Miskolc Mathematical Notes
Vol. 16 (2015), No 1, pp. 575-586
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2015.1258

Monotone iterative technique by upper and lower solutions with initial time difference

Coşkun Yakar, Ismet Arslan, and Muhammed Çiçek

Miskolc Mathematical Notes
Vol. 16 (2015), No. 1, pp. 575–586

HU e-ISSN 1787-2413

MONOTONE ITERATIVE TECHNIQUE BY UPPER AND LOWER SOLUTIONS WITH INITIAL TIME DIFFERENCE

COŞKUN YAKAR, İSMET ARSLAN, AND MUHAMMED ÇİÇEK

Received 21 May, 2014

Abstract. In this work, the monotone iterative technique have been investigated by choosing upper and lower solutions with initial time difference that start at different initial times for the initial value problem. This method offers a way of proving existence of maximal and minimal solutions in addition to obtaining solutions in closed sectors.

2010 *Mathematics Subject Classification:* 34A99; 34C60; 34A12

Keywords: initial time difference, monotone iterative technique, existence theorems, comparison results

1. INTRODUCTION

The original method of monotone iterative technique provides an explicit analytic representation for the solution of nonlinear differential equations which yields pointwise upper and lower estimates for the solution of problem whenever the functions involved are monotone nondecreasing and nonincreasing [1–3, 8, 10–13]. As a result, the method has been popular in applied areas [1, 2],[4–6],[9, 12, 13]. The monotone iterative technique [2, 6–8], uncoupled with the method of upper and lower solutions, offers monotone sequences that converge uniformly and monotonically to the extremal solutions of the given nonlinear problem. Since each member of such a sequence is the solution of a certain (ODEs) which can be explicitly computed, the advantage and the importance of this technique needs no special emphasis. Moreover, this method can successfully be employed to generate two sided pointwise bounds on solutions of initial value problems of ODEs from which qualitative and quantitative behavior can be investigated. In this paper especially we employed monotone iterative technique for ODEs with initial time difference.

2. PRELIMINARIES

In this section we will give some basic definition and theorems by [4] which are very useful to use in our future references.

We consider the following initial value problem

$$x'(t) = f(t,x(t)), x(t_0) = x_0 \text{ for } t \geq t_0, t_0 \in R_+ \tag{2.1}$$

where $f \in C[R_+ \times R, R]$ and $t \in [t_0, t_0 + T]$.

Definition 1. (i) Let $r(t)$ be a solution of the (2.1) on $t \in [t_0, t_0 + T]$. Then $r(t)$ is said to be a maximal solution of (2.1) if, for every solution $x(t)$ of (2.1) existing on $[t_0, t_0 + T]$ the inequality

$$x(t) \leq r(t), t \in [t_0, t_0 + T] \tag{2.2}$$

holds.

(ii) Let $\rho(t)$ be a solution of the (2.1) on $t \in [t_0, t_0 + T]$. Then $\rho(t)$ is said to be a minimal solution of (2.1) if, for every solution $x(t)$ of (2.1) existing on $[t_0, t_0 + T]$ the inequality

$$x(t) \geq \rho(t), t \in [t_0, t_0 + T] \tag{2.3}$$

holds.

Definition 2. (i) A function $\beta \in C^1[[t_0, t_0 + T], R]$ is said to be an upper solution of (2.1) if

$$\beta' \geq f(t,\beta), \beta(t_0) \geq x_0, t \in [t_0, t_0 + T] \tag{2.4}$$

(ii) A function $\alpha \in C^1[[t_0, t_0 + T], R]$ is said to be a lower solution of (2.1) if

$$\alpha' \leq f(t,\alpha), \alpha(t_0) \leq x_0, t \in [t_0, t_0 + T] \tag{2.5}$$

Theorem 1 ([4])**.** *Let $\alpha, \beta \in C^1[[t_0, t_0 + T], R]$ be lower and upper solutions of* (2.1) *respectively. Suppose that $x \geq y$, f satisfies the inequality*

$$f(t,x) - f(t,y) \leq M(x - y) \tag{2.6}$$

where M is a positive constant .Then $\alpha(t_0) \leq \beta(t_0)$ implies that $\alpha(t) \leq \beta(t), t \in [t_0, t_0 + T]$.

Remark 1 ([4]). Let the assumptions of Theorem 1 hold. Then every solution $x(t)$ of (2.1) such that $\alpha(t_0) \leq x(t_0) \leq \beta(t_0)$ satisfies the estimate

$$\alpha(t) \leq x(t) \leq \beta(t), t \in [t_0, t_0 + T]. \tag{2.7}$$

Theorem 2 ([4])**.** *Let $f \in C[[t_0, t_0 + T] \times R, R]$ and $|f(t,x)| \leq L$. Then there exist a solution of the IVP* (2.1) *on $[t_0, t_0 + T]$.*

3. Comparison Theorems and Existence Results relative to Initial Time Difference

In this section, we will give some basic comparison theorems and existence results relative to initial time difference.

Theorem 3 ([5]). *Assume that* $f \in C[R_+ \times R, R]$ *and*
(i) $\alpha \in C^1[[\tau_0, \tau_0 + T], R], \tau_0 \geq 0, T > 0, \beta \in C^1[[\eta_0, \eta_0 + T], R], \eta_0 > 0,$ *and*

$$\alpha'(t) \leq f(t, \alpha(t)) \text{ for } t \in [\tau_0, \tau_0 + T] \tag{3.1}$$

$$\beta'(t) \geq f(t, \beta(t)) \text{ for } t \in [\eta_0, \eta_0 + T] \tag{3.2}$$

with $\alpha(\tau_0) \leq \beta(\eta_0)$;
(ii) $f(t,x) - f(t,y) \leq M(x-y), x \geq y, M > 0$;
(iii) $\tau_0 < t_0 < \eta_0$ *and* $f(t,x)$ *is nondecreasing in* t *for each* x.
Then (A) $\alpha(t) \leq \beta(t + (\sigma + \xi))$ *for* $t \geq \tau_0$ *where* $\sigma = t_0 - \tau_0, \xi = \eta_0 - t_0$.
(B) $\alpha(t - \sigma) \leq \beta(t + \xi)$ *for* $t \geq t_0$,*where* $\sigma = t_0 - \tau_0$ *and* $\xi = \eta_0 - t_0$.
(C) $\alpha(t - (\sigma + \xi) \leq \beta(t), t \geq \eta_0$ *where* $\sigma = t_0 - \tau_0, \xi = \eta_0 - t_0$.

Proof of Theorem 3. Please see [5] for the details of the proof by simple modification of (A), (B) and (C). □

The following theorem is the existence result in the closed sectors.

Theorem 4. *Assume that* $f \in C[R_+ \times \Omega, R]$ *and*
(i) $\alpha \in C^1[[\tau_0, \tau_0 + T], R], \tau_0 \geq 0, T > 0, \beta \in C^1[[\eta_0, \eta_0 + T], R], \eta_0 > 0$

$$\alpha'(t) \leq f(t, \alpha(t)) \text{ for } t \in [\tau_0, \tau_0 + T]$$

$$\beta'(t) \geq f(t, \beta(t)) \text{ for } t \in [\eta_0, \eta_0 + T]$$

with $\alpha(\tau_0) \leq \beta(\eta_0)$;
(ii) $\tau_0 < t_0 < \eta_0$ *and* $f(t,x)$ *is nondecreasing in* t *for each* x;
(iii) $\alpha(t - \sigma) \leq \beta(t + \xi)$ *for* $t_0 \leq t \leq t_0 + T$ *where* $\sigma = t_0 - \tau_0, \xi = \eta_0 - t_0$.
Then there exist a solution of initial value problem (2.1) *satisfying*

$$\alpha(t - \sigma) \leq x(t) \leq \beta(t + \xi)$$

for $t_0 \leq t \leq t_0 + T$.

Proof of Theorem 4. Let $\beta_0(t) = \beta(t + \xi)$ and $\alpha_0(t) = \alpha(t - \sigma)$ for $t \in [t_0, t_0 + T] = I$. Then we get $\beta_0(t_0) = \beta(\eta_0) \geq \alpha_0(t_0) = \alpha(\tau_0)$ and

$$\beta_0'(t) \geq f(t + \xi, \beta_0(t)), \alpha_0'(t) \leq f(t - \sigma, \alpha_0(t)).$$

Assume that $\alpha_0(t_0) \leq x_0 \leq \beta_0(t_0)$ and $p : [t_0, t_0 + T] \times R \to R$ such that

$$p(t,x) = \max[\alpha_0(t), \min[x, \beta_0(t)]]$$

Then $f(t, p(t,x))$ defines a continuous extension of f to $[t_0, t_0 + T] \times R$ which is also bounded since f is bounded on Ω, where

$$\Omega = \{(t,x) \in R_+ \times R : \alpha(t - \sigma) \leq x \leq \beta(t + \xi) \text{ for } t_0 \leq t \leq t_0 + T\}. \tag{3.3}$$

Therefore, the initial value problem $x' = f(t, p(t,x)), x(t_0) = x_0$ according to Theorem 2 has a solution on I. For sufficiently small $\varepsilon > 0$, consider

$$\alpha_{0_\varepsilon}(t) = \alpha_0(t) - \varepsilon(1 + t) \tag{3.4}$$

$$\beta_{0_\varepsilon}(t) = \beta_0(t) + \varepsilon(1+t). \tag{3.5}$$

Clearly

$$\alpha_{0_\varepsilon}(t_0) < \alpha_0(t_0) \leq x_0 \leq \beta_0(t_0) < \beta_{0_\varepsilon}(t_0)$$

and hence $\alpha_{0_\varepsilon}(t_0) < x_0 < \beta_{0_\varepsilon}(t_0)$. We wish to show that

$$\alpha_{0_\varepsilon}(t) < x(t) < \beta_{0_\varepsilon}(t) \text{ on } I$$

Suppose that it is not true, then there exists a $t_1 \in (t_0, t_0+T]$ such that

$$\alpha_{0_\varepsilon}(t) < x(t) < \beta_{0_\varepsilon}(t) \text{ on } [t_0, t_1) \text{ and } \beta_{0_\varepsilon}(t_1) = x(t_1).$$

Then $x(t_1) > \beta_0(t_1)$ and so

$$p(t_1, x(t_1)) = \beta_0(t_1)$$

Also $\alpha_0(t_1) \leq p(t_1, x(t_1)) \leq \beta_0(t_1)$. Hence

$$\beta_0'(t_1) \geq f(t_1, \beta_0(t_1)) = f(t_1, p(t_1, x(t_1))) = x'(t_1)$$

Since $\beta_{0_\varepsilon}'(t_1) > \beta_0'(t_1) \geq x'(t_1)$ we have $\beta_{0_\varepsilon}'(t_1) > x'(t_1)$. This contradicts $x(t) < \beta_{0_\varepsilon}(t)$ for $t \in [t_0, t_1)$. The other case can be proved similarly. Consequently, we obtain $\alpha_{0_\varepsilon}(t) < x(t) < \beta_{0_\varepsilon}(t)$ on I. Letting $\varepsilon \to 0$ we get

$$\alpha_0(t) \leq x(t) \leq \beta_0(t) \text{ on } I.$$

Therefore the proof is completed. □

Remark 2. Assume that $f \in C[R_+ \times R, R]$, assumptions (i), (ii) of Theorem 4 hold and

$(iii)^*$ $\alpha(t) \leq \beta(t + (\sigma + \xi))$ for $\tau_0 \leq t \leq \tau_0 + T$ where $\sigma = t_0 - \tau_0, \xi = \eta_0 - t_0$. Then there exist a solution satisfying

$$\alpha(t) \leq x(t+\sigma) \leq \beta(t + (\sigma + \xi))$$

for $\tau_0 \leq t \leq \tau_0 + T$.

Remark 3. Assume that $f \in C[R_+ \times R, R]$, assumptions (i), (ii) of Theorem 4 hold and

$(iii)^{**}$ $\alpha(t - (\sigma + \xi)) \leq \beta(t)$ for $\eta_0 \leq t \leq \eta_0 + T$ where $\sigma = t_0 - \tau_0, \xi = \eta_0 - t_0$. Then there exist a solution satisfying

$$\alpha(t - (\sigma + \xi)) \leq x(t - \xi) \leq \beta(t)$$

for $\eta_0 \leq t \leq \eta_0 + T$.

4. Monotone Iterative Technique with the different initial data

In this section we have applied the monotone iterative technique for the nonlinear initial value problem of (2.1) by choosing lower and upper solutions with known at the different initial data.

Theorem 5. *Assume that* $f \in C[R_+ \times R, R]$ *and*
(i) $\alpha \in C^1[[\tau_0, \tau_0+T], R], \tau_0 \geq 0, T > 0, \beta \in C^1[[\eta_0, \eta_0+T], R], \eta_0 > 0$

$$\alpha'(t) \leq f(t, \alpha(t)) \text{ for } t \in [\tau_0, \tau_0+T]$$
$$\beta'(t) \geq f(t, \beta(t)) \text{ for } t \in [\eta_0, \eta_0+T]$$

with $\alpha(\tau_0) \leq \beta(\eta_0)$;
(ii) $\tau_0 < t_0 < \eta_0$ *and* $f(t,x)$ *is nondecreasing in* t *for each* x;
(iii) $\alpha(t-\sigma) \leq \beta(t+\xi)$ *for* $t_0 \leq t \leq t_0+T, \sigma = t_0 - \tau_0, \xi = \eta_0 - t_0$;
(iv) $f(t,x) - f(t,y) \geq -M(x-y)$ *where* $M > 0$ *and* $\alpha(t-\sigma) \leq y \leq x \leq \beta(t+\xi)$ *for* $t \in [t_0, t_0+T]$.
Then there exist monotone sequences $\{\widetilde{\alpha}_n\}$ *and* $\{\widetilde{\beta}_n\}$ *which converge uniformly and monotonically on* $[t_0, t_0+T]$ *such that* $\widetilde{\alpha}_n \to \rho$ *and* $\widetilde{\beta}_n \to r$ *as* $n \to \infty$. *Moreover,* ρ *and* r *are minimal and maximal solutions such that* ρ *is the minimal solution of the initial value problem of* $x' = f(t,x), x(\tau_0) = x_0$ *on* $[\tau_0, \tau_0+T]$ *and* r *is the maximal solution of the initial value problem of* $x' = f(t,x), x(\eta_0) = x_0$ *on* $[\eta_0, \eta_0+T]$ *respectively where* $\widetilde{\beta}_0(t) = \beta(t+\xi), \widetilde{\alpha}_0(t) = \alpha(t-\sigma)$.

Proof of Theorem 5. Since $\widetilde{\beta}_0(t) = \beta(t+\xi), \widetilde{\beta}_0(t_0) = \beta(t_0+\xi) = \beta(\eta_0) \geq \alpha_0(\tau_0)$ and $\widetilde{\beta}_0'(t) \geq f(t+\xi, \widetilde{\beta}_0(t)), \widetilde{\alpha}_0'(t) \leq f(t-\sigma, \widetilde{\alpha}_0(t)), t \in [t_0, t_0+T]$. Consider the following linear initial value problems

$$\widetilde{\alpha}_{n+1}'(t) = f(t-\sigma, \widetilde{\alpha}_n(t)) - M(\widetilde{\alpha}_{n+1}(t) - \widetilde{\alpha}_n(t)), \widetilde{\alpha}_{n+1}(t_0) = x_0 \tag{4.1}$$
$$\widetilde{\beta}_{n+1}'(t) = f(t+\xi, \widetilde{\beta}_n(t)) - M(\widetilde{\beta}_{n+1}(t) - \widetilde{\beta}_n(t)), \widetilde{\beta}_{n+1}(t_0) = x_0 \tag{4.2}$$

Setting $p(t) = \widetilde{\beta}_1(t) - \widetilde{\beta}_0(t)$ where $p(t_0) \leq 0$ for $t \in I$.

$$\begin{aligned} p'(t) &= \widetilde{\beta}_1'(t) - \widetilde{\beta}_0'(t) \\ &= f(t+\xi, \widetilde{\beta}_0(t)) - M(\widetilde{\beta}_1(t) - \widetilde{\beta}_0(t)) - \widetilde{\beta}_0'(t) \\ &\leq f(t+\xi, \widetilde{\beta}_0(t)) - M(\widetilde{\beta}_1(t) - \widetilde{\beta}_0(t)) - f(t+\xi, \widetilde{\beta}_0(t)) \\ p'(t) &\leq -Mp(t) \end{aligned}$$

This shows that $p(t) \le p(t_0)e^{-Mt} \le 0$ since we have $p(t_0) \le 0$. Hence $\widetilde{\beta}_1(t) \le \widetilde{\beta}_0(t)$ on I. Similarly we can show that $\widetilde{\alpha}_0(t) \le \widetilde{\alpha}_1(t)$ on I. Setting $p(t) = \widetilde{\alpha}_1(t) - \widetilde{\alpha}_0(t)$ and $p(t_0) \ge 0$ for $t \in [t_0, t_0+T]$.

$$
\begin{aligned}
p'(t) &= \widetilde{\alpha}_1'(t) - \widetilde{\alpha}_0'(t) \\
&= f(t-\sigma, \widetilde{\alpha}_0(t)) - M(\widetilde{\alpha_1}(t) - \widetilde{\alpha}_0(t)) - \widetilde{\alpha}_0'(t) \\
&\ge f(t-\sigma, \widetilde{\alpha}_0(t)) - M(\widetilde{\alpha}_1(t) - \widetilde{\alpha}_0(t)) - f(t-\sigma, \widetilde{\alpha}_0(t)) \\
p'(t) &\ge -Mp(t)
\end{aligned}
$$

This shows that $p(t) \ge p(t_0)e^{-Mt} \ge 0$. Hence $\widetilde{\alpha}_0(t) \le \widetilde{\alpha}_1(t)$ on I. Now we can show $\widetilde{\alpha}_1(t) \le \widetilde{\beta}_1(t)$. Setting $p(t) = \widetilde{\alpha}_1(t) - \widetilde{\beta}_1(t)$ where $p(t_0) \le 0$ for $t \in [t_0, t_0+T]$.

$$
\begin{aligned}
p'(t) &= \widetilde{\alpha}_1'(t) - \widetilde{\beta}_1'(t) \\
&= f(t-\sigma, \widetilde{\alpha}_0(t)) - M(\widetilde{\alpha}_1(t) - \widetilde{\alpha}_0(t)) - f(t+\xi, \widetilde{\beta}_0(t)) + M(\widetilde{\beta}_1(t) - \widetilde{\beta}_0(t)) \\
&\le f(t, \widetilde{\alpha}_0(t)) - M(\widetilde{\alpha}_1(t) - \widetilde{\alpha}_0(t)) - f(t, \widetilde{\beta}_0(t)) + M(\widetilde{\beta}_1(t) - \widetilde{\beta}_0(t)) \\
&= f(t, \widetilde{\alpha}_0(t)) - f(t, \widetilde{\beta}_0(t)) - M(\widetilde{\alpha}_1(t) - \widetilde{\alpha}_0(t)) + M(\widetilde{\beta}_1(t) - \widetilde{\beta}_0(t)) \\
&\le -\Big[-M(\widetilde{\beta}_0(t) - \widetilde{\alpha}_0(t))\Big] - M(\widetilde{\alpha}_1(t) - \widetilde{\alpha}_0(t)) + M(\widetilde{\beta}_1(t) - \widetilde{\beta}_0(t)) \\
&= M(\widetilde{\beta}_1(t) - \widetilde{\alpha}_1(t)) \\
p'(t) &\le -Mp(t)
\end{aligned}
$$

This shows that $p(t) \le p(t_0)e^{-Mt} \le 0$ on I. Hence $\widetilde{\alpha}_1(t) \le \widetilde{\beta}_1(t)$ on $[t_0, t_0+T]$. Consequently, we have

$$\widetilde{\alpha}_0(t) \le \widetilde{\alpha}_1(t) \le \widetilde{\beta}_1(t) \le \widetilde{\beta}_0(t) \text{ on } I.$$

To employ the method of mathematical induction, assume that for some $k > 1$

$$\widetilde{\alpha}_{k-1}(t) \le \widetilde{\alpha}_k(t) \le \widetilde{\beta}_k(t) \le \widetilde{\beta}_{k-1}(t) \text{ on } I$$

we then show that

$$\widetilde{\alpha}_k(t) \le \widetilde{\alpha}_{k+1}(t) \le \widetilde{\beta}_{k+1}(t) \le \widetilde{\beta}_k(t) \text{ on } I.$$

where $\widetilde{\alpha}_{k+1}(t)$ and $\widetilde{\beta}_{k+1}(t)$ are the solutions of the linear IVPs

$$
\begin{aligned}
\widetilde{\alpha}_{k+1}'(t) &= f(t-\sigma, \widetilde{\alpha}_k(t)) - M(\widetilde{\alpha}_{k+1}(t) - \widetilde{\alpha}_k(t)),\ \widetilde{\alpha}_{k+1}(t_0) = x_0 \\
\widetilde{\beta}_{k+1}'(t) &= f(t+\xi, \widetilde{\beta}_k(t)) - M(\widetilde{\beta}_{k+1}(t) - \widetilde{\beta}_k(t)),\ \widetilde{\beta}_{k+1}(t_0) = x_0.
\end{aligned}
$$

As we have done before, we set $p(t) = \widetilde{\alpha}_{k+1}(t) - \widetilde{\alpha}_k(t)$ where $p(t_0) \geq 0$.

$$
\begin{aligned}
p'(t) &= \widetilde{\alpha}'_{k+1}(t) - \widetilde{\alpha}'_k(t)\\
&= f(t-\sigma, \widetilde{\alpha}_k(t)) - M(\widetilde{\alpha}_{k+1}(t)\\
&\quad - \widetilde{\alpha}_k(t)) - f(t-\sigma, \widetilde{\alpha}_{k-1}(t)) + M(\widetilde{\alpha}_k(t) - \widetilde{\alpha}_{k-1}(t))\\
&= f(t-\sigma, \widetilde{\alpha}_k(t)) - f(t-\sigma, \widetilde{\alpha}_{k-1}(t))\\
&\quad - M(\widetilde{\alpha}_{k+1}(t) - \widetilde{\alpha}_k(t)) + M(\widetilde{\alpha}_k(t) - \widetilde{\alpha}_{k-1}(t))\\
&\geq -M(\widetilde{\alpha}_k(t) - \widetilde{\alpha}_{k-1}(t)) - M(\widetilde{\alpha}_{k+1}(t) - \widetilde{\alpha}_k(t)) - M(\widetilde{\alpha}_k(t) - \widetilde{\alpha}_{k-1}(t))\\
&= -M(\widetilde{\alpha}_{k+1}(t) - \widetilde{\alpha}_k(t))\\
p'(t) &\geq -Mp(t)
\end{aligned}
$$

This shows that $p(t) \geq p(t_0)e^{-Mt} \geq 0$ on I. Hence $\widetilde{\alpha}_k(t) \leq \widetilde{\alpha}_{k+1}(t)$ on I. Setting $p(t) = \widetilde{\beta}_{k+1}(t) - \widetilde{\beta}_k(t)$ where $p(t_0) \leq 0$.

$$
\begin{aligned}
p'(t) &= \widetilde{\beta}'_{k+1}(t) - \widetilde{\beta}'_k(t)\\
&= f(t+\xi, \widetilde{\beta}_k(t)) - M(\widetilde{\beta}_{k+1}(t) - \widetilde{\beta}_k(t))\\
&\quad - f(t+\xi, \widetilde{\beta}_{k-1}(t)) + M(\widetilde{\beta}_k(t) - \widetilde{\beta}_{k-1}(t))\\
&= f(t+\xi, \widetilde{\beta}_k(t)) - f(t+\xi, \widetilde{\beta}_{k-1}(t))\\
&\quad - M(\widetilde{\beta}_{k+1}(t) - \widetilde{\beta}_k(t)) + M(\widetilde{\beta}_k(t) - \widetilde{\beta}_{k-1}(t))\\
&\leq -\left[-M(\widetilde{\beta}_{k-1}(t) - \widetilde{\beta}_k(t))\right] - M(\widetilde{\beta}_{k+1}(t) - \widetilde{\beta}_k(t)) + M(\widetilde{\beta}_k(t) - \widetilde{\beta}_{k-1}(t))\\
p'(t) &\leq -Mp(t)
\end{aligned}
$$

This shows that $p(t) \leq p(t_0)e^{-Mt} \leq 0$ on I. Hence $\widetilde{\beta}_{k+1}(t) \leq \widetilde{\beta}_k(t)$ on I. Now setting $p(t) = \widetilde{\alpha}_{k+1}(t) - \widetilde{\beta}_{k+1}(t)$ where $p(t_0) \leq 0$.

$$
\begin{aligned}
p'(t) &= \widetilde{\alpha}'_{k+1}(t) - \widetilde{\beta}'_{k+1}(t)\\
&= f(t-\sigma, \widetilde{\alpha}_k(t)) - M(\widetilde{\alpha}_{k+1}(t) - \widetilde{\alpha}_k(t))\\
&\quad - f(t+\xi, \widetilde{\beta}_k(t)) + M(\widetilde{\beta}_{k+1}(t) - \widetilde{\beta}_k(t))\\
&\leq f(t, \widetilde{\alpha}_k(t)) - f(t, \widetilde{\beta}_k(t)) - M(\widetilde{\alpha}_{k+1}(t) - \widetilde{\alpha}_k(t)) + M(\widetilde{\beta}_{k+1}(t) - \widetilde{\beta}_k(t))
\end{aligned}
$$

$$\leq -\left[-M(\widetilde{\beta}_k(t)-\widetilde{\alpha}_k(t))\right]-M(\widetilde{\alpha}_{k+1}(t)-\widetilde{\alpha}_k(t))+M(\widetilde{\beta}_{k+1}(t)-\widetilde{\beta}_k(t))$$

$$p'(t)\leq -Mp(t)$$

This shows that $p(t)\leq p(t_0)e^{-Mt}\leq 0$ on I. Hence $\widetilde{\alpha}_{k+1}(t)\leq \widetilde{\beta}_{k+1}(t)$ on I. Consequently, for all $k\in N$ and for $t\in I$. We get

$$\widetilde{\alpha}_k(t)\leq \widetilde{\alpha}_{k+1}(t)\leq \widetilde{\beta}_{k+1}(t)\leq \widetilde{\beta}_k(t) \text{ on } I.$$

Hence it follows that for all $n\in N$ and $t\in I$, we have

$$\widetilde{\alpha}_0(t)\leq \widetilde{\alpha}_1(t)\leq ...\leq \widetilde{\alpha}_n(t)\leq \widetilde{\beta}_n(t)\leq ...\leq \widetilde{\beta}_1(t)\leq \widetilde{\beta}_0(t) \text{ on } I. \tag{4.3}$$

It is clear that the sequences $\left\{\widetilde{\alpha}_n\right\}$ and $\left\{\widetilde{\beta}_n\right\}$ are uniformly bounded and equicontinuous sequence of functions on $[t_0,t_0+T]$ and consequently by Ascoli-Arzela's theorem there exist subsequences $\left\{\widetilde{\alpha}_{n_k}\right\}$ and $\left\{\widetilde{\beta}_{n_k}\right\}$ that converge uniformly on $[t_0,t_0+T]$. In view of (4.3) it follows that the entire sequences $\left\{\widetilde{\alpha}_n\right\}$ and $\left\{\widetilde{\beta}_n\right\}$ converge uniformly and monotonically to $\widetilde{\rho}$ and $\widetilde{r}$,respectively, as $n\to\infty$. We have obtained the following corresponding Volterra integral equation for (4.1) and (4.2)

$$\widetilde{\alpha}_{n+1}(t)=x_0+\int_{t_0}^{t}(f(s-\sigma,\widetilde{\alpha}_n(s))-M(\widetilde{\alpha}_{n+1}(s)-\widetilde{\alpha}_n(s))ds$$

$$\widetilde{\beta}_{n+1}(t)=x_0+\int_{t_0}^{t}(f(s+\xi,\widetilde{\beta}_n(s))-M(\widetilde{\beta}_{n+1}(s)-\widetilde{\beta}_n(s))ds.$$

Therefore, we get

$$\widetilde{\rho}'(t)=f(t-\sigma,\widetilde{\rho}(t)),\widetilde{\rho}(t_0)=x_0 \tag{4.4}$$

$$\widetilde{r}'(t)=f(t+\xi,\widetilde{r}(t)),\widetilde{r}(t_0)=x_0 \tag{4.5}$$

as $n\to\infty$ where $\widetilde{\rho}(t)=\rho(t-\sigma)$ and $\widetilde{r}(t)=r(t+\xi)$, respectively. Finally, we must show that $\widetilde{\rho}$ and $\widetilde{r}$ are the minimal and maximal solutions of the IVP (4.4) and (4.5), respectively. Let $x(t)$ be any solution of (2.1) such that

$$\widetilde{\alpha}_0(t)\leq x(t)\leq \widetilde{\beta}_0(t) \text{ on } [t_0,t_0+T].$$

Then we need to prove

$$\widetilde{\alpha}_0(t)\leq \widetilde{\rho}\leq x(t)\leq \widetilde{r}\leq \widetilde{\beta}_0(t) \text{ on } [t_0,t_0+T].$$

Suppose that for some n,

$$\widetilde{\alpha}_n(t) \le x(t) \le \widetilde{\beta}_n(t).$$

Then, we set $p(t) = \widetilde{\alpha}_{n+1}(t) - x(t)$ where $p(t_0) = 0$. Thus

$$\begin{aligned} p'(t) &= \widetilde{\alpha}'_{n+1}(t) - x'(t) \\ &= f(t-\sigma, \widetilde{\alpha}_n(t)) - M(\widetilde{\alpha}_{n+1}(t) - \widetilde{\alpha}_n(t)) - f(t,x) \\ &\le f(t, \widetilde{\alpha}_n(t)) - f(t,x) - M(\widetilde{\alpha}_{n+1}(t) - \widetilde{\alpha}_n(t)) \\ &\le -M(\widetilde{\alpha}_n(t) - x(t)) - M(\widetilde{\alpha}_{n+1}(t) - \widetilde{\alpha}_n(t)) \\ &= -M(\widetilde{\alpha}_{n+1}(t) - x(t)) \\ &= -Mp(t). \end{aligned}$$

This shows that $p(t) \le p(t_0)e^{-Mt} \le 0$ on I since we have $p(t_0) \le 0$. Hence $\widetilde{\alpha}_{n+1}(t) \le x(t)$ on I. In a similar manner, we can show that $x(t) \le \widetilde{\beta}_n(t)$ on $[t_0, t_0 + T]$. This proves by induction that $\widetilde{\alpha}_n(t) \le x(t) \le \widetilde{\beta}_n(t)$ for all n taking limit as $n \to \infty$ we arrive at $\widetilde{\rho} \le x(t) \le \widetilde{r}$ on $[t_0, t_0 + T]$. Therefore the proof is completed. □

Corollary 1. *If in addition to the assumption of Theorem 5, we assume*

$$f(t,x) - f(t,y) \le M(x-y), \alpha(t-\sigma) \le y \le x \le \beta(t+\xi), M > 0$$

then we have unique solution of (2.1) such that $\widetilde{\rho} = x = \widetilde{r}$.

Proof. If we set $p = r - \rho$ then $p' = r' - \rho' = f(t,r) - f(t,\rho) \le M(r - \rho)$, which gives $p' \le Mp$ and $p(t_0) = 0$.Hence we get $p(t) \le 0$ on $[t_0, t_0 + T]$ which implies $r \le \rho$. Also , utilizing the fact that $\rho \le r$, we have $\rho = x = r$ is the unique solution of (2.1). □

Corollary 2. *If in addition to the assumption* $(i), (ii)$ *of Theorem 5, we assume* (iii) $\alpha(t) \le \beta(t + (\sigma + \xi))$ *for* $\tau_0 \le t \le \tau_0 + T$ *where* $\sigma = t_0 - \tau_0, \xi = \eta_0 - t_0$; (iv) $f(t,x) - f(t,y) \ge -M(x-y)$ *where* $M > 0$ *and* $\alpha(t) \le y \le x \le \beta(t + (\sigma + \xi))$ *for* $t \in [\tau_0, \tau_0 + T]$.

Then there exist monotone sequences $\{\alpha_n\}$ *and* $\left\{\widetilde{\beta}_n\right\}$ *which converge uniformly and monotonically on* $[\tau_0, \tau_0 + T]$ *such that* $\alpha_n \to \rho$ *and* $\widetilde{\beta}_n \to \widetilde{r}$ *as* $n \to \infty$. *Moreover,* ρ *and* r *are minimal and maximal solutions such that* ρ *is the minimal solution of the initial value problem of* $x' = f(t,x), x(\tau_0) = x_0$ *on* $[\tau_0, \tau_0 + T]$ *and* r *is the maximal solution of the initial value problem of*

$$\widetilde{x}'(t) = f(t+\sigma, \widetilde{x}(t)), \widetilde{x}(\tau_0) = x_0, t \in [\tau_0, \tau_0 + T] \tag{4.6}$$

respectively where $\widetilde{\beta}_0(t) = \beta(t + (\sigma + \xi)), \widetilde{\alpha}_0(t) = \alpha_0(t)$.

Corollary 3. *If in addition to the assumption* $(i), (ii)$ *of Theorem 5, we assume* (iii) $\alpha(t - (\sigma + \xi)) \le \beta(t)$ *for* $\eta_0 \le t \le \eta_0 + T$ *where* $\sigma = t_0 - \tau_0, \xi = \eta_0 - t_0$; (iv) $f(t,x) - f(t,y) \ge -M(x-y)$ *where* $M > 0$ *and* $\alpha(t - (\sigma + \xi)) \le y \le x \le \beta(t)$ *for* $t \in [\eta_0, \eta_0 + T]$.

Then there exist monotone sequences $\left\{\widetilde{\alpha}_n\right\}$ *and* $\{\beta_n\}$ *which converge uniformly and monotonically on* $[\eta_0, \eta_0 + T]$ *such that* $\widetilde{\alpha}_n \to \widetilde{\rho}$ *and* $\beta_n \to r$ *as* $n \to \infty$. *Moreover,* $\widetilde{\rho}$ *and* r *are minimal and maximal solutions such that* $\widetilde{\rho}$ *is the minimal solution of the initial value problem of*

$$\widetilde{x}'(t) = f(t - \xi, \widetilde{x}(t)), \widetilde{x}(\eta_0) = x_0, t \in [\eta_0, \eta_0 + T] \tag{4.7}$$

and r *is the maximal solution of the initial value problem of* $x' = f(t,x), x(\eta_0) = x_0$ *on* $[\eta_0, \eta_0 + T]$ *where* $\widetilde{\beta}_0(t) = \beta_0(t), \widetilde{\alpha}_0(t) = \alpha_0(t - (\sigma + \xi))$.

5. Examples

Example 1. Consider the nonlinear initial value problem

$$x'(t) = e^t x^2, \ x(1) = -1 \text{ for } t \ge 1 \tag{5.1}$$

where $f(t,x) = e^t x^2 \in C[R_+ \times R, R]$ and $t \in [1,4]$.

(E_1) $\alpha(t) = -\frac{2}{e^t}$, $\alpha(0) = -2$, $\alpha(t) \in C^1[[0,3], R]$ and $\beta(t) = -\frac{1}{2e^t}$, $\beta(2) = -\frac{1}{2e^2}$, $\beta(t) \in C^1[[2,5], R]$, then we get for $T = 3$

$$\alpha'(t) = \frac{2}{e^t} \text{ and } f(t,\alpha) = \frac{4}{e^t} \text{ then } \alpha'(t) \le f(t,\alpha) \text{ for } t \in [0,3]$$
$$\beta'(t) = \frac{1}{2e^t} \text{ and } f(t,\beta) = \frac{1}{4e^t} \text{ then } \beta'(t) \ge f(t,\alpha) \text{ for } t \in [2,5].$$

Therefore, $\alpha(t)$ and $\beta(t)$ are lower and upper solutions, respectively and

$$\alpha(\tau_0) = \alpha(0) = -2 < x(t_0) = x(1) = -1 < \beta(\eta_0) = \beta(2) = -\frac{1}{2e^2}.$$

(E_2) $0 < 1 < 2$ and $f(t,x)$ is nondecreasing in t for each x and $\alpha(1) = \frac{-2}{e} \le \beta(1) = -\frac{1}{2e}$.

(E_3) $f(t,x) - f(t,y) \ge -M(x-y)$ where $M = 4e^4 > 0$ is the Lipschitz constant for $\alpha(1) \le y \le x \le \beta(1)$, $t \in [0,5]$. Also $\widetilde{\alpha}_0(t) = \alpha(t-1)$ and $\widetilde{\beta}_0(t) = \beta(t+1)$ for $t \in [1,4]$.

Therefore, $\widetilde{\alpha}_{n+1}$ is a lower solution and $\widetilde{\beta}_{n+1}$is an upper solution of (5.1) for $t \in [1,3]$. Thus $\widetilde{\alpha}_{n+1}(t) \le \widetilde{\beta}_{n+1}(t)$ for $t \in [1,4]$.

Consequently, we have for all n,

$$\widetilde{\alpha}_0(t) \le \widetilde{\alpha}_1(t) \le \cdots \le \widetilde{\alpha}_n(t) \le \widetilde{\beta}_n(t) \le \cdots \le \widetilde{\beta}_1(t) \le \widetilde{\beta}_0(t) \ \text{ for } t \in [1,4]$$

Employing the standard monotone technique it can be shown that the monotone sequence $\left\{\widetilde{\alpha}_n(t)\right\}$ converges to $\widetilde{\rho}$ which is the minimal solution of (5.1) as $n \to \infty$ and monotone sequence $\left\{\widetilde{\beta}_n(t)\right\}$ converges to $\widetilde{r}$ which is the maximal solution of (5.1) as $n \to \infty$. We arrive at $\widetilde{\rho} \le x(t) \le \widetilde{r}$ on $[1,3]$. In this example,since the solution $x(t) = -\frac{1}{e^t+1-e}$ of (5.1) is unique, $\widetilde{\rho} = x(t) = \widetilde{r}$.

ACKNOWLEDGEMENT

The authors would like to thank the referees for their insightful comments and detailed suggestions which improved the quality of the paper.

REFERENCES

[1] G. Deekshitulu, "Generalized monotone iterative technique for fractional R-L differential equations," *Nonlinear Studies*, vol. 16, no. 1, pp. 85–94, 2009.

[2] S. Hristova and A. Golev, "Monotone-iterative method for the initial value problem with initial time difference for differential equations with "Maxima"," *Abstract and Applied Analysis*, vol. 2012, no. 493271, 2012.

[3] S. Koksal and C. Yakar, "Generalized quasilinearization method with initial time difference," *International Journal of Electrical, Electronic and other Physical Systems*, vol. 24, no. 5, 2002.

[4] V. Ladde, G. S. Lakshmikantham and V. A. S., *Monotone Iterative Technique for Nonlinear Differential Equations*, 2nd ed., ser. Series is books. Boston: Pitman Publishing Inc., 1985, vol. III.

[5] V. Lakshmikantham and A. S. Vatsala, "Differential inequalities with initial time difference and applications," *Journal of Inequalities and Applications*, vol. 2, no. 3, pp. 233–244, 1999.

[6] V. Lakshmikantham and A. S. Vatsala, "General uniqueness and monotone iterative technique for fractional differential equations," *Applied Mathematics Letters*, vol. 21, no. 8, pp. 828–834, 2008.

[7] F. McRae, "Monotone iterative technique and existence results for fractional differential equations," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 71, no. 12, pp. 6093–6096, 2009.

[8] J. Vasundhara Devi, "Generalized monotone technique for periodic boundary value problems of fractional differential equations," *Communications in Applied Analysis*, vol. 12, no. 4, pp. 399–406, 2008.

[9] C. Yakar, B. Bal, and A. Yakar, "Monotone technique in terms of two monotone functions in finite systems," *Journal of Concrete and Applicable Mathematics*, pp. 233–239, 2011.

[10] C. Yakar and A. Yakar, "An extension of the quasilinearization method with initial time difference," *Dynamics of Continuous, Discrete and Impulsive Systems (Series A: Mathematical Analysis)*, vol. 1-305, no. 14, pp. 275–279, 2007.

[11] C. Yakar and A. Yakar, "Further generalization of quasilinearization method with initial time difference," *J. of Appl. Funct. Anal.*, vol. 4, no. 4, pp. 714–727, 2009.

[12] C. Yakar and A. Yakar, “A refinement of quasilinearization method for Caputo sense fractional order differential equations,” *Abstract and Applied Analysis*, vol. 2010, no. Article ID 704367, p. 10 pages, 2010.

[13] C. Yakar and A. Yakar, “Monotone iterative techniques for fractional order differential equations with initial time difference,” *Hacettepe Journal of Mathematics and Statistics*, no. 14, pp. 331–340, 2011.

Authors' addresses

Coşkun Yakar

Gebze Technical University Faculty of Sciences Department of Mathematics Applied Mathematics Gebze-Kocaeli 141-41400

E-mail address: cyakar@gtu.edu.tr

İsmet Arslan

Gebze Technical University Faculty of Sciences Department of Mathematics Applied Mathematics Gebze-Kocaeli 141-41400

E-mail address: iarslan@gtu.edu.tr

Muhammed Çiçek

Gebze Technical University Faculty of Sciences Department of Mathematics Applied Mathematics Gebze-Kocaeli 141-41400

E-mail address: mcicek@gtu.edu.tr