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PRINCIPAL FUNCTIONS OF MATRIX STURM–LIOUVILLE OPERATORS WITH BOUNDARY CONDITIONS DEPENDENT ON THE SPECTRAL PARAMETER

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Abstract. Let L denote operator generated in $L_2(\mathbb{R}_+, E)$ by the differential expression

$$l(y) = -y'' + Q(x)y,\ x \in \mathbb{R}_+ := [0, \infty),$$

and the boundary condition $(A_0 + A_1\lambda)Y'(0,\lambda) - (B_0 + B_1\lambda)Y(0,\lambda) = 0$, where Q is a matrix-valued function and A_0, A_1, B_0, B_1 are non-singular matrices, with $A_0B_1 - A_1B_0 \neq 0$. In this paper, we investigate the principal functions corresponding to the eigenvalues and the spectral singularities of L.

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Keywords: eigenvalues, spectral singularities, spectral analysis, Sturm–Liouville operator, non-selfadjoint matrix operator, principal functions

1. Introduction

Let us consider the boundary value problem (BVP)

$$-u'' + q(x)u = \lambda^2 u\ , \quad x \in \mathbb{R}_+\ , \tag{1.1}$$

$$u(0) = 0\ , \tag{1.2}$$

in $L^2(\mathbb{R}_+)$, where q is a complex-valued function. The spectral theory of the BVP (1.1)–(1.2) with continuous and point spectrum was investigated by Naimark [20]. He showed the existence of the spectral singularities in the continuous spectrum of the BVP (1.1)–(1.2). Note that the eigenfunctions and the associated functions (principal functions) corresponding to the spectral singularities are not the elements of $L^2(\mathbb{R}_+)$. Also, the spectral singularities belong to the continuous spectrum and are the poles of the resolvent's kernel, but are not the eigenvalues of the BVP (1.1)–(1.2). The spectral singularities in the spectral expansion of the BVP (1.1)–(1.2) in terms of the principal functions have been investigated in [19]. The spectral analysis of the quadratic pencil of Schrödinger, Dirac and Klein-Gordon operators with spectral singularities were studied in [2–18]. The spectral analysis of the non-selfadjoint operator, generated in $L^2(\mathbb{R}_+)$ by (1.1) and the boundary condition

$$\frac{y'(0)}{y(0)} = \frac{\beta_1\lambda+\beta_0}{\alpha_1\lambda+\alpha_0},$$

where $\alpha_i,\ \beta_i \in \mathbb{C},\ i=0,\ 1$ with $\alpha_0\beta_1-\alpha_1\beta_0 \neq 0$ were investigated in detail by Bairamov et al. [9]. The all above mentioned papers related with the differential and difference equations are scalar coefficients. Spectral analysis of the self-adjoint differential and difference equations with matrix coefficients are studied in [3,10–13].

Let E be an n-dimensional $(n<\infty)$ Euclidian space with the norm $\|.\|$ and let us introduce the Hilbert space $L^2(\mathbb{R}_+,E)$ consisting of vector-valued functions with the values in E. We will consider the BVP

$$-y''+Q(x)\,y=\lambda^2 y\ ,\quad x\in\mathbb{R}_+\ , \tag{1.3}$$

$$y(0)=0, \tag{1.4}$$

in $L^2(\mathbb{R}_+,E)$ where Q is a non-selfadjoint matrix-valued function (i. e., $Q\neq Q^*$). It is clear that, the BVP (1.3)–(1.4) is nonselfadjoint. In [14, 21] discrete spectrum of the non-selfadjoint matrix Sturm–Liouville operator and properties of the principal functions corresponding to the eigenvalues and the spectral singularities were investigated.

Let us consider the BVP in $L_2(\mathbb{R}_+,E)$

$$-y''+Q(x)y=\lambda^2 y,\quad x\in\mathbb{R}_+, \tag{1.5}$$

$$(A_0+A_1\lambda)y'(0,\lambda)-(B_0+B_1\lambda)y(0,\lambda)=0, \tag{1.6}$$

where Q is a non-singular matrix-valued function and $A_0,\ A_1,\ B_0,\ B_1$ are non-selfadjoint matrices such $A_0B_1-A_1B_0\neq 0$. In this paper, we aim to investigate the properties of the principal functions corresponding to the eigenvalues and the spectral singularities of the BVP (1.5)–(1.6).

2. Jost Solution of (1.5)

We will denote the solution of (1.5) satisfying the condition

$$\lim_{x\to\infty} y(x,\lambda)e^{-i\lambda x}=I,\qquad \lambda\in\overline{\mathbb{C}}_+ := \{\lambda:\lambda\in\mathbb{C},\operatorname{Im}\lambda\geq 0\} \tag{2.1}$$

by $E(x,\lambda)$. The solution $E(x,\lambda)$ is called the Jost solution of (1.5).

Under the condition

$$\int_0^\infty x\,\|Q(x)\|\,dx<\infty \tag{2.2}$$

the Jost solution has a representation

$$E(x,\lambda)=e^{i\lambda x}I+\int_x^\infty K(x,t)e^{i\lambda t}dt, \tag{2.3}$$

for $\lambda \in \bar{\mathbb{C}}_+$, where the kernel matrix function $K(x,t)$ satisfies

$$K(x,t) = \frac{1}{2}\int_{\frac{x+t}{2}}^{\infty} Q(s)ds + \frac{1}{2}\int_{x}^{\frac{x+t}{2}}\int_{t+x-s}^{t+s-x} Q(s)K(s,v)dvds$$

$$+\frac{1}{2}\int_{\frac{x+t}{2}}^{\infty}\int_{s}^{t+s-x} Q(s)K(s,v)dvds. \tag{2.4}$$

Moreover, $K(x,t)$ is continuously differentiable with respect to its arguments and

$$\|K(x,t)\| \leq c\sigma(\frac{x+t}{2}), \tag{2.5}$$

$$\|K_x(x,t)\| \leq \frac{1}{4}\left\|Q(\frac{x+t}{2})\right\| + c\sigma(\frac{x+t}{2}), \tag{2.6}$$

$$\|K_t(x,t)\| \leq \frac{1}{4}\left\|Q(\frac{x+t}{2})\right\| + c\sigma(\frac{x+t}{2}), \tag{2.7}$$

where $\sigma(x) = \int_x^{\infty} \|Q(s)\|\, ds$ and $c > 0$ is a constant. Therefore, $E(x,\lambda)$ is analytic with respect to λ in $\mathbb{C}_+ := \{\lambda : \lambda \in \mathbb{C}, \operatorname{Im}\lambda > 0\}$ and continuous on the real axis [1].

Let $\hat{E}^{\pm}(x,\lambda)$ denote the solutions of (1.5) subject to the conditions

$$\lim_{x\to\infty} \hat{E}^{\pm}(x,\lambda)e^{\pm i\lambda x} = I, \quad \lim_{x\to\infty} \hat{E}^{\pm}_x(x,\lambda)e^{\pm i\lambda x} = \pm i\lambda I, \quad \lambda \in \bar{\mathbb{C}}_{\pm}. \tag{2.8}$$

Then

$$W\left[E(x,\lambda), \hat{E}^{\pm}(x,\lambda)\right] = \mp 2i\lambda I, \quad \lambda \in \mathbb{C}_{\pm}, \tag{2.9}$$

$$W\left[E(x,\lambda), E(x,-\lambda)\right] = -2i\lambda I, \quad \lambda \in \mathbb{R}, \tag{2.10}$$

where $W[f_1, f_2]$ is the Wronskian of f_1 and f_2.

Let $\varphi(x,\lambda)$ denote the solution of (1.5) subject to the initial conditions $\varphi(0,\lambda) = A_0 + A_1\lambda$, $\varphi'(0,\lambda) = B_0 + B_1\lambda$. Therefore $\varphi(x,\lambda)$ is an entire function of λ.

Let us define the following functions:

$$D_{\pm}(\lambda) = \varphi(0,\lambda)E_x(0,\pm\lambda) - \varphi'(0,\lambda)E(0,\pm\lambda) \quad \lambda \in \bar{\mathbb{C}}_{\pm}, \tag{2.11}$$

where $\bar{\mathbb{C}}_{\pm} = \{\lambda : \lambda \in \mathbb{C},\ \pm\operatorname{Im}\lambda \geq 0\}$. It is obvious that the functions $D_+(\lambda)$ and $D_-(\lambda)$ are analytic in $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively, and continuous on the real axis. The functions D_+ and D_- are called Jost functions of L.

3. Eigenvalues and Spectral Singularities of L

The resolvent of L defined by

$$R_\lambda(L)f = \int_0^\infty G(x,t;\lambda)g(t)dt, \qquad g \in L_2(\mathbb{R}_+, E), \tag{3.1}$$

where

$$G(x,t;\lambda) = \begin{cases} G_+(x,t;\lambda), & \lambda \in \mathbb{C}_+ \\ G_-(x,t;\lambda), & \lambda \in \mathbb{C}_-, \end{cases} \tag{3.2}$$

and

$$G_\pm(x,t;\lambda) = \begin{cases} -E(x,\pm\lambda)D_\pm^{-1}(\lambda)\varphi^T(t,\lambda),\ 0 \le t \le x \\ -\varphi(x,\lambda)\left[D_+^T(\pm\lambda)\right]^{-1} E^T(t,\pm\lambda),\ x \le t < \infty. \end{cases} \tag{3.3}$$

We will show the set of eigenvalues and the set of spectral singularities of the operator L by σ_d and σ_{ss}, respectively.

Let us suppose that

$$H_\pm(\lambda) = \det D_\pm(\lambda). \tag{3.4}$$

From (2.3) and (3.1)–(3.4), we get

$$\begin{aligned} \sigma_d &= \{\lambda : \lambda \in \mathbb{C}_+,\ H_+(\lambda) = 0\} \cup \{\lambda : \lambda \in \mathbb{C}_-,\ H_-(\lambda) = 0\} \\ \sigma_{ss} &= \left\{\lambda : \lambda \in \mathbb{R}^*,\ H_+(\lambda) = 0\right\} \cup \left\{\lambda : \lambda \in \mathbb{R}^*,\ H_-(\lambda) = 0\right\}, \end{aligned} \tag{3.5}$$

where $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$.

We see from that, the functions

$$K^+(\lambda) = \frac{\hat{D}_+(\lambda)}{2i\lambda}E(x,\lambda) - \frac{D_+(\lambda)}{2i\lambda}\hat{E}^+(x,\lambda),\ \ \lambda \in \mathbb{C}_+, \tag{3.6}$$

$$K^-(\lambda) = \frac{\hat{D}_-(\lambda)}{2i\lambda}E(x,-\lambda) - \frac{D_-(\lambda)}{2i\lambda}\hat{E}^-(x,\lambda),\ \ \lambda \in \mathbb{C}_-, \tag{3.7}$$

$$K(\lambda) = \frac{D_+(\lambda)}{2i\lambda}E(x,-\lambda) - \frac{D_-(\lambda)}{2i\lambda}E(x,\lambda),\ \ \lambda \in \mathbb{R}^*, \tag{3.8}$$

are the solutions of the boundary problem (1.5)–(1.6), where

$$\hat{D}_\pm(\lambda) = (A_0 + A_1\lambda)\hat{E}_x^\pm(0,\lambda) - (B_0 + B_1\lambda)\hat{E}^\pm(0,\lambda). \tag{3.9}$$

Now let us assume that

$$Q \in AC(\mathbb{R}_+)\ ,\ \lim_{x\to\infty} Q(x) = 0\ ,\ \sup_{x\in[0,\infty)}\left[e^{\varepsilon\sqrt{x}}\left\|Q'(x)\right\|\right] < \infty,\ \ \varepsilon > 0. \tag{3.10}$$

Theorem 1. *Under the condition (3.10), the operator L has a finite number of eigenvalues and spectral singularities, and each of them is of finite multiplicity.*

4. Principal Functions of L

Under the condition (3.10), let $\lambda_1,...,\lambda_j$ and $\lambda_{j+1},...,\lambda_k$ denote the zeros H^+ in $\mathbb{C}_+$ and H^- in $\mathbb{C}_-$ (which are the eigenvalues of L) with multiplicities $m_1,...,m_j$ and $m_{j+1},...,m_k$, respectively. It is obvious that from the definition of the Wronskian

$$\left\{\frac{d^n}{d\lambda^n}W\left[K^+(x,\lambda),E(x,\lambda)\right]\right\}_{\lambda=\lambda_p}=\left\{\frac{d^n}{d\lambda^n}D_+(\lambda)\right\}_{\lambda=\lambda_p}=0 \tag{4.1}$$

for $n=0,1,...,m_p-1,\ p=1,2,...,j$, and

$$\left\{\frac{d^n}{d\lambda^n}W\left[K^-(x,\lambda),E(x,-\lambda)\right]\right\}_{\lambda=\lambda_p}=\left\{\frac{d^n}{d\lambda^n}D_-(\lambda)\right\}_{\lambda=\lambda_p}=0 \tag{4.2}$$

for $n=0,1,...,m_p-1,\ p=j+1,...,k$.

Theorem 2. *The following formulae:*

$$\left\{\frac{\partial^n}{\partial\lambda^n}K^+(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{m=0}^{n}F_m(\lambda_p)\left\{\frac{\partial^m}{\partial\lambda^m}E(x,\lambda)\right\}_{\lambda=\lambda_p}, \tag{4.3}$$

$n=0,1,...,m_p-1,\ p=1,2,...,j$, *where*

$$F_m(\lambda_p)=\binom{n}{m}\left\{\frac{\partial^{n-m}}{\partial\lambda^{n-m}}\hat{D}_+(\lambda)\right\}_{\lambda=\lambda_p}, \tag{4.4}$$

$$\left\{\frac{\partial^n}{\partial\lambda^n}K^-(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{m=0}^{n}N_m(\lambda_p)\left\{\frac{\partial^m}{\partial\lambda^m}E(x,-\lambda)\right\}_{\lambda=\lambda_p}, \tag{4.5}$$

$n=0,1,...,m_p-1,\ p=j+1,...,k$, *where*

$$N_m(\lambda_p)=\binom{n}{m}\left\{\frac{\partial^{n-m}}{\partial\lambda^{n-m}}\hat{D}_-(\lambda)\right\}_{\lambda=\lambda_p} \tag{4.6}$$

hold.

Proof. We will prove only (4.3) using the method of induction, because the case of (4.5) is similar. Let be $n=0$. Since $K^+(x,\lambda)$ and $E(x,\lambda)$ are linearly dependent from (4.1), we get

$$K^+(x,\lambda_p)=f_0(\lambda_p)E(x,\lambda_p), \tag{4.7}$$

where $f_0(\lambda_p)\neq 0$. Let us assume that $1\le n_0\le m_p-2$, (4.7) holds; that is,

$$\left\{\frac{\partial^{n_0}}{\partial\lambda^{n_0}}K^+(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{m=0}^{n_0}F_m(\lambda_p)\left\{\frac{\partial^m}{\partial\lambda^m}E(x,\lambda)\right\}_{\lambda=\lambda_p}. \tag{4.8}$$

We will prove that (4.3) holds for n_0+1. If $Y(x,\lambda)$ is a solution of (1.5), then $\frac{\partial^n}{\partial\lambda^n}Y(x,\lambda)$ satisfies

$$\left[-\frac{d^2}{dx^2}+Q(x)-\lambda^2\right]\frac{\partial^n}{\partial\lambda^n}Y(x,\lambda)=2\lambda n\frac{\partial^{n-1}}{\partial\lambda^{n-1}}Y(x,\lambda)+n(n-1)\frac{\partial^{n-2}}{\partial\lambda^{n-2}}Y(x,\lambda). \tag{4.9}$$

Writing (4.9) for $K^+(x,\lambda)$ and $E(x,\lambda)$, and using (4.8), we find

$$\left[-\frac{d^2}{dx^2}+Q(x)-\lambda^2\right]g_{n_0+1}(x,\lambda_p)=0, \tag{4.10}$$

where

$$g_{n_0+1}(x,\lambda_p)=\left\{\frac{\partial^{n_0+1}}{\partial\lambda^{n_0+1}}K^+(x,\lambda)\right\}_{\lambda=\lambda_p}-\sum_{m=0}^{n_0+1}F_m(\lambda_p)\left\{\frac{\partial^m}{\partial\lambda^m}E(x,\lambda)\right\}_{\lambda=\lambda_p}. \tag{4.11}$$

From (4.1), we have

$$W\left[g_{n_0+1}(x,\lambda_p),E(x,\lambda_p)\right]=\left\{\frac{d^{n_0+1}}{d\lambda^{n_0+1}}W\left[K^+(x,\lambda),E(x,\lambda)\right]\right\}_{\lambda=\lambda_p}=0. \tag{4.12}$$

Hence there exists a constant $f_{n_0+1}(\lambda_p)$ such that

$$g_{n_0+1}(x,\lambda_p)=f_{n_0+1}(\lambda_p)E(x,\lambda_p). \tag{4.13}$$

This shows that (4.3) holds for $n=n_0+1$. □

Using (4.3) and (4.5), define the functions

$$U_{n,p}(x)=\left\{\frac{\partial^n}{\partial\lambda^n}K^+(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{m=0}^{n}F_m(\lambda_p)\left\{\frac{\partial^m}{\partial\lambda^m}E(x,\lambda)\right\}_{\lambda=\lambda_p}, \tag{4.14}$$

$n=0,1,...,m_p-1,\ p=1,2,...,j$ and

$$U_{n,p}(x)=\left\{\frac{\partial^n}{\partial\lambda^n}K^-(x,\lambda)\right\}_{\lambda=\lambda_p}=\sum_{m=0}^{n}N_m(\lambda_p)\left\{\frac{\partial^m}{\partial\lambda^m}E(x,-\lambda)\right\}_{\lambda=\lambda_p}, \tag{4.15}$$

$n=0,1,...,m_p-1,\ p=j+1,...,k.$

Then for $\lambda=\lambda_p,\ p=1,2,...,j,j+1,...,k,$

$$\begin{aligned} l(U_{0,p})&=0,\\ l(U_{1,p})+\frac{1}{1!}\frac{\partial}{\partial\lambda}l(U_{0,p})&=0,\\ l(U_{n,p})+\frac{1}{1!}\frac{\partial}{\partial\lambda}l(U_{n-1,p})+\frac{1}{2!}\frac{\partial^2}{\partial\lambda^2}l(U_{n-2,p})&=0,\end{aligned} \tag{4.16}$$

$n = 2, 3, ..., m_p - 1,$
hold, where $l(u) = -u'' + Q(x)u - \lambda^2 u$ and $\frac{\partial^m}{\partial \lambda^m} l(u)$ denote the differential expressions whose coefficients are the m-th derivatives with respect to λ of the corresponding coefficients of the differential expression $l(u)$. (4.16) shows that $U_{0,p}$ is the eigenfunction corresponding to the eigenvalue $\lambda = \lambda_p$; $U_{1,p}, U_{2,p}, ...U_{m_p-1,p}$ are the associated functions of $U_{0,p}$ [16, 17].

$U_{0,p}, U_{1,p}, ...U_{m_p-1,p},\ p = 1, 2, ..., j, j+1, ..., k$ are called the principal functions corresponding to the eigenvalue $\lambda = \lambda_p,\ p = 1, 2, ..., j, j+1, ..., k$ of L.

Theorem 3.

$$U_{n,p} \in L_2(\mathbb{R}_+, E), \quad n = 0, 1, ...m_p - 1,\ p = 1, 2, ..., j, j+1, ..., k. \tag{4.17}$$

Proof. Let be $0 \leq n \leq m_p - 1$ and $1 \leq p \leq j$. Using (2.5) and (3.10), we obtain that

$$\|K(x,t)\| \leq c e^{-\sqrt{\frac{x+t}{2}}}. \tag{4.18}$$

From (2.3), we get

$$\left\| \left\{ \frac{\partial^n}{\partial \lambda^n} E(x,\lambda) \right\}_{\lambda=\lambda_p} \right\| \leq x^n e^{-x \operatorname{Im} \lambda_p} + c \int_x^\infty t^n e^{-\sqrt{\frac{x+t}{2}}} e^{-t \operatorname{Im} \lambda_p} dt, \tag{4.19}$$

where $c > 0$ is a constant. Since $\operatorname{Im} \lambda_p > 0$ for the eigenvalues $\lambda_p,\ p = 1, 2, ..., j,$ of L, implies that

$$\left\{ \frac{\partial^n}{\partial \lambda^n} E(x,\lambda) \right\}_{\lambda=\lambda_p} \in L_2(\mathbb{R}_+, E), \quad n = 0, 1, ...m_p - 1,\ p = 1, 2, ..., j. \tag{4.20}$$

So we get $U_{n,p} \in L_2(\mathbb{R}_+, E)$ from (4.14) and (4.20) Similarly we prove the results for $0 \leq n \leq m_p - 1,\ j+1 \leq p \leq k$. This completes the proof. □

Let $\mu_1, ..., \mu_v$ and $\mu_{v+1}, ..., \mu_l$ be the zeros of D_+ and D_- in $\mathbb{R}^*$ with multiplicities $n_1, ..., n_v$ and $n_{v+1}, ..., n_l$, respectively. We can show

$$\left\{ \frac{\partial^n}{\partial \lambda^n} K(x,\lambda) \right\}_{\lambda=\mu_p} = \sum_{m=0}^{n} C_m(\lambda_p) \left\{ \frac{\partial^m}{\partial \lambda^m} E(x,\lambda) \right\}_{\lambda=\mu_p}, \tag{4.21}$$

$n = 0, 1, ..., n_p - 1,\ p = 1, 2, ..., v,$ where

$$C_m(\mu_p) = -\binom{n}{m} \left\{ \frac{\partial^{n-m}}{\partial \lambda^{n-m}} D_-(\lambda) \right\}_{\lambda=\mu_p}, \tag{4.22}$$

$$\left\{ \frac{\partial^n}{\partial \lambda^n} K(x,\lambda) \right\}_{\lambda=\mu_p} = \sum_{m=0}^{n} R_m(\mu_p) \left\{ \frac{\partial^m}{\partial \lambda^m} E(x,-\lambda) \right\}_{\lambda=\mu_p},$$

$n = 0,1,...,n_p - 1,\ p = \upsilon + 1,...,l,$
where

$$R_m(\mu_p) = \binom{n}{m} \left\{ \frac{\partial^{n-m}}{\partial \lambda^{n-m}} D_+(\lambda) \right\}_{\lambda=\mu_p}. \tag{4.23}$$

Now define the generalized eigenfunctions and generalized associated functions corresponding to the spectral singularities of L by the following :

$$V_{n,p}(x) = \left\{ \frac{\partial^n}{\partial \lambda^n} K(x,\lambda) \right\}_{\lambda=\mu_p} = \sum_{m=0}^{n} C_m(\mu_p) \left\{ \frac{\partial^m}{\partial \lambda^m} E(x,\lambda) \right\}_{\lambda=\mu_p}, \tag{4.24}$$

$n = 0,1,...,n_p - 1,\ p = 1,2,...,\upsilon,$

$$V_{n,p}(x) = \left\{ \frac{\partial^n}{\partial \lambda^n} K(x,\lambda) \right\}_{\lambda=\mu_p} = \sum_{m=0}^{n} R_m(\mu_p) \left\{ \frac{\partial^m}{\partial \lambda^m} E(x,-\lambda) \right\}_{\lambda=\mu_p}, \tag{4.25}$$

$n = 0,1,...,n_p - 1,\ p = \upsilon + 1,...,l.$

Then $V_{n,p}$, $n = 0,1,...,n_p - 1,\ p = 1,2,...,\upsilon,\upsilon + 1,...,l$, also satisfy the equations analogous to (4.16).

$V_{0,p}, V_{1,p},..., V_{n_p-1,p},\ p = 1,2,...,\upsilon,\upsilon+1,...,l$ are called the principal functions corresponding to the spectral singularities $\lambda = \mu_p, p = 1,2,...,\upsilon,\upsilon+1,...,l$ of L.

Theorem 4.

$$V_{n,p} \notin L_2(\mathbb{R}_+, E), \quad n = 0,1,...n_p - 1,\ p = 1,2,...,\upsilon,\upsilon+1,...,l.$$

Proof. For $0 \le n \le n_p - 1$ and $1 \le p \le \upsilon$ using (2.3), we obtain

$$\left\| \left\{ \frac{\partial^n}{\partial \lambda^n} E(x,\lambda) \right\}_{\lambda=\mu_p} \right\| \le \left\| (ix)^n e^{i\mu_p x} I + \int_x^\infty (it)^n K(x,t) e^{i\mu_p t} dt \right\|$$

since $\operatorname{Im} \mu_p = 0,\ p = 1,2,...,\upsilon$, we find that

$$\int_0^\infty \left\| (ix)^n e^{i\mu_p x} I \right\|^2 dx = \int_0^\infty x^{2n} = \infty.$$

So we obtain $V_{n,p} \notin L_2(\mathbb{R}_+, E)$, $n = 0,1,...n_p - 1,\ p = 1,2,...,\upsilon$. Using the similar way, we may also prove the results for $0 \le n \le n_p - 1,\ \upsilon + 1 \le p \le l$. □

Now define the Hilbert spaces of vector-valued functions with values in E by

$$H_n := \left\{ f : \int_0^\infty (1 + |x|)^{2n} \|f(x)\|^2 dx < \infty \right\}, \quad n = 1,2,..., \tag{4.26}$$

$$H_{-n} := \left\{ g : \int_0^\infty (1+|x|)^{-2n} \|g(x)\|^2 \, dx < \infty \right\}, \quad n = 1,2,..., \tag{4.27}$$

with the norms

$$\|f\|_n^2 := \int_0^\infty (1+|x|)^{2n} \|f(x)\|^2 \, dx,$$

and

$$\|g\|_{-n}^2 := \int_0^\infty (1+|x|)^{-2n} \|g(x)\|^2 \, dx,$$

respectively. Then

$$H_{n+1} \subsetneqq H_n \subsetneqq L_2(\mathbb{R}_+, E) \subsetneqq H_{-n} \subsetneqq H_{-(n+1)}, \quad n = 1,2,..., \tag{4.28}$$

and H_{-n} is isomorphic to the dual of H_n.

Theorem 5.

$$V_{n,p} \in H_{-(n+1)}, \quad n = 0,1,...n_p - 1, \; p = 1,2,...,v,v+1,...,l.$$

Proof. For $0 \le n \le n_p - 1$ and $1 \le p \le v$ using (2.3) and (4.24), we get

$$\int_0^\infty (1+|x|)^{-2(n+1)} \left\| V_{n,p} \right\|^2 dx$$

$$\le M \int_0^\infty (1+|x|)^{-2(n+1)} \left\{ \left\| \{E(x,\lambda)\}_{\lambda=\mu_p} \right\|^2 + ... + \left\| \left\{ \frac{\partial^n}{\partial \lambda^n} E(x,\lambda) \right\}_{\lambda=\mu_p} \right\|^2 \right\}$$

where $M > 0$ is a constant. Using (2.3), we have

$$\int_0^\infty (1+|x|)^{-2(n+1)} \left\| (ix)^n \, e^{i\mu_p x} I \right\|^2 dx < \infty,$$

and

$$\int_0^\infty (1+|x|)^{-2(n+1)} \left\| \int_x^\infty (it)^n \, K(x,t) e^{i\mu_p t} dt \right\|^2 dx < \infty.$$

Consequently $V_{n,p} \in H_{-(n+1)}$ for $0 \le n \le n_p - 1$ and $1 \le p \le v$. Similarly, we obtain $V_{n,p} \in H_{-(n+1)}$ for $0 \le n \le n_p - 1$ and $v+1 \le p \le l$. □

Let us choose

$$n_0 = \max\{n_1, n_2, ..., n_v, n_{v+1}, ..., n_l\}.$$

By (4.28), we get the following

Theorem 6.

$$V_{n,p} \in H_{-n_0}, \quad n = 0, 1, ...n_p - 1, p = 1, 2, ..., v, v+1, ..., l.$$

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