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PRACTICAL STABILITY OF IMPULSIVE FUNCTIONAL DIFFERENTIAL SYSTEMS VIA ANALYSIS TECHNIQUES

SHUN-XUAN CHEN AND ZHI-QIANG ZHU

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Abstract. In this paper we consider the practical stability for a class of functional differential system with impulses. By making use of the analysis techniques, specially, the Bernoulli inequality, we obtain some criteria to guarantee the practical stability of our system, including the finite-time stability and infinite-time stability.

2010 *Mathematics Subject Classification:* 34K45; 34k20

Keywords: impulse, functional differential system, finite-time stability, infinite-time stability, Bernoulli inequality

1. INTRODUCTION AND PRELIMINARIES

Since the system of functional differential equations with impulses can better describe the erupted process such as population, economics, control models and so on, there have been many research activities concerning the qualitative theory for the equations of this type, see, for example, the recent literature [2, 3, 5–7] and the others cited therein. This paper is concerned with the practical stability of the following system

$$\begin{cases} x'(t) = A(t)x(t) + f(t, x(t), x_t), \ t \geq 0, t \neq t_k, \\ x(t_k) = I_k x(t_k^-), \ k = 1, 2, 3, \ldots, \end{cases} \tag{1.1}$$

where $t_1 < t_2 < t_3 < \ldots$ and $t_k \to \infty$ as $k \to \infty$, x_t stands for the delay functions $x_t : [-\tau, 0] \to \mathbb{R}^n$ for a given positive constant $\tau > 0$ and any fixed t, and

$$x(t_k) = x(t_k^+) = \lim_{h \to 0^+} x(t_k + h), \ \ x(t_k^-) = \lim_{h \to 0^-} x(t_k + h).$$

Further, $I_k \in \mathbb{R}^{n \times n}$ is invertible for each k, $A \in C(\mathbb{R}^{n \times n}, \mathbb{R}^n)$, and $f : C([0, \infty) \times \mathbb{R}^n \times \mathcal{D}, \mathbb{R}^n)$, here $\mathcal{D}$ denotes the set of functions $\phi : [-\tau, 0] \to \mathbb{R}^n$ with the properties that $\phi(t)$ is continuous everywhere except for a countable number of points $\widetilde{t}$ at which $\phi(\widetilde{t} + 0)$ and $\phi(\widetilde{t} - 0)$ exist and $\phi(\widetilde{t} + 0) = \phi(\widetilde{t})$.

Note that the practical stability is different from the classical Lyapunov stability, see [1] for details. Roughly speaking, the practical stability means that the solutions

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of our system does not exceed some bounds on an advanced interval. Our motivation in this paper stems from the work by Stamova [6], who imposed the direct Lyapunov method to study the practical stability of (1.1) with special impulsive effects. We remark that, in general, it is difficult to find Lyapunov functions. In this paper we intend to avoid the trouble. For this purpose we require some hypotheses and notations as follows:

(H1) there exist two continuous functions $b_i : [0,\infty) \to (0,\infty)$ for $i = 1,2$, and a constant $\lambda > 0$ such that

$$|f(t,x,\phi)| \le b_1(t)|x|^\lambda + b_2(t)||\phi||^\lambda \text{ for all } (t,x,\phi) \in [0,\infty)\times\mathbb{R}^n\times\mathcal{D},$$

where $|\cdot|$ represents the norm of $\mathbb{R}^n$ and $||\cdot||$ the norm of $\mathcal{D}$ defined by $||\phi|| = \sup_{-\tau\le t\le 0}|\phi(t)|$;

(H2)

$$\int_0^\infty ||A(s)||\mathrm{d}s < \infty \text{ and } \int_0^\infty (b_1(s)+b_2(s))\mathrm{d}s < \infty,$$

where $||A(s)||$ is the norm of $A(s)$ induced by $|\cdot|$;

(H3) the functions $E(t)$ and $p(t)$ are defined, respectively, by

$$E(t) := \prod_{k:t_k\in[0,t]} I_k \text{ and } p(t) := \max\{||A(t)||,\ b_1(t)+b_2(t)\},$$

where $E(t)$ reduces an identity matrix when $t < t_1$;

(H4) there exists a constant $M_\infty > 0$ such that

$$M_\infty = \sup_{0\le t<\infty}\{||E(t)||, ||E^{-1}(t)||\},$$

where $E^{-1}(t)$ denotes the invertible of $E(t)$.

Let $\zeta > 0$ and $\phi \in \mathcal{D}$. By a solution $x(t) := x(t,\phi)$ of (1.1) defined on $[-\tau,\zeta)$ we mean that $x(t)$ is right continuous with $x(t) = \phi(t)$ on $[-\tau,0]$ and $x(t_k) = I_k x(t_k^-)$ for each possible k, and that $x(t)$ satisfies

$$x'(t) = A(t)x(t) + f(t,x(t),x_t),\ t \ne t_k \text{ and a.e. } t \ge 0. \tag{1.2}$$

Next we consider the relation of solutions between (1.1) and the following initial problem

$$\begin{cases} y'(t) = E^{-1}(t)[A(t)E(t)y(t) + f(t,E(t)y(t),(Ey)_t)], t\ge 0,\ t\ne t_k, \\ y(t) = \phi(t),\ t\in[-\tau,0], \end{cases} \tag{1.3}$$

where $(Ey)_t \in \mathcal{D}$ defined by $(Ey)_t(\theta) = E(t+\theta)y(t+\theta)$ for all $\theta\in[-\tau,0]$.

By a solution $y(t)$ of (1.3) we mean that $y(t)$ is continuous at points t_k, coincides with $\phi(t)$ on $[-\tau,0]$ and satisfies

$$y'(t) = E^{-1}(t)[A(t)E(t)y(t) + f(t,E(t)y(t),(Ey)_t)],\ t\ne t_k \text{ and a.e. } t\ge 0. \tag{1.4}$$

Let $x(t) := x(t,\phi)$ be a solution of (1.1) and $y(t) = E^{-1}(t)x(t)$. Then, a straightforward verification shows that $y(t)$ is continuous at points t_k for each possible k, and that $y(t)$ is a solution of (1.3). To the contrary, for a solution $y(t)$ of (1.3) we can verify that $x(t) = E(t)y(y)$ meets (1.2) and the conditions $x(t_k) = I_k x(t_k^-)$. Hence we get a preliminary result as follows.

Lemma 1. *The solution $x(t) := x(t,\phi)$ of (1.1) implies that $y(t) = E^{-1}(t)x(t)$ is a solution of (1.3). Conversely, the solution $y(t)$ of (1.3) implies that $x(t) = E(t)y(t)$ is a solution of (1.1) satisfying the condition $x(t) = \phi(t)$ for all $t \in [-\tau, 0]$.*

With the preliminaries in hand we can now give the precise definition of practical stability. The system(1.1) is said to be practically stable with respect to (α, β, T) if for given $\beta > 0$ and $T > 0$, there exists a positive number $\alpha = \alpha(\beta, T) \leq \beta$ such that $||\phi|| \leq \alpha$ implies the solution $x(t) := x(t,\phi)$ of (1.1) fulfils $|x(t)| \leq \beta$ for all $t \in [0, T]$. Furthermore, if there exists a positive number $\alpha = \alpha(\beta) \leq \beta$ such that $||\phi|| \leq \alpha$ implies $|x(t)| \leq \beta$ for all $t \in [0, \infty)$, then the system (1.1) is said to be practically stable with respect to (α, β).

2. Main Results

We are now in a position to establish the practical stability criteria for (1.1). Referring to Hale's monograph[4, Chapter 2, Theorem 2.1], we note that the solution $x(t) := x(t,\phi)$ of (1.1) exists locally for each $\phi \in \mathcal{D}$. Further, the following holds.

Lemma 2. *Under Assumptions (H1)–(H4) and $\phi \in \mathcal{D}$, the solution $x(t) := x(t,\phi)$ of (1.1) exists on $[-\tau, \infty)$.*

Proof. Suppose that the solution $x(t) := x(t,\phi)$ of (1.1) exists on $[-\tau, \zeta)$, here $0 < \zeta \leq \infty$. Then, Lemma 1 implies that $y(t) = E^{-1}(t)x(t)$ is a solution of (1.3). We assert $x(t)$ is bounded on $[-\tau, \zeta)$. Indeed, from (1.4) we have

$$y(t) = \phi(0) + \int_0^t E^{-1}(s)[A(s)E(s)y(s) + f(s, E(s)y(s), (Ey)_s)]\mathrm{d}s, \; t \geq 0 \quad (2.1)$$

and hence, with the help of $x(t) = E(t)y(t)$ and Assumptions (H1) and (H4), it follows that

$$|x(t)| \quad (2.2)$$
$$\leq M_\infty ||\phi|| + M_\infty^2 \int_0^t (||A(s)|| \cdot |x(s)| + b_1(s)|x(s)|^\lambda + b_2(s)||x_s||^\lambda)\mathrm{d}s, \; t \in [0, \zeta),$$

which infers that

$$||x_t|| \leq L + M_\infty^2 \int_0^t (||A(s)|| \cdot |x(s)| + b_1(s)|x(s)|^\lambda + b_2(s)||x_s||^\lambda)\mathrm{d}s, \; t \in [0, \zeta), \quad (2.3)$$

where $L := M_\infty ||\phi||$. Let $u(t)$, $v(t)$ and $w(t)$ be defined, respectively, by

$$u(t) := t^\lambda, \; v(t) := L + M_\infty^2 \int_0^t (||A(s)|| \cdot |x(s)| + b_1(s)|x(s)|^\lambda + b_2(s)||x_s||^\lambda)\mathrm{d}s$$

and

$$w(t) := \int_L^t \frac{\mathrm{d}s}{u(s)},\ t \geq L.$$

Without loss of generality we can set $L \geq 1$. Then, from (2.2) and (2.3) we have

$$\frac{|x(t)|}{u(v(t))} \leq 1,\ \frac{u(|x(t)|)}{u(v(t))} \leq 1,\ \frac{u(||x_t||)}{u(v(t))} \leq 1,\ t \in [0,\zeta)$$

and hence, it follows that

$$\frac{||A(t)|| \cdot |x(s)| + b_1(t)u(|x(t)|) + b_2(t)u(||x_t||)}{u(|v(t)|)} \leq ||A(t)|| + b_1(t) + b_2(t),$$

which, with the aid of $w(t)$, produces

$$\frac{\mathrm{d}}{\mathrm{d}t} w(v(t)) \leq M_\infty^2(||A(t)|| + b_1(t) + b_2(t)). \tag{2.4}$$

Now integrating (2.4) from 0 to $t \in [0,\zeta)$ we obtain

$$w(v(t)) \leq M_\infty^2 \int_0^t (||A(s)|| + b_1(s) + b_2(s))\mathrm{d}s$$

and this, with the invertibility and monotonicity of w, results in

$$v(t) \leq w^{-1}\left(M_\infty^2 \int_0^t (||A(s)|| + b_1(s) + b_2(s))\mathrm{d}s\right),\ t \in [0,\zeta). \tag{2.5}$$

Invoking Assumption (H2), together with (2.2) and (2.5), it is clear that the solution $x(t) := x(t,\phi)$ of (1.1) is bounded on $[-\tau,\zeta)$.

Now for the solution $x(t) := x(t,\phi)$ of (1.1) we can set

$$|A(t)x(t)| \leq M_a \ \text{ and } \ |f(t,x(t),x_t)| \leq M_\phi \ \text{ for all } \ t \in [-\tau,\zeta).$$

We assert that the solution $x(t) := x(t,\phi)$ exists on $[-\tau,t_1]$. Otherwise, $\zeta \leq t_1$ and, from (2.1) we have

$$|x(s_1) - x(s_2)| \leq M_a|s_1 - s_2| + M_\phi|s_1 - s_2|,\ s_1,s_2 \in [0,\zeta),$$

which implies that $\lim_{t\to\zeta^-} x(t)$ exists as a finite number. Thus $\varphi := x_\zeta \in \mathcal{D}$. Consequently, we can find a solution $z(t) := z(t;\zeta,\varphi)$ of (1.1) with $z_\zeta \equiv \varphi$, that is, the solution $x(t) := x(t,\phi)$ of (1.1) can be extended to a much larger interval than $[-\tau,\zeta)$, which leads to a contradiction.

Similarly, we can show that the solution $x(t) := x(t,\phi)$ of (1.1) exists on $[-\tau,t_k]$ for $k > 1$. Since $t_k \to \infty$ as $k \to \infty$, the desired result holds and the proof is complete. □

Theorem 1. *Suppose that Assumptions (H1) and (H3) are satisfied. Suppose further that $\beta > 0$, $T > 0$ and $M_T = \sup_{t\in[0,T]}\{||E(t)||, ||E^{-1}(t)||\}$. Then system (1.1) is practically stable with respect to (α,β,T) if one of the following conditions holds:*

(i) $\lambda = 1$ *and*

$$\alpha = \frac{\beta}{M_T e^{2M_T^2 \int_0^T p(s)\mathrm{d}s}}; \tag{2.6}$$

(ii) $\lambda > 1$ *and*

$$\alpha = \frac{1}{M_T \left(\left((\frac{1}{\beta})^{\lambda-1} + 1 \right) e^{M_T^2 (\lambda-1) \int_0^T p(s)\mathrm{d}s} - 1 \right)^{\frac{1}{\lambda-1}}}; \tag{2.7}$$

(iii) $0 < \lambda < 1$ *and*

$$\alpha = \frac{1}{M_T} \left(\left(\beta^{1-\lambda} + 1 \right) e^{M_T^2 (\lambda-1) \int_0^T p(s)\mathrm{d}s} - 1 \right)^{\frac{1}{1-\lambda}} \tag{2.8}$$

with

$$\beta > \left(e^{M_T^2 (1-\lambda) \int_0^T p(s)\mathrm{d}s} - 1 \right)^{\frac{1}{1-\lambda}}.$$

Proof. Let $J := [0,T]$. Note that from (2.1) it follows that

$$|x(t)| \leq M_T ||\phi|| + M_T^2 \int_0^t (||A(s)|| \cdot |x(s)| + b_1(s)|x(s)|^\lambda + b_2(s)||x_s||^\lambda)\mathrm{d}s,\ t \in J, \tag{2.9}$$

which induces

$$||x_t|| \leq M_T ||\phi|| + M_T^2 \int_0^t (||A(s)|| \cdot |x(s)| + b_1(s)|x(s)|^\lambda + b_2(s)||x_s||^\lambda)\mathrm{d}s,\ t \in J. \tag{2.10}$$

Now if we set

$$R(t) = M_T ||\phi|| + M_T^2 \int_0^t (||A(s)|| \cdot |x(s)| + b_1(s)|x(s)|^\lambda + b_2(s)||x_s||^\lambda)\mathrm{d}s,\ \ t \in J,$$

then $R(0) = M_T ||\phi||$ and, together with (2.9) and (2.10),

$$R'(t) \leq M_T^2 p(t) R(t) + M_T^2 p(t) R(t)^\lambda,\ t \in J, \tag{2.11}$$

where we have invoked the Assumption (H3). Next we proceed in steps.
(i) Case $\lambda = 1$. From (2.11) it follows that

$$R(t) \leq R(0) e^{2M_T^2 \int_0^t p(s)\mathrm{d}s},\ t \in J.$$

Since $|x(t)| \leq R(t)$, to fulfill $|x(t)| \leq \beta$ on J we turn to

$$R(0) e^{2M_T^2 \int_0^T p(s)\mathrm{d}s} \leq \beta,$$

which implies that

$$R(0) = M_T||\phi|| \leq \frac{\beta}{e^{2M_T^2 \int_0^T p(s)\mathrm{d}s}}.$$

That is, if we choose α as in (2.6), then system (1.1) is practically stable with respect to (α, β, T).

(ii) Case $\lambda > 1$. In this case we multiply (2.11) by $(1-\lambda)R(t)^{-\lambda}e^{M_T^2(\lambda-1)\int_0^t p(s)\mathrm{d}s}$ and obtain

$$\Big((R(t)^{1-\lambda}+1)e^{M_T^2(\lambda-1)\int_0^t p(s)\mathrm{d}s}\Big)' \geq 0,\ t \in J,$$

which means that

$$R(t)^{1-\lambda}+1 \geq (R(0)^{1-\lambda}+1)e^{M_T^2(1-\lambda)\int_0^t p(s)\mathrm{d}s},\ t \in J$$

and hence, when

$$R(0) < \frac{1}{\Big(e^{M_T^2(\lambda-1)\int_0^T p(s)\mathrm{d}s}-1\Big)^{\frac{1}{\lambda-1}}}, \tag{2.12}$$

it follows that

$$R(t) \leq \frac{1}{\Big((R(0)^{1-\lambda}+1)e^{M_T^2(1-\lambda)\int_0^t p(s)\mathrm{d}s}-1\Big)^{\frac{1}{\lambda-1}}},\ t \in J.$$

Now, to meet $|x(t)| \leq \beta$ on J we consider

$$\frac{1}{\Big((R(0)^{1-\lambda}+1)e^{M_T(1-\lambda)\int_0^T p(s)\mathrm{d}s}-1\Big)^{\frac{1}{\lambda-1}}} \leq \beta,$$

which deduces

$$R(0) \leq \frac{1}{\Big(\Big((\frac{1}{\beta})^{\lambda-1}+1\Big)e^{M_T^2(\lambda-1)\int_0^T p(s)\mathrm{d}s}-1\Big)^{\frac{1}{\lambda-1}}}. \tag{2.13}$$

Thus, from (2.12) and (2.13) we learn that when $\lambda > 1$ and α as in (2.7), system (1.1) is practically stable with respect to (α, β, T).

(iii) Case $0 < \lambda < 1$. Similarly, multiplying (2.11) by

$$(1-\lambda)R(t)^{-\lambda}e^{M_T^2(\lambda-1)\int_0^t p(s)\mathrm{d}s}$$

we have

$$\Big((R(t)^{1-\lambda}+1)e^{M_T^2(\lambda-1)\int_0^t p(s)\mathrm{d}s}\Big)' \leq 0,\ t \in J,$$

which means that

$$R(t) \leq \Big((R(0)^{1-\lambda}+1)e^{M_T^2(1-\lambda)\int_0^t p(s)\mathrm{d}s}-1\Big)^{\frac{1}{1-\lambda}},\ t \in J.$$

Solving

$$\left((R(0)^{1-\lambda}+1)e^{M_T^2(1-\lambda)\int_0^T p(s)\mathrm{d}s}-1\right)^{\frac{1}{1-\lambda}}\le\beta,$$

we obtain

$$R(0)\le\left(\left(\beta^{1-\lambda}+1\right)e^{M_T^2(\lambda-1)\int_0^T p(s)\mathrm{d}s}-1\right)^{\frac{1}{1-\lambda}},$$

where we require

$$\beta>\left(e^{M_T^2(1-\lambda)\int_0^T p(s)\mathrm{d}s}-1\right)^{\frac{1}{1-\lambda}}.$$

Hence, if we let α be defined as in (2.8), then system (1.1) can realize the practical stability with respect to (α,β,T).

In summary, the proof is complete. □

Note that by Assumptions (H2) and (H4) it holds that $\int_0^\infty p(s)\mathrm{d}s<\infty$. In addition, the Lemma 2 implies that for each $\phi\in\mathcal{D}$, the solution $x(t,\phi)$ of (1.1) exists on $[-\tau,\infty)$. Hence, it is reasonable to consider the practical stability on the interval $[-\tau,\infty)$. Indeed, The proof of Theorem 1 is valid for $T=\infty$. So the following results are clear and we give it to end our discussions.

Theorem 2. *Suppose that Assumptions (H1)–(H4) are satisfied and $\beta>0$. Then system (1.1) is practically stable with respect to (α,β) if one of the following conditions holds:*

(i) $\lambda=1$ *and*

$$\alpha=\frac{\beta}{M_\infty e^{2M_\infty^2\int_0^\infty p(s)\mathrm{d}s}};$$

(ii) $\lambda>1$ *and*

$$\alpha=\frac{1}{M_\infty\left(\left((\frac{1}{\beta})^{\lambda-1}+1\right)e^{M_\infty^2(\lambda-1)\int_0^\infty p(s)\mathrm{d}s}-1\right)^{\frac{1}{\lambda-1}}};$$

(iii) $0<\lambda<1$ *and*

$$\alpha=\frac{1}{M_\infty}\left(\left(\beta^{1-\lambda}+1\right)e^{M_\infty^2(\lambda-1)\int_0^\infty p(s)\mathrm{d}s}-1\right)^{\frac{1}{1-\lambda}}$$

with

$$\beta>\left(e^{M_\infty^2(1-\lambda)\int_0^\infty p(s)\mathrm{d}s}-1\right)^{\frac{1}{1-\lambda}}.$$

Next we conclude this paper with an example.

Example 1. Let $\lambda > 0$ be constant, $x = (x_1, x_2)^T \in \mathbb{R}^2$ with the norm $|x| = |x_1| + |x_2|$, and $\phi \in \mathcal{D}$ with $\phi : [-1, 0] \to \mathbb{R}^2$ and $\phi = (\phi_1, \phi_2)^T$. Suppose in (1.1) that

$$A(t) = \begin{pmatrix} te^{-t} & 0 \\ 0 & e^{-t} \end{pmatrix}, f(t,x,\phi) = \frac{1}{1+t^2}\begin{pmatrix} \sin^\lambda(x_1+x_2) \\ \sin^\lambda(\phi_1(-1)+\phi_2(-1/2)) \end{pmatrix}$$

as well as

$$I_k = \begin{pmatrix} e^{\frac{(-1)^k}{k}} & 0 \\ 0 & e^{-\frac{(-1)^k}{k}} \end{pmatrix}, \; k = 1, 2, 3, \ldots.$$

Then

$$|f(t,x,\phi)| \le \frac{1}{1+t^2}(|x|^\lambda + ||\phi||^\lambda), \;\; \text{for all } (t,x,\phi) \in [0,\infty) \times \mathbb{R}^2 \times \mathcal{D}$$

and hence, the Assumption (H1) is met. In this case the functions b_i and $p(t)$ in Assumptions (H1) and (H3) can be taken as

$$b_i(t) = \frac{1}{1+t^2}, \; p(t) = \frac{2}{1+t^2} \;\; \text{for } \; i = 1, 2 \; \text{ and } \; t \ge 0.$$

On the other hand, we have

$$E(t) = \begin{pmatrix} c(t) & 0 \\ 0 & d(t) \end{pmatrix},$$

where

$$c(t) = e^{\sum_{k=1}^{[t]} \frac{(-1)^k}{k}}, \; d(t) = e^{-\sum_{k=1}^{[t]} \frac{(-1)^k}{k}}, \; t \ge 0$$

and $[t]$ indicates the integral part of t. Thus, a simple verification shows that

$$\frac{1}{3} \le c(t) \le 1 \;\text{ and }\; \frac{3}{2} \le d(t) \le 3 \;\text{ for }\; t \ge 0.$$

In other words, we can take $M_\infty = 3$ for the Assumption (H4). Note that $\int_0^\infty p(s)\mathrm{d}s = \pi$, for $\beta > 0$ we choose

$$\alpha = \frac{\beta}{3e^{18\pi}} \;\text{ when }\; \lambda = 1, \;\text{ or}$$

$$\alpha = \frac{1}{3\left(\left((\frac{1}{\beta})^{\lambda-1}+1\right)e^{9\pi(\lambda-1)}-1\right)^{\frac{1}{\lambda-1}}} \;\text{ when }\; \lambda > 1,$$

or, when $\lambda \in (0,1)$,

$$\alpha = \frac{1}{3}\left(\left(\beta^{1-\lambda}+1\right)e^{9\pi(\lambda-1)}-1\right)^{\frac{1}{1-\lambda}} \;\text{ with }\; \beta > \left(e^{9\pi(1-\lambda)}-1\right)^{\frac{1}{1-\lambda}}.$$

Then, by Theorem 2 we see that under our considerations, system (1.1) is practically stable with respect to (α, β).

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Authors' addresses

Shun-Xuan Chen

Guangdong Polytechnic Normal University, College of Computer Science, 510665 Guangzhou, P. R. China

E-mail address: chenshunxuan@126.com

Zhi-Qiang Zhu

Guangdong Polytechnic Normal University, Department of Computer Science, 510665 Guangzhou, P. R. China

E-mail address: z3825@163.com (corresponding author: Zhi-Qiang Zhu)