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SOME ITERATED CONVERGENCE AND FIXED POINT THEOREMS IN REAL LINEAR n-NORMED SPACES

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Abstract. In this paper, we prove some iterated convergence theorems to fixed points of self mappings satisfying Z-type conditions and Z-operators by considering the base space as a real linear $n(>1)$-normed space. We also establish some common fixed point theorems for quadruple of self mappings satisfying certain weak conditions and Φ- contraction. The paper also demonstrates a way to construct convergence and fixed point theorems in real linear n-normed spaces as base space.

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1. INTRODUCTION AND PRELIMINARIES

In mathematics, a fixed point (also known as an invariant point) of a function is a point that is mapped to itself by the function. In many fields, equilibria or stability are fundamental concepts that can be described in terms of fixed points. Many applications of fixed point theorems can be found both on the theoretical side and on the applied side.

In the last three decades many papers have been published on the iterative approximation of fixed points for certain classes of operators, by using Picard, Krasnoselskij, Mann and Ishikawa iteration methods. They do basically differ due to their speed of convergence depending on the position of parameters involved.

The concept of 2-normed spaces was initially developed by Gähler [11] in the mid of 1960's, while that of n-normed spaces can be found in Misiak [19]. Since then, many others have studied this concept and obtained various results; see for instance Gunawan and Mashadi [12], Dutta [8, 9], and Dutta, Reddy and Cheng [10] etc.

Let $n \in N$ and X be a real vector space of dimension d, where $n \leq d$. A real-valued function $||.,\ldots,.||$on X^n satisfying the following four conditions:

(N1) $||x_1, x_2, \ldots, x_n|| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent,

(N2) $||x_1, x_2, \ldots, x_n||$is invariant under permutation,

(N3) $||\alpha x_1, x_2, \ldots, x_n|| = |\alpha| ||x_1, x_2, \ldots, x_n||$, for any $\alpha \in R$, (N4) $||x+x^{/}, x_2, \ldots, x_n|| \leq ||x, x_2, \ldots, x_n|| + ||x^{/}, x_2, \ldots, x_n||$ is called an *n-norm* on X, and the pair (X, $||.,\ldots,.||$) is called an n-normed space.

A trivial example of an n-normed space is X=R^n equipped with the following Euclidean n-norm:

$$||x_1,x_2,\ldots,x_n||_E = \text{abs}\left(\begin{vmatrix} x_{11} \cdots x_{1n} \\ \vdots \ \ddots \ \vdots \\ x_{n1} \cdots x_{nn} \end{vmatrix}\right),$$ where $x_i=(x_{i1},\ldots,x_{in}) \in R^n$ for each i=1,2,....,n.

If (X,$||.,\ldots,.||$) be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \ldots, a_n\}$ be a linearly independent set in X. Then the following function $||.,\ldots,.||_\infty$ on X^{n-1} defined by

$||x_1,x_2,\ldots,x_{n-1}||_\infty = \max\{||x_1,x_2,\ldots,x_{n-1},a_i||: i = 1, 2, \ldots, n\}$ defines an (n-1) norm on X with respect to $\{a_1, a_2, \ldots, a_n\}$.

The standard n- norm on X, a real inner product space of dimension $d = n$ is as follows:

$$||x_1,\ldots,x_n||_S = \begin{vmatrix} \langle x_1,x_1\rangle \cdots \langle x_1,x_n\rangle \\ \vdots \ \ \ddots \ \ \vdots \\ \langle x_n,x_1\rangle \cdots \langle x_n,x_n\rangle \end{vmatrix}^{\frac{1}{2}},$$ where $\langle .,. \rangle$ denotes the inner product on X. If X=R^n, then this n-norm is exactly the same as the Euclidean n-norm $||x_1,x_2,\ldots,x_n||_E$ mentioned earlier. For $n = 1$, this n-norm is the usual norm $||x_1|| = \langle x_1,x_1\rangle^{\frac{1}{2}}$.

A sequence (x_k) in an n-normed space (X, $||.,\ldots,.||$) is said to *converge* to some $L \in X$ in the n-norm if

$\lim_{k\to\infty} \|x_k - L, u_2, \ldots\ldots, u_n\| = 0$, for every $u_2, \ldots, u_n \in X$.

A sequence (x_k) in an n-normed space (X, $||.,\ldots,.||$) is said to be *Cauchy* with respect to the n-norm if

$\lim_{k,l\to\infty} \|x_k - x_l, u_2, \ldots\ldots, u_n\| = 0$, for every $u_2, \ldots, u_n \in X$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

If a term in the definition of n-norm represents the change of shape, and the n-norm stands for the associated area or center of gravity of the term, may be we can think of some applications of the notion of n-norm, and then the generalized convergence make sense. This is considered as the main issue in the use of n-normed structures here. Keeping this in mind, we consider real linear n-normed space as the base space.

The following iterative scheme was introduced by Mann [18] in 1953.

Let K be a non empty, closed, convex subset of a normed linear space E and $T : K \to K$ be a self map. For $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, n = 0,1,2,..., \tag{1.1}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a real sequence in $[0,1]$, is called the Mann iteration process. Many converging theorems and approximation results have been proved using the Mann iteration process. A two-step iteration process was introduced by Ishikawa [13] in 1974 as defined below:

For $x_0 \in K$, the sequence $\{x_n\}_{n=0}^{\infty}$ is given by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, y_n = (1-\beta_n)x_n + \beta_n T x_n, n = 0,1,2,..., \quad (1.2)$$

where K be a non empty, closed, convex subset of a normed linear space E and $T : K \to K$ be a self map and $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0,1]$. This iteration process is called as Ishikawa iteration process. When $\beta_n = 0$, this iteration process reduces to Mann iteration scheme given by (1.1).

We procure the following definitions in a metric space (X,d):

A mapping $T : X \to X$ is called an $a-$contraction if there exists $a \in [0,1)$ such that

(z_1) $d(Tx,Ty) \leq ad(x,y)$, for all $x,y \in X$.

The map T is called a Kannan mapping [16] if there exists $b \in [0,\frac{1}{2})$ such that

(z_2) $d(Tx,Ty) \leq b[d(x,Tx) + d(y,Ty)]$, for all $x,y \in X$.

A similar definition is due to Chatterjea [7] if there exists $c \in [0,\frac{1}{2})$ such that (z_3) $d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)]$, for all $x,y \in X$.

The conditions (z_1), (z_2) and (z_3) are independent contractive conditions [24].

Combining these three definitions, Zamfirescu [27] obtained the following important fixed point theorem in 1972.

Theorem 1. *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping for which there exist real numbers a,b and c with $a \in [0,1)$, $b \in [0,\frac{1}{2})$and $c \in [0,\frac{1}{2})$ such that for all $x,y \in X$ at least one of the following conditions holds:*

$$(z_1) : d(Tx,Ty) \leq ad(x,y)$$

$$(z_2) : d(Tx,Ty) \leq b[d(x,Tx) + d(y,Ty)]$$

$$(z_3) : d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)] \quad (1.3)$$

Then T has a unique fixed point x^* and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, n \in N$$

converges to x^* for any arbitrary $x_0 \in X$.

An operator T which satisfies at least one of the contractive conditions (z_1), (z_2) and (z_3) is called a Zamfirescu operator (Z-operator).

In 2004, Berinde [2] proved the strong convergence of Ishikawa iterative process given by (1.2) to approximate fixed points of Zamfirescu operators in an arbitrary Banach space E. In proving the theorem, he made use of the conditions,

$$\|Tx - Ty\| \leq \delta \|x,y\| + 2\delta \|x - Tx\| \quad (1.4)$$

which holds for any $x,y \in E$, where $0 \leq \delta < 1$.

In this paper, we define the following condition in order to establish our results of this paper in real linear n-normed spaces:

Let K be a non empty, closed, convex subset of a real linear n- normed space E and $T : K \to K$ be a self map. There exists a constant $L \geq 0$ such that for all $x, y, z_2, ..., z_n \in K$, we have

$$\|Tx - Ty, z_2, ..., z_n\| \leq e^{L\|x-Tx,z_2,...,z_n\|} (2\delta \|x - Tx, z_2, ..., z_n\| + \delta \|x - y, z_2, ..., z_n\|) \quad (1.5)$$

where $0 \leq \delta < 1$ and e^x denotes the exponential function of $x \in K$. This condition will be recalled in the paper as generalized Z-type condition.

The above condition is similar to a condition introduced in [5] by generalizing (1.4) in normed linear spaces.

If $L = 0$, in the above condition, we obtain

$$\|Tx - Ty, z_2, ..., z_n\| \leq \delta \|x - y, z_2, ..., z_n\| + 2\delta \|x - Tx, z_2, ..., z_n\|$$

which is an n-norm extension of the Zamfirescu condition used by Berinde [3], where

$$\delta = \max\left\{a, \frac{b}{1-b}, \frac{c}{1-c}\right\}, 0 \leq \delta < 1,$$

where constants a, b and c are as defined in Theorem 1.

In a similar fashion, we can extend the Theorem 1 to real liner n-normed spaces by replacing the base space (metric spaces). For some relevant results in normed linear spaces, we refer to [5, 23], [26], etc. We extend some such results in the next section of this paper.

We shall use the following lemma in proving some results of this paper.

Lemma 1 (Berinde [4])**.** *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$ be sequences of non-negative numbers satisfying $a_{n+1} \leq (1 - w_n)a_n + b_n + c_n$, for all $n \geq 0$, where $\{w_n\}_{n=0}^{\infty} \subset [0, 1]$.*
If $\sum_{n=0}^{\infty} w_n = \infty$, $b_n = O(w_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n = 0$.

2. Convergence theorems

In this section, we shall prove some convergence theorems to fixed points. Throughout this section, K is taken as a non empty, closed, convex subset of an n-normed linear space E.

Theorem 2. *Let $T : K \to K$ and $S : K \to K$ be two self mappings with $F_T \bigcap F_S \neq \phi$ where F_T and F_S are the sets of fixed points of T and S respectively satisfying generalized Z - type conditions given by (1.5). For arbitrary $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by generalized Mann type iteration scheme (introduced by Owojori and Imoru [20])*

$$x_{n+1} = a_n x_n + b_n T x_n + c_n S x_n, n = 0, 1, 2, ...,$$

where $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are real sequences in* $[0,1]$ *with* $a_n+b_n+c_n=1$ *and* $b_n+c_n=\alpha_n$ *and* $\sum_{n=0}^{\infty}\alpha_n=\infty$*. Then* $\{x_n\}_{n=0}^{\infty}$ *converges to a common fixed point of* T *and* S*.*

Proof. Since $F_T\bigcap F_S\neq\phi$, let us take $x^*\in K$ be a common fixed point of T and S. Since T and S satisfy generalized Z-type condition given by (1.5), we have that for all $x,y,z_2,...,z_n\in K$, the following inequalities hold

$$\|Tx-Ty,z_2,...,z_n\| \leq e^{L\|x-Tx,z_2,...,z_n\|}\left(2\delta\|x-Tx,z_2,...,z_n\|+\delta\|x-y,z_2,...,z_n\|\right) \tag{2.1}$$

and

$$\|Sx-Sy,z_2,...,z_n\| \leq e^{L\|x-Tx,z_2,...,z_n\|}\left(2\delta\|x-Sx,z_2,...,z_n\|+\delta\|x-y,z_2,...,z_n\|\right) \tag{2.2}$$

where $L\geq 0$, $0\leq\delta<1$ and $\delta=\max\left\{a,\frac{b}{1-b},\frac{c}{1-c}\right\}$.

Now let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by generalized Mann type iteration scheme and $x_0\in K$ be arbitrary. Then for all $z_2,...,z_n\in K$

$$\begin{aligned}
\left\|x_{n+1}-x^*,z_2,...,z_n\right\| &= \left\|a_nx_n+b_nTx_n+c_nSx_n-x^*,z_2,...,z_n\right\| \\
&= \left\|(1-\alpha_n)x_n+b_nTx_n+c_nSx_n-(a_n+b_n+c_n)x^*,z_2,...,z_n\right\| \\
&= \left\|(1-\alpha_n)x_n+b_nTx_n+c_nSx_n-(1-\alpha_n)x^*-b_nx^*-c_nx^*,z_2,...,z_n\right\| \\
&= \left\|(1-\alpha_n)(x_n-x^*)+b_n(Tx_n-x^*)+c_n(Sx_n-x^*),z_2,...,z_n\right\| \\
&\leq (1-\alpha_n)\left\|x_n-x^*,z_2,...,z_n\right\|+b_n\left\|Tx_n-x^*,z_2,...,z_n\right\| \\
&\quad +c_n\left\|Sx_n-x^*,z_2,...,z_n\right\|
\end{aligned} \tag{2.3}$$

Taking $x=x^*$ and $y=x_n$ in (2.1), we get

$$\begin{aligned}
&\left\|Tx^*-Tx_n,z_2,...,z_n\right\| \\
&\leq e^{L\|x^*-Tx^*,z_2,...,z_n\|}\left(2\delta\left\|x^*-Tx^*,z_2,...,z_n\right\|+\delta\left\|x^*-x_n,z_2,...,z_n\right\|\right) \\
&= e^{L\|x^*-x^*,z_2,...,z_n\|}\left(2\delta\left\|x^*-x^*,z_2,...,z_n\right\|+\delta\left\|x^*-x_n,z_2,...,z_n\right\|\right) \\
&= e^{L(0)}\left(2\delta(0)+\delta\left\|x^*-x_n,z_2,...,z_n\right\|\right),
\end{aligned}$$

This implies that

$$\left\|Tx_n-x^*,z_2,...,z_n\right\|\leq\delta\left\|x_n-x^*,z_2,...,z_n\right\| \tag{2.4}$$

Similarly by taking $x=x^*$ and $y=x_n$ in (2.2), we obtain

$$\left\|Sx_n-x^*,z_2,...,z_n\right\|\leq\delta\left\|x_n-x^*,z_2,...,z_n\right\| \tag{2.5}$$

Now using (2.4) and (2.5) in (2.3), we have

$$\begin{aligned}
\left\|x_{n+1}-x^*,z_2,...,z_n\right\| &\leq (1-\alpha_n)\left\|x_n-x^*,z_2,...,z_n\right\| \\
&+b_n\delta\left\|x_n-x^*,z_2,...,z_n\right\|+c_n\delta\left\|cx_n-x^*,z_2,...,z_n\right\| \\
&= (1-\alpha_n+b_n\delta+c_n\delta)\left\|x_n-x^*,z_2,...,z_n\right\|
\end{aligned}$$

$$= (1-\alpha_n+\alpha_n\delta)\left\|x_n-x^*,z_2,...,z_n\right\|$$

Thus we have,

$$\left\|x_{n+1}-x^*,z_2,...,z_n\right\| \leq [1-\alpha_n(1-\delta)]\left\|x_n-x^*,z_2,...,z_n\right\|, n=0,1,2,...$$

Since $\{\alpha_n\}$ and δ satisfy the conditions of Lemma 1, we have our expected condition $\lim_{n\to\infty}\|x_{n+1}-x^*,z_2,...,z_n\| = 0$, for every $z_2,...,z_n \in K$.

Thus $\{x_n\}_{n=0}^{\infty}$ converges to x^* which is the common fixed point of T and S. □

Corollary 1. *Let $T : K \to K$ and $S : K \to K$ be two Z-operators with $F_T \bigcap F_S \neq \phi$ where F_T and F_S are the sets of fixed points of T and S respectively. For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence as defined in the statement of Theorem 2. Then $\{x_n\}_{n=0}^{\infty}$ converges to a common fixed point of T and S.*

Theorem 3. *Let $T : K \to K$ be a self maps satisfying generalized Z-type condition given by (1.5) with $F_T \neq \phi$, where F_T is the set of fixed points of T. For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the following two step iteration scheme (introduced by Thianwan [25])*

$$x_{n+1} = (1-\alpha_n)y_n+\alpha_n Ty_n,$$
$$y_n = (1-\beta_n)x_n+\beta_n Tx_n, n=0,1,2,...,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0,1]$ satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges to the unique fixed point of T.

Proof. The assumption $F_T \neq \phi$ guarantees that T has a fixed point in K, say x^*. Since T satisfies generalized Z-type condition given by (1.5), we have the inequality (2.1) holds for all $x,y,z_2,...,z_n \in K$. Now let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated as given in the statement and $x_0 \in K$ be arbitrary. Then for all $z_2,...,z_n \in K$,

$$\begin{aligned}\left\|x_{n+1}-x^*,z_2,...,z_n\right\| &= \left\|(1-\alpha_n)y_n+\alpha_n Ty_n-x^*,z_2,...,z_n\right\|\\ &= \left\|(1-\alpha_n)(y_n-x^*)+\alpha_n(Ty_n-x^*),z_2,...,z_n\right\|\\ &\leq (1-\alpha_n)\left\|y_n-x^*,z_2,...,z_n\right\|+\alpha_n\left\|Ty_n-x^*,z_2,...,z_n\right\|\\ &= (1-\alpha_n)\left\|y_n-x^*,z_2,...,z_n\right\|+\alpha_n\left\|Tx^*-Ty_n,z_2,...,z_n\right\|\end{aligned}$$

Considering (2.1) by taking $x = x^*$ and $y = x_n$, the above inequality becomes

$$\left\|x_{n+1}-x^*,z_2,...,z_n\right\| \leq (1-\alpha_n)\left\|y_n-x^*,z_2,...,z_n\right\|$$
$$+\alpha_n\left[e^{L\|x^*-Tx^*,z_2,...,z_n\|}\left(2\delta\left\|x^*-Tx^*,z_2,...,z_n\right\|+\delta\left\|x^*-y_n,z_2,...,z_n\right\|\right)\right]$$
$$= (1-\alpha_n)\left\|y_n-x^*,z_2,...,z_n\right\|$$
$$+\alpha_n\left[e^{L\|x^*-x^*,z_2,...,z_n\|}\left(2\delta\left\|x^*-x^*,z_2,...,z_n\right\|+\delta\left\|x^*-y_n,z_2,...,z_n\right\|\right)\right]$$
$$= (1-\alpha_n)\left\|y_n-x^*,z_2,...,z_n\right\|+\alpha_n\left[e^{L(0)}\left(2\delta(0)+\delta\left\|x^*-y_n,z_2,...,z_n\right\|\right)\right]$$
$$= (1-\alpha_n)\left\|y_n-x^*,z\right\|+\alpha_n\delta\left\|y_n-x^*\right\|,$$

So,

$$\left\|x_{n+1}-x^*,z_2,...,z_n\right\| \leq (1-\alpha_n+\alpha_n\delta)\left\|y_n-x^*,z_2,...,z_n\right\| \tag{2.6}$$

We have

$$\left\|y_n-x^*,z_2,...,z_n\right\| = \left\|(1-\beta_n)x_n+\beta_n Tx_n-x^*,z_2,...,z_n\right\|$$
$$= \left\|(1-\beta_n)(x_n-x^*)+\beta_n(Tx_n-x^*),z_2,...,z_n\right\|$$
$$\leq (1-\beta_n)\left\|x_n-x^*,z_2,...,z_n\right\| + \beta_n\left\|Tx_n-x^*,z_2,...,z_n\right\| \tag{2.7}$$

Using (2.7) and then (2.4) in (2.6), we get

$$\left\|x_{n+1}-x^*,z_2,...,z_n\right\| \leq (1-\alpha_n+\alpha_n\delta)\left\|y_n-x^*,z_2,...,z_n\right\|$$
$$\leq (1-\alpha_n+\alpha_n\delta)\left[(1-\beta_n)\left\|x_n-x^*,z_2,...,z_n\right\| + \beta_n\left\|Tx_n-x^*,z_2,...,z_n\right\|\right]$$
$$\leq (1-\alpha_n+\alpha_n\delta)\left[(1-\beta_n)\left\|x_n-x^*,z_2,...,z_n\right\| + \beta_n\delta\left\|x_n-x^*,z_2,...,z_n\right\|\right]$$
$$= (1-\alpha_n+\alpha_n\delta)(1-\beta_n+\beta_n\delta)\left\|x_n-x^*,z\right\|.$$

Since,

$$(1-\alpha_n+\alpha_n\delta)(1-\beta_n+\beta_n\delta) = 1-\alpha_n(1-\delta)-\beta_n(1-\delta)[1-\alpha_n(1-\delta)]$$
$$\leq 1-\alpha_n(1-\delta),$$

we have the inequality

$$\left\|x_{n+1}-x^*,z_2,...,z_n\right\| \leq [1-\alpha_n(1-\delta)]\left\|x_n-x^*,z_2,...,z_n\right\|, n=0,1,2,...$$

Since $0 \leq \delta < 1$, $\alpha_n \in [0,1]$ and $\sum_{n=0}^{\infty}\alpha_n = \infty$, setting $a_n = \|x_n-x^*,z_2,...,z_n\|$ (for arbitrarily fixed $z_2,...,z_n \in K$), $w_n = \alpha_n(1-\delta)$ and applying Lemma 1, we have $\lim_{n\to\infty}\|x_{n+1}-x^*,z_2,...,z_n\| = 0$, for every $z_2,...,z_n \in K$.

Thus it follows that $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point of T.

To show the uniqueness of fixed point, let us assume that x_1^* and x_2^* be two distinct fixed points of T. Applying generalized Z-type condition given by (1.5) and using the fact that $0 \leq \delta < 1$, we obtain for every $z_2,...,z_n \in K$:

$$\left\|x_1^*-x_2^*,z_2,...,z_n\right\| = \left\|Tx_1^*-Tx_2^*,z_2,...,z_n\right\|$$
$$\leq e^{L\left\|x_1^*-Tx_1^*,z_2,...,z_n\right\|}\left(2\delta\left\|x_1^*-Tx_1^*,z_2,...,z_n\right\| + \delta\left\|x_1^*-x_2^*,z_2,...,z_n\right\|\right)$$
$$= e^{L\left\|x_1^*-x_1^*,z_2,...,z_n\right\|}\left(2\delta\left\|x_1^*-x_1^*,z_2,...,z_n\right\| + \delta\left\|x_1^*-x_2^*,z_2,...,z_n\right\|\right)$$
$$= e^{L(0)}\left(2\delta(0) + \delta\left\|x_1^*-x_2^*,z_2,...,z_n\right\|\right) = \delta\left\|x_1^*-x_2^*,z_2,...,z_n\right\|$$
$$< \left\|x_1^*-x_2^*,z_2,...,z_n\right\|,$$

a contradiction.

Therefore $x_1^* = x_2^*$. Thus $\{x_n\}_{n=0}^{\infty}$ converges to the unique fixed point of T. □

Corollary 2. *Let E be an arbitrary n-Banach space and $T : K \to K$ a Z-operator. For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be defined by two-step iteration process as in the statement of Theorem 3 with $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0,1]$ satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges to the unique fixed point of T.*

Theorem 4. *Let $T : K \to K$ and $S : K \to K$ be two self mappings of K satisfying generalized Z-type condition given by (1.5) with $F_T \bigcap F_S \neq \phi$ where F_T and F_S are the sets of fixed points of T and S respectively. For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence defined by following two-step iteration scheme (introduced by Raphel and Pulickakunnel [23])*

$$x_{n+1} = (1-\alpha_n)y_n + \alpha_n S y_n,$$
$$y_n = (1-\beta_n)x_n + \beta_n T x_n, n = 0,1,2,...,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0,1]$ satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges to the common fixed point of T and S.

Proof. Since S satisfies generalized Z-type condition given by (1.5), we get that the inequality (2.2) holds for all $x, y, z_2, ..., z_n \in K$. As $F_T \bigcap F_S \neq \phi$, let x^* be the common fixed point of S and T in K. Now for every $z_2, ..., z_n \in K$,

$$\begin{aligned}
&\left\|x_{n+1} - x^*, z_2, ..., z_n\right\| = \left\|(1-\alpha_n)y_n + \alpha_n S y_n - x^*, z_2, ..., z_n\right\| \\
&= \left\|(1-\alpha_n)(y_n - x^*) + \alpha_n(S y_n - x^*), z_2, ..., z_n\right\| \\
&\leq (1-\alpha_n)\left\|y_n - x^*, z_2, ..., z_n\right\| + \alpha_n\left\|S y_n - x^*, z_2, ..., z_n\right\| \\
&= (1-\alpha_n)\left\|y_n - x^*, z_2, ..., z_n\right\| + \alpha_n\left\|S x^* - S y_n, z_2, ..., z_n\right\|.
\end{aligned}$$

Taking $x = x^*$ and $y = y_n$ in (2.2), we obtain

$$\begin{aligned}
&\left\|x_{n+1} - x^*, z_2, ..., z_n\right\| \leq (1-\alpha_n)\left\|y_n - x^*, z_2, ..., z_n\right\| \\
&\quad + \alpha_n\left[e^{L\|x^* - Sx^*, z_2, ..., z_n\|}\left(2\delta\left\|x^* - Sx^*, z_2, ..., z_n\right\| + \delta\left\|x^* - y_n, z_2, ..., z_n\right\|\right)\right] \\
&= (1-\alpha_n)\left\|y_n - x^*, z_2, ..., z_n\right\| \\
&\quad + \alpha_n\left[e^{L\|x^* - x^*, z_2, ..., z_n\|}\left(2\delta\left\|x^* - x^*, z_2, ..., z_n\right\| + \delta\left\|x^* - y_n, z_2, ..., z_n\right\|\right)\right] \\
&= (1-\alpha_n)\left\|y_n - x^*, z_2, ..., z_n\right\| + \alpha_n\left[e^{L(0)}\left(2\delta(0) + \delta\left\|x^* - y_n, z_2, ..., z_n\right\|\right)\right] \\
&= (1-\alpha_n)\left\|y_n - x^*, z_2, ..., z_n\right\| + \alpha_n\delta\left\|y_n - x^*, z_2, ..., z_n\right\|.
\end{aligned}$$

Thus,

$$\left\|x_{n+1} - x^*, z_2, ..., z_n\right\| \leq (1 - \alpha_n + \alpha_n\delta)\left\|y_n - x^*, z_2, ..., z_n\right\|.$$

Now, applying iteration process as given in the statement and proceeding similarly as in the proof of Theorem 3, we obtain

$\lim_{n\to\infty}\|x_{n+1} - x^*, z_2, ..., z_n\| = 0$, for every $z_2, ..., z_n \in K$.

This completes the proof. □

Corollary 3. *Let $T : K \to K$ and $S : K \to K$ be two Z-operators with $F_T \bigcap F_S \neq \phi$ where F_T and F_S are the sets of fixed points of T and S respectively. For any $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence as defined in the statement of Theorem 4 with $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of positive numbers in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$ converges to a common fixed point of T and S.*

3. Fixed point theorems

In main aim of this section is to prove some common fixed point theorems for a quadruple of weak compatible self mappings on a real linear n-normed space satisfying a common $(E.A)$ property and a generalized Φ- contraction.

Definition 1. Two self mappings A and S of an n-normed space $(X, ||.,\ldots,||)$ are called compatible if $\lim_{m\to\infty} \|ASx_m, -SAx_m, z_2, ..., z_n\| = 0$ (for every z_2 ,...,$z_n \in X$), whenever $\{x_m\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_m = \lim_{n\to\infty} Sx_m = t$, for some t in X.

In 1986, the notion of compatible mappings which generalized commuting mappings, was introduced by Jungck [14]. This has proven useful for generalization of results in metric fixed point theory for single-valued as well as multi-valued mappings. Further in 1998, the more general class of mappings called weakly compatible mappings was introduced by Jungck and Rhoades [15]. Recall that self mappings S and T of a metric space (X,d) are called weakly compatible if $Sx = Tx$ for some $x \in X$ implies that $STx = TSx$. i.e., weakly compatible pair commute at coincidence points. For some other results on fixed point theory, one may refer to [21, 22], etc.

Aamri and Moutawakil [1] introduced the following notion of *the property* $(E.A)$ for a pair of self maps in metric spaces. We define this notion for real linear n-normed spaces as follows:

Definition 2. Two self mappings S and T on a real linear n-normed space $(X, ||.,\ldots,||)$ are said to satisfy the property $(E.A)$, if there exists a sequence $\{x_m\}$ in X such that for every z_2 ,...,$z_n \in X$ $\lim_{m\to\infty} \|Tx_m, z_2, ..., z_n\| = \lim_{m\to\infty} \|Sx_m, z_2, ..., z_n\|$ = t, for some $t \in X$.

Definition 3. Let $A, B, S, T : (X, \|.,...,.\|) \to (X, \|.,...,.\|)$. The pairs (A,S) and (B,T) are said to satisfy a common property $(E.A)$, if there exist two sequences $\{x_m\}$ and $\{y_m\}$ such that for every z_2 ,...,$z_n \in X$

$$\lim_{m\to\infty} \|Ax_m, z_2, ..., z_n\| = \lim_{m\to\infty} \|Sx_m, z_2, ..., z_n\| = \lim_{m\to\infty} \|By_m, z_2, ..., z_n\|$$

$$= \lim_{m\to\infty} \|Ty_m, z_2, ..., z_n\| = t \in X.$$

This definition reduces to the previous one, if we take, $B = A$ and $S = T$. Further this definition is given by Liu, Wu and Liu [17] for metric spaces.

We denote by Φ the collection of all functions $\varphi : [0,\infty) \to [0,\infty)$ which are upper semi-continuous from the right, non-decreasing and satisfy $\lim_{s\to t+} \sup\varphi(s) < t$, $\varphi(t) < t$ for all $t > 0$.

Let A, B, S and T be self-mappings on a real linear n-normed space $(X, ||.,\ldots,.||)$ such that for every $z_2 ,...,z_n \in X$

$$\begin{aligned}
&[\|Ax - By, z_2, ..., z_n\|^p + a\,\|Sx - Ty, z_2, ..., z_n\|^p]\,\|Ax - By, z_2, ..., z_n\|^p \\
&\le a\max\{\|Ax - Sx, z_2, ..., z_n\|^p\,\|By - Ty, z_2, ..., z_n\|^p, \\
&\|Ax - Ty, z_2, ..., z_n\|^q\,\|By - Sx, z_2, ..., z_n\|^{q'}\} \\
&+ \max\{\varphi_1(\|Sx - Ty, z_2, ..., z_n\|^{2p}), \\
&\varphi_2(\|Ax - Sx, z_2, ..., z_n\|^r\,\|By - Ty, z_2, ..., z_n\|^{r'}), \\
&\varphi_3(\|Ax - Ty, z_2, ..., z_n\|^s\,\|By - Sx, z_2, ..., z_n\|^{s'}), \\
&\varphi_4(\frac{1}{2}[\|Ax - Ty, z_2, ..., z_n\|^l\,\|Ax - Sx, z_2, ..., z_n\|^{l'} \\
&+ \|By - Sx, z_2, ..., z_n\|^l\,\|By - Ty, z_2, ..., z_n\|^{l'}])\},
\end{aligned} \tag{3.1}$$

for all $x, y, z_2, ..., z_n \in X$, $\varphi_i \in \Phi(i = 1,2,3,4)$, $a, p, q, q', r, r', s, s', l, l' \ge 0$ and $2p = q + q' = r + r' = s + s' = l + l'$. The condition (3.1) is commonly called a generalized Φ- contraction.

Now we first prove a common fixed point theorem for a quadruple of weak compatible self mappings satisfying a common $(E.A)$ property and a generalized Φ-contraction and then present some other results as corollaries. Finally, we present an extension of this common fixed point theorem.

Theorem 5. *Let A, B, S and T be self mappings on a real linear n-normed space $(X, \|.,...,.\|)$ satisfying (3.1). If the pairs (A,S) and (B,T) satisfy a common $(E.A)$ property, weakly compatible and that $T(X)$ and $S(X)$ are closed subsets of X, then A, B, S and T have a unique common fixed point in X.*

Proof. Since (A,S) and (B,T) satisfy a common property $(E.A)$, we can find two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

for some $z \in X$.

Let us assume that $S(X)$ and $T(X)$ are closed subspaces of X. Then, $z = Su = Tv$ for some $u, v \in X$. Then using (3.1) by considering x in place of x_n and v in place of y, we have for all $z_2,...,z_n \in X$

$$\begin{aligned}
&[\|Ax_n - Bv, z_2, ..., z_n\|^p + a\,\|Sx_n - Tv, z_2, ..., z_n\|^p]\,\|Ax_n - Bv, z_2, ..., z_n\|^p \\
&\le a\max\{\|Ax_n - Sx_n, z_2, ..., z_n\|^p\,\|Bv - Tv, z_2, ..., z_n\|^p,
\end{aligned}$$

$$\|Ax_n - Tv, z_2, ..., z_n\|^q \, \|Bv - Sx_n, z_2, ..., z_n\|^{q'}\}$$
$$+ \max\{\varphi_1(\|Sx_n - Tv, z_2, ..., z_n\|^{2p}),$$
$$\varphi_2(\|Ax_n - Sx_n, z_2, ..., z_n\|^r \, \|Bv - Tv, z_2, ..., z_n\|^{r'}),$$
$$\varphi_3(\|Ax_n - Tv, z_2, ..., z_n\|^s \, \|Bv - Sx_n, z_2, ..., z_n\|^{s'}),$$
$$\varphi_4(\frac{1}{2}[\|Ax_n - Tv, z_2, ..., z_n\|^l \, \|Ax_n - Sx_n, z_2, ..., z_n\|^{l'}$$
$$+ \|Bv - Sx_n, z_2, ..., z_n\|^l \, \|Bv - Tv, z_2, ..., z_n\|^{l'}])\}.$$

Taking limit as $n \to \infty$, and $Tv = z$, we obtain

$$[\|z - Bv, z_2, ..., z_n\|^p + a\|z - z, z_2, ..., z_n\|^p] \, \|z - Bv, z_2, ..., z_n\|^p$$
$$\leq a \max\{\|z - z, z_2, ..., z_n\|^p \, \|Bv - z, z_2, ..., z_n\|^p,$$
$$\|z - z, z_2, ..., z_n\|^q \, \|Bv - z, z_2, ..., z_n\|^{q'}\} + \max\{\varphi_1(\|z - z, z_2, ..., z_n\|^{2p}),$$
$$\varphi_2(\|z - z, z_2, ..., z_n\|^r \, \|Bv - z, z_2, ..., z_n\|^{r'}),$$
$$\varphi_3(\|z - z, z_2, ..., z_n\|^s \, \|Bv - z, z_2, ..., z_n\|^{s'}),$$
$$\varphi_4(\frac{1}{2}[\|z - z, z_2, ..., z_n\|^l \, \|z - z, z_2, ..., z_n\|^{l'}$$
$$+ \|Bv - z, z_2, ..., z_n\|^l \, \|Bv - z, z_2, ..., z_n\|^{l'}])\}$$

or

$$\|z - Bv, z_2, ..., z_n\|^{2p}$$
$$\leq \max\{\varphi_1(0), \varphi_2(0), \varphi_3(0), \varphi_4(\frac{1}{2}\|Bv - z, z_2, ..., z_n\|^{l+l'})\},$$

or

$$\|z - Bv, z_2, ..., z_n\|^{2p}$$
$$\leq \max\{\varphi_1(\|z - Bv, z_2, ..., z_n\|^{2p}), \varphi_2(\|z - Bv, z_2, ..., z_n\|^{r+r'}),$$
$$\varphi_3(\|z - Bv, z_2, ..., z_n\|^{s+s'}), \varphi_4(\frac{1}{2}\|Bv - z, z_2, ..., z_n\|^{l+l'})\}.$$

If $\varphi_i \in \Phi$ where $i \in I$ (some index set), then there exists a $\varphi \in \Phi$ such that $\max\{\varphi_i, i \in I\} \leq \varphi(t)$ for all $t > 0$ (Chang [6]). This along with the above argument and the property $\varphi(t) < t$ for all $t > 0$; we have $\|z - Bv, z_2, ..., z_n\|^{2p} \leq \varphi(\|z - Bv, z_2, ..., z_n\|^{2p}) < \|z - Bv, z_2, ..., z_n\|^{2p}$, a contradiction. This implies that $z = Bv$. Therefore $Tv = z = Bv$. Hence it follows by the weak compatibility of the pair (B, T) that $BTv = TBv$, that is $Bz = Tz$.

Now, we shall show that z is a common fixed point of B and T. For this, considering x_n in place of x and z in place of y, (3.1) becomes

$$[\|Ax_n - Bz, z_2, ..., z_n\|^p + a\|Sx_n - Tz, z_2, ..., z_n\|^p] \, \|Ax_n - Bz, z_2, ..., z_n\|^p$$
$$\leq a \max\{\|Ax_n - Sx_n, z_2, ..., z_n\|^p \, \|Bz - Tz, z_2, ..., z_n\|^p,$$
$$\|Ax_n - Tz, z_2, ..., z_n\|^q \, \|Bz - Sx_n, z_2, ..., z_n\|^{q'}\}$$

$$
\begin{aligned}
&+\max\{\varphi_1(\|Sx_n - Tz, z_2, ..., z_n\|^{2p}),\\
&\varphi_2(\|Ax_n - Sx_n, z_2, ..., z_n\|^{r}\,\|Bz - Tz, z_2, ..., z_n\|^{r'}),\\
&\varphi_3(\|Ax_n - Tz, z_2, ..., z_n\|^{s}\,\|Bz - Sx_n, z_2, ..., z_n\|^{s'}),\\
&\varphi_4(\frac{1}{2}[\|Ax_n - Tz, z_2, ..., z_n\|^{l}\,\|Ax_n - Sx_n, z_2, ..., z_n\|^{l'}\\
&+\|Bz - Sx_n, z_2, ..., z_n\|^{l}\,\|Bz - Tz, z_2, ..., z_n\|^{l'}])\}
\end{aligned}
$$

Letting $n \to \infty$, and using the fact that $\lim_{n\to\infty} Ax_n = z = \lim_{n\to\infty} Sx_n$ and $Bz = Tz$, we get

$$
\begin{aligned}
&[\|z - Bz, z_2, ..., z_n\|^{p} + a\,\|z - Tz, z_2, ..., z_n\|^{p}]\,\|z - Bz, z_2, ..., z_n\|^{p}\\
&\leq a\max\{\|z - z, z_2, ..., z_n\|^{p}\,\|Bz - Tz, z_2, ..., z_n\|^{p},\\
&\|z - Tz, z_2, ..., z_n\|^{q}\,\|Bz - z, z_2, ..., z_n\|^{q'}\} + \max\{\varphi_1(\|z - Tz, z_2, ..., z_n\|^{2p}),\\
&\varphi_2(\|z - z, z_2, ..., z_n\|^{r}\,\|Bz - Tz, z_2, ..., z_n\|^{r'}),\\
&\varphi_3(\|z - Tz, z_2, ..., z_n\|^{s}\,\|Bz - z, z_2, ..., z_n\|^{s'}),\\
&\varphi_4(\frac{1}{2}[\|z - Tz, z_2, ..., z_n\|^{l}\,\|z - z, z_2, ..., z_n\|^{l'}\\
&+\|Bz - z, z_2, ..., z_n\|^{l}\,\|Bz - Tz, z_2, ..., z_n\|^{l'}])\},
\end{aligned}
$$

or $\quad \|z - Bz, z_2, ..., z_n\|^{2p} + a\,\|z - Bz, z_2, ..., z_n\|^{2p}$

$$
\begin{aligned}
&\leq a\,\|Bz - z, z_2, ..., z_n\|^{q+q'} + \max\{\varphi_1(\|z - Bz, z_2, ..., z_n\|^{2p}), \varphi_2(0),\\
&\varphi_3(\|z - Bz, z_2, ..., z_n\|^{s+s'}), \varphi_4(0)\},
\end{aligned}
$$

or $\quad (1+a)\,\|z - Bz, z_2, ..., z_n\|^{2p} \leq a\,\|Bz - z, z_2, ..., z_n\|^{q+q'}$

$$
\begin{aligned}
&+\max\{\varphi_1(\|z - Bz, z_2, ..., z_n\|^{2p}),\\
&\varphi_2(0), \varphi_3(\|z - Bz, z_2, ..., z_n\|^{s+s'}), \varphi_4(0)\},
\end{aligned}
$$

or $\quad \|z - Bz, z_2, ..., z_n\|^{2p} \leq \frac{a}{1+a}\,\|Bz - z, z_2, ..., z_n\|^{q+q'}$

$$
\begin{aligned}
&+\frac{1}{1+a}\max\{\varphi_1(\|z - Bz, z_2, ..., z_n\|^{2p}),\\
&\varphi_2(0), \varphi_3(\|z - Bz, z_2, ..., z_n\|^{s+s'}), \varphi_4(0)\}
\end{aligned}
$$

Since $2p = q + q'$, using the same argument as applied earlier, we have $\|z - Bz, z_2, ..., z_n\|^{2p} < \|z - Bz, z_2, ..., z_n\|^{2p}$ (for every $z_2, ..., z_n \in X$), a contradiction. So, $z = Bz = Tz$. Thus z is a common fixed point of B and T.

Similarly, we can prove that z is a common fixed point of A and S. Thus z is the common fixed point of A, B, S and T. The uniqueness of z as a common fixed point of A, B, S and T can easily be verified. □

Now we deduce some other relevant results as corollaries.

Corollary 4. *Let A, B and S be self-mappings of X such that (A,S) and (B,S) satisfy a common $(E.A)$ property and*

$$\begin{aligned}
\|Ax-By,z_2,...,z_n\|^{2p} \le a\max\{&\|Ax-Sx,z_2,...,z_n\|^{p}\,\|By-Sy,z_2,...,z_n\|^{p},\\
&\|Ax-Sy,z_2,...,z_n\|^{q}\,\|By-Sx,z_2,...,z_n\|^{q'}\}+\\
&\max\{\varphi_2(\|Ax-Sx,z_2,...,z_n\|^{r}\,\|By-Sy,z_2,...,z_n\|^{r'}),\\
&\varphi_3(\|Ax-Sy,z_2,...,z_n\|^{s}\,\|By-Sx,z_2,...,z_n\|^{s'}),\\
&\varphi_4(\frac{1}{2}[\|Ax-Sy,z_2,...,z_n\|^{l}\,\|Ax-Sx,z_2,...,z_n\|^{l'}\\
&+\|By-Sx,z_2,...,z_n\|^{l}\,\|By-Sy,z_2,...,z_n\|^{l'}])\}, \qquad (3.2)
\end{aligned}$$

for all $x,y,z_2,...,z_n \in X$, $\varphi_i \in \Phi$ $(i=2,3,4)$, $a,p,q,q',r,r',s,s',l,l' \ge 0$ and $2p = q+q' = r+r' = s+s' = l+l'$. If the pairs (A,S) and (B,S) are weakly compatible and that $S(X)$ is closed, then A, B and S have a unique common fixed point in X.

Proof. This is an immediate consequence of Theorem 5 with $S=T$. □

Corollary 5. *Let A, B, S and T be self mappings of a real linear n-normed space $(X,\|.,...,.\|)$. If the pairs (A,S) and (B,T) satisfy a common $(E.A)$ property and*

$$\begin{aligned}
\|Ax-By,z_2,...,z_n\|^{2p} \le h\max\{&\|Sx-Ty,z_2,...,z_n\|^{2p},\\
&\|Ax-Sx,z_2,...,z_n\|^{r}\,\|By-Ty,z_2,...,z_n\|^{r'},\\
&\|Ax-Ty,z_2,...,z_n\|^{s}\,\|By-Sx,z_2,...,z_n\|^{s'},\\
&\frac{1}{2}[\|Ax-Ty,z_2,...,z_n\|^{l}\,\|Ax-Sx,z_2,...,z_n\|^{l'}\\
&+\|By-Sx,z_2,...,z_n\|^{l}\,\|By-Ty,z_2,...,z_n\|^{l'}]\}, \qquad (3.3)
\end{aligned}$$

for all $x,y,z_2,...,z_n \in X$, $p,q,q',r,r',s,s',l,l' \ge 0$ and $2p = q+q' = r+r' = s+s' = l+l'$. If the pairs (A,S) and (B,T) are weakly compatible and that $T(X)$ and $S(X)$ are closed, then A, B, S and T have a unique common fixed point in X.

Proof. If we take $a=0$ and $\varphi_i(t)=ht$ $(i=1,2,3,4)$, where $0<h<1$ in Theorem 5, we have the desired result. □

Corollary 6. *Let A and B be self mappings of a complete real linear n-normed space $(X,\|.,...,.\|)$ satisfying the following condition:*

$$\begin{aligned}
&[\|Ax-By,z_2,...,z_n\|^{p}+a\,\|x-y,z_2,...,z_n\|^{p}]\,\|Ax-By,z_2,...,z_n\|^{p}\\
&\le a\max\{\|Ax-x,z_2,...,z_n\|^{p}\,\|By-y,z_2,...,z_n\|^{p},\\
&\quad\|Ax-y,z_2,...,z_n\|^{q}\,\|By-x,z_2,...,z_n\|^{q'}\}+\\
&\max\{\varphi_1(\|x-y,z_2,...,z_n\|^{2p}),\varphi_2(\|Ax-x,z_2,...,z_n\|^{r}\,\|By-y,z_2,...,z_n\|^{r'}),\\
&\quad\varphi_3(\|Ax-y,z_2,...,z_n\|^{s}\,\|By-x,z_2,...,z_n\|^{s'}),
\end{aligned}$$

$$\varphi_4(\frac{1}{2}[\|Ax-y,z_2,...,z_n\|^l \|Ax-x,z_2,...,z_n\|^{l'} + \|By-x,z_2,...,z_n\|^l \|By-y,z_2,...,z_n\|^{l'}])\},$$

for all $x,y,z_2,...,z_n \in X$, $\varphi_i \in \Phi$ $(i=1,2,3,4)$, $a,p,q,q',r,r',s,s',l,l' \geq 0$ *and* $2p = q+q' = r+r' = s+s' = l+l'$, *then* A *and* B *have a unique common fixed point in* X.

Proof. The proof is immediate, if we take $S = T = I_X$ (the identity mapping on X) in Theorem 5. □

Finally, we give the following result as an extension of Theorem 5.

Theorem 6. *Let* S, T *and* A_n $(n \in N)$ *be self mappings of a real linear n-normed space* $(X, \|.,...,.\|)$. *Suppose further that the pairs* (A_{2n-1}, S) *and* (A_{2n}, T) *are weakly compatible for any* $n \in N$ *and satisfying a common* $(E.A)$ *property. If* $S(X)$ *and* $T(X)$ *are closed and that for any* $i \in N$, *the following condition is satisfied for all* $x,y,z_2,...,z_n \in X$

$$[\|A_ix-A_{i+1}y,z_2,...,z_n\|^p + a\|Sx-Ty,z_2,...,z_n\|^p]\,\|A_ix-A_{i+1}y,z_2,...,z_n\|^p$$
$$\leq a\max\{\|A_ix-Sx,z_2,...,z_n\|^p \|A_{i+1}y-Ty,z_2,...,z_n\|^p \|A_ix-Ty,z_2,...,z_n\|^q \|A_{i+1}y-Sx,z_2,...,z_n\|^{q'}\}$$
$$+\max\{\varphi_1(\|Sx-Ty,z_2,...,z_n\|^{2p}),$$
$$\varphi_2(\|A_ix-Sx,z_2,...,z_n\|^r \|A_{i+1}y-Ty,z_2,...,z_n\|^{r'}),$$
$$\varphi_3(\|A_ix-Ty,z_2,...,z_n\|^s \|A_{i+1}y-Sx,z_2,...,z_n\|^{s'}),$$
$$\varphi_4(\frac{1}{2}[\|A_ix-Ty,z_2,...,z_n\|^l \|A_ix-Sx,z_2,...,z_n\|^{l'} + \|A_{i+1}y-Sx,z_2,...,z_n\|^l \|A_{i+1}y-Ty,z_2,...,z_n\|^{l'}])\},$$

where $\varphi_i \in \Phi$ $(i=1,2,3,4)$, $a,p,q,q',r,r',s,s',l,l' \geq 0$ *and* $2p = q+q' = r+r' = s+s' = l+l'$, *then* S, T *and* A_n $(n \in N)$ *have a common fixed point in* X.

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