(\sin(x) - x\cos(x)) > 0,$$

as $\sin(x) > x\cos(x)$ (i.e. $\tan(x) > x$). Thus we get

$$s''(x) > s''(0) = 0,\ s'(x) > s'(0) = 0,$$

and finally $s(x) > s(0) = 0$. □

3. An other proof of (1.1)

Lemma 2. *Let $f(x)$ be two times differentiable function on $(0, \pi)$. Define $g(x) = f(x)/\sin(x)$, $h(x) = \sin(x)^2 g'(x)$ and $F(x) = f(x) + f''(x)$. Then the sign of $h'(x)$ depends on the sign of $F(x)$.*

Proof. One has

$$\sin(x)^2 g'(x) = f'(x)\sin(x) - f(x)\cos(x) = h(x),$$

and

$$h'(x) = (f(x) + f''(x))\sin(x) = F(x)\sin(x).$$

As $\sin(x) > 0$ for all $x \in (0, \pi)$, the result follows. □

Theorem 2. *For $x \in (0, \pi)$, we have*

$$\frac{\pi^2 - x^2}{\pi^2 + x^2} < \frac{\sin(x)}{x} < c_1 \frac{\pi^2 - x^2}{\pi^2 + x^2},$$

where $c_1 = 1.07514$.

Proof. Let $g(x) = f(x)/\sin(x)$ for $x \in (0,\pi)$, where

$$f(x) = \frac{x(\pi^2 - x^2)}{\pi^2 + x^2}.$$

One has

$$f'(x) = \frac{\pi^4 - 4\pi^2 x^2 - x^4}{(\pi^2 + x^2)^2},$$

so we get

$$F(x) = f(x) + f''(x) = \frac{xp(x)}{(\pi^2 + x^2)^3},$$

i.e, by Lemma 2 the sign of $h'(x)$ depends on the sign of $p(x)$. Here

$$\begin{aligned} h(x) &= f'(x)\sin(x) - f(x)\cos(x) \\ &= \frac{x\left(\pi^4 - x^4\right)\cos(x) + \left(-x^4 - 4\pi^2 x^2 + \pi^4\right)\sin(x)}{\left(x^2 + \pi^2\right)^2} \\ &= \frac{k(x)}{\left(x^2 + \pi^2\right)^2} \end{aligned}$$

and

$$p(x) = -x^6 - \pi^2 x^4 + \pi^2.x^2(\pi^2 + 4) - 12\pi^4 + \pi^6.$$

Here $h(0) = h(\pi) = 0$ and $h(x_0) = 0$.

Elementary computation gives $k(2\pi/3) < 0$, while with the use of a computer we get $k(17\pi/24) > 0$. So $k(x)$ has a root x_0 between $2\pi/3 \approx 2.0944$ and $17\pi/24 \approx 2.22529$.

Now, by letting $y = x^2$, we get

$$p(x) = q(y) = -y^3 - \pi^2 y^2 + \pi^2 y(\pi^2 + 4) + \pi^6 - 12\pi^4.$$

Since

$$q'(y) = -3y^2 - 2\pi^2 y + \pi^2(\pi^2 + 4),$$

and $y > 0$, the only root of $q'(y) = 0$ is $y^* = (2\pi\sqrt{3 + \pi^2 + 3} - \pi^2)/3$, which lies between π and π^2. One can verify that $q(0) < 0$ and $q(\pi) > 0$. As $q'(y) > 0$ for $y \in (0, y^*)$ and < 0 in $y \in (y^*, \pi^2)$, and as $q(\pi^2) < 0$, we get the following: q is increasing from $q(0)$ to $q(y^*)$ and decreasing from $q(y^*)$ to $q(\pi^2)$. Thus there exist only two roots in $(0,\pi^2)$ to $q(y)$, let them z_1 and z_2. Clearly, z_1 is in $(0,\pi)$ and y_2 in (y^*, π^2). More precisely, as $q(6) < 0$ and $q(7) > 0$, one finds that $y_2 > 6$. These imply that $q(y) < 0$ in $(0, z_1)$, > 0 in (z_1, z_2) and $<$ in (z_2, π^2). In terminology of $P(x)$, we get that $p(x) < 0$ in $(0, \sqrt{z_1})$; > 0 in $(\sqrt{z_1}, \sqrt{z_2})$, and < 0 in $(\sqrt{z_2}, \pi)$. In $(0, \sqrt{z_1})$ clearly $h'(x) < 0$, so $h(x) < h(0) = 0$; similarly in (z_2, π) one has $h'(x) < 0$, so $h(x) > h(\pi) = 0$. Remains the interval $(\sqrt{z_1}, \sqrt{z_2})$. As $z_1 < \pi$ and $z_2 > 6$, we get that

$$\sqrt{z_1} < \sqrt{\pi} < 2 < x_0 < \sqrt{6} \approx 2.44949 < \sqrt{z_2},$$

so we find that x_0 lies between $\sqrt{z_1}$ and $\sqrt{z_2}$. Then clearly $h(x) < h(x_0) = 0$ in $(\sqrt{z_1}, x_0)$, and $h(x) > h(x_0) = 0$ in $(x_0, \sqrt{z_2})$, so all is done.

Thus the minimum point of g is at x_0. As $g(x)$ tends to 1 when x tends to 0 or π, thus the Redheffer's inequality follows. On the other hand, we get also $g(x) \geq g(x_0)$, i.e. the best possible converse to Redheffer's inequality. Now, with the aid of a computer one can find the more precise approximation $x_0 \approx 2.12266$, giving $g(x_0) \approx 0.93012 = 1/c_1$ so the converse to the Redheffer's inequality holds true. □

Lemma 3. *For $a = 2.175$, the function*

$$Q(x) = -3ax^{2a+2} - 2a\pi^a x^{a+2}$$
$$+2a(a-1)(2a-1)\pi^a x^a + a\pi^{2a}x^a - 2a(a+1)(a-2)\pi^{2a}$$

has exactly two roots y_1 and y_2 in $(0,\pi)$.

Proof. We have $Q(1/2) < 0$, $Q(\pi/2) > 0$, and $Q(\pi) < 0$, so Q has two roots y_1 in $(1/2, \pi/2)$, resp. y_2 in $(\pi/2, \pi)$. To show that Q has no other zeros, we have to consider $Q'(x) = xR(x)$, where

$$R(x) = -3a(2a+2)x^{2a} - 2a(a+2)\pi^a x^a - 2a^2(a-a)(2a-1)\pi^a x^{a-2} + 2a\pi^{2a}.$$

One has further

$$R'(x) = 2a^2 x^{a-3} T(x),$$

here

$$T(x) = -3(2a+2)x^{a+2} - (a+2)\pi^a x^2 - 2a^2(a-1)(a-2)(2a-1)\pi^a.$$

Since $a > 2$, we get $T(x) < 0$, so $R'(x) < 0$. One has $Q'(x) = xR(x)$, where $R'(x) < 0$. Since $R(0) > 0$ and $R(\pi) < 0$, and $R(x)$ is strictly decreasing, $R(x) = 0$ can have exactly one root r in $(0,\pi)$. Therefore, $Q(x)$ has exactly one extremal point. Since $Q(0) < 0$, $Q(\pi) < 0$ and Q takes also positive values, clearly $Q(r)$ will be a maximum of $Q(x)$. This shows that Q has exactly two roots in $(0,\pi)$: one in $(0,r)$ and the other one in (r,π).

□

Theorem 3. *For $x \in (0,\pi)$, the following inequality holds*

$$\frac{\sin(x)}{x} < \frac{\pi^a - x^a}{\pi^a + x^a},$$

where $a = 87/40 = 2.175$.

Proof. Inequality can be written as $g(x) = f(x)/\sin(x) > 1$, where

$$f(x) = \frac{x(\pi^a - x^a)}{\pi^a + x^a}.$$

First of all, similarly to the proof of Theorem 2, one has

$$f'(x) = \frac{\pi^{2a} - 2a\pi^a x^a - x^{2a}}{(\pi^a + x^a)^2},$$

so we get

$$h(x) = \frac{K(x)}{(\pi^a + x^a)^2},$$

where

$$K(x) = (\pi^2 - 2a\pi^a x^a - x^{2a})\sin(x) - x(\pi^{2a} - x^{2a})\cos(x).$$

One finds

$$F(x) = \frac{xP(x)}{(\pi^a + x^a)^3},$$

where

$$P(x) = -x^{3a} - \pi^a x^{2a} + 2a(a-1)\pi^a x^{2a-2} + \pi^{2a} x^a - 2a(a+1)\pi^{2a} x^{a-2} + \pi^{3a}.$$

Now the proof of Theorem runs as follows: Since $K(1/2) > 0$ and $K(\pi/4) < 0$, $K(\pi/2) > 0$ and $K(2\pi/3) < 0$, we get x_1 in $(1/2, \pi/4)$ such that $K(x_1) = 0$ and x_2 in $(\pi/2, 2\pi/3)$ such that $K(x_2) = 0$. As $P(0) > 0$ and $P(\pi/4) < 0$. One has

$$P'(x) = xQ(x),$$

where $Q(x)$ is as in Lemma 3. It follows from Lemma 3 that that, $Q(x) < 0$ for x in $(0, y_1)$ and (y_2, π), and $Q(x) > 0$ for x in (y_1, y_2). This shows that $P(x)$ is strictly decreasing in $(0, y_1)$ and (y_2, π) and strictly increasing in (p_1, p_2). This implies that $P(x)$ has a unique root p_1 in $(0, y_1)$, as well as a unique p_2 in (y_1, y_2) and p_3 in (y_2, π). By approximate computation we can see that $p_1 < 2/3$, $p_2 > 1$ and $p_3 > 2$, and $p_1 < p_2 < p_3$. This shows that for the roots x_1 and x_2 of function $k(x)$ one has that x_1 is in (p_1, p_2) and x_2 in (p_2, p_3). As $h(0) = h(x_1) = h(x_2) = h(\pi) = 0$, we get the following;

(1) for $x \in (0, p_1)$ one has $h(x) > h(0) = 0$,
(2) for $x \in (p_1, x_1)$ one has $h(x) > h(x_1) = 0$,
(3) for $x \in (x_1, p_2)$ one has $h(x) < h(x_1) = 0$,
(4) for $x \in (p_2, x_2)$ one has $h(x) < h(x_2) = 0$,
(5) for $x \in (x_2, p_3)$ one has $h(x) > h(x_2) = 0$,
(6) for $x \in (p_3, \pi)$ one has $h(x) > h(\pi) = 0$.

From the above it follows that x_1 is a local minimum point , while x_2 a local maximum point of $g(x)$. Clearly, $g(x_1) > \lim g(x)$, when x tends to zero, $= 1$ and $\lim g(x)$ at $x = \pi$ is $a/2 > 1$. This completes the proof.

$\square$

4. A HYPERBOLIC ANALOGUE

Lemma 4. *For $x \in (0, \infty)$,*

$$x + \sinh(x) > x^2 \coth(x/2).$$

Let $h_2(x) = x + \sinh(x) - x^2 \coth(x/2)$.

Proof. One has

$$h_2'(x) = \frac{(\sinh(x)-x)^2}{2\sinh(x/2)^2} > 0,$$

and $h_2(x) > \lim_{x\to 0} h_2(x) = 0$. Thus, inequality holds. □

Theorem 4. *For $x \in (0,\infty)$, we have*

$$\frac{\sinh(x)+x}{\sinh(x)-x} > \frac{12}{x^2}. \tag{4.1}$$

Proof. Let

$$f_2(x) = \frac{x^2(\sinh(x)+x)}{\sinh(x)-x},$$

for $x \in (0,\infty)$. We get

$$\frac{(\sinh(x)-x)^2}{2x} f_2'(x) = (\sinh(x/2))((x+\sinh(x)) - x^2\cosh(x/2)^2),$$

which is positive by Lemma 4. Thus, f_2 is strictly increasing in $x \in (0,\infty)$, and the inequality (4.1) follows from $f_2(x) > \lim_{x\to 0} f_2(x) = 12$. Therefore, we have an analogue of the second inequality in the circular case, and this is (4.1). When $x^2 < 12$, then (4.1) becomes

$$\frac{\sinh(x)}{x} < \frac{12+x^2}{12-x^2}.$$

□

It is interesting to observe that

$$\frac{12+x^2}{12-x^2} < \frac{\pi^2+x^2}{\pi^2-x^2},$$

if $x < \pi$. Indeed this becomes: $\pi^2 < 12$.

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