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MULTIVARIABLE MATRIX GENERALIZATION OF GOULD-HOPPER POLYNOMIALS

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Abstract. The main object of this investigation is to define a multivariable matrix generalization of Gould-Hopper polynomials and to reveal some relations such as matrix generating function, matrix recurrence relation, matrix differential equation for them. Furthermore, more general families of bilinear and bilateral matrix generating functions are obtained for these matrix polynomials.

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1. INTRODUCTION

Matrix polynomials and special matrix functions, which have many applications on statistics, group representation theory, scattering theory, interpolation and quadrature, splines and medical imaging, comprise an emerging field of study with important results in literature .

In the recent papers, the matrix generalizations of many polynomials were introduced by many authors and their various properties were given from the scalar case, see for example [1, 3–6, 9–13].

Throughout this paper, for a matrix A in $\mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of A. If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z, which are defined in an open set Ω of the complex plane and A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A)\subset\Omega$, then from the properties of the matrix functional calculus in [7], it follows that:

$$f(A)g(A)=g(A)f(A).$$

If D is the complex plane cut along the negative real axis and $log(z)$ denotes the principle logarithm of z, then $z^{1/2}$ represents $\exp((1/2)log(z))$. If A is a matrix in $\mathbb{C}^{N\times N}$ with $\sigma(A)\subset D$, then $A^{1/2}=\sqrt{A}$ denotes the image by $z^{1/2}$ of the matrix functional calculus acting on the matrix A.

Let $A \in \mathbb{C}^{N\times N}$ so that $Re(z) > 0$, $\forall z \in \sigma(A)$, then we say that A is a positive stable matrix in $\mathbb{C}^{N\times N}$. Hermite matrix polynomials $H_n(x, A)$ are defined by [9]:

$$H_n(x,A) = n!\sum_{k=0}^{[\frac{n}{2}]} \frac{(-1)^k}{k!(n-2k)!}(x\sqrt{2A})^{n-2k} \quad , \quad n \geq 0$$

and satisfy the three term matrix recurrence relation:

$$\begin{aligned}
&cH_n(x,A) = xI\sqrt{2A}H_{n-1}(x,A) - 2(n-1)H_{n-2}(x,A); \;\; n \geq 1 \\
&H_{-1}(x,A) = \theta, \;\; H_0(x,A) = I
\end{aligned}$$

where I is unit matrix and θ is zero matrix in $\mathbb{C}^{N\times N}$. Also, for these matrix polynomials, it follows

$$\begin{aligned}
\frac{d}{dx}H_n(x,A) &= n\sqrt{2A}H_{n-1}(x,A), \;\; n \geq 1 \\
\exp\left(xt\sqrt{2A} - t^2 I\right) &= \sum_{n=0}^{\infty} \frac{1}{n!}H_n(x,A)t^n, \;\; |t| < +\infty.
\end{aligned}$$

In this paper, we deal with matrix version of the multivariable extension of Gould-Hopper polynomials which are generalization of Hermite polynomials.

We organize the paper as follows:

In Section 2, we construct matrix extension of Gould-Hopper polynomials and give some matrix recurrence relations and matrix differential equation for these polynomials. In Section 3, multivariable generalization of the matrix polynomials presented in Section 2 is defined and their properties are examined. In the last section, bilinear and bilateral generating matrix functions are derived for the multivariable Gould-Hopper matrix polynomials and some applications of our results are presented.

We recall that the Gould–Hopper polynomials $g_n^m(x, y)$ are specified by the generating function

$$\exp(xt + yt^m) = \sum_{n=0}^{\infty} g_n^m(x,y)\frac{t^n}{n!} \tag{1.1}$$

where m is positive integer [8] (see also [14]).

In the special case $m = 2$, we have $g_n^2(x, y) = H_n(x, y)$ where $H_n(x, y)$ denotes the two-variable Hermite–Kampé de Fériet polynomials generated by (see [2])

$$\exp(xt + yt^2) = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{1.2}$$

Furhermore, we note that $H_n(\nu x,-1) = H_{n,m,\nu}(x)$,where $H_{n,m,\nu}(x)$ is the generalized Hermite polynomials defined by Lahiri [15] and they are generated by

$$\exp(\nu xt - t^m) = \sum_{n=0}^{\infty} H_{n,m,\nu}(x)\frac{t^n}{n!}. \tag{1.3}$$

From the equations (1.1), (1.2) and (1.3) we have the special cases $g_n^2(2x,-1) = H_n(x)$, $H_n(2x,-1) = H_n(x)$ and $H_{n,2,2}(x) = H_n(x)$, respectively where $H_n(x)$ denotes Hermite polynomials.

2. Matrix generalization of Gould-Hopper polynomials

We define a matrix version of Gould-Hopper polynomials as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} g_n^m(x,y;A,B)\frac{t^n}{n!} &= G(x,y,t)\\ &= \exp(xt\sqrt{2A})\exp(Byt^m)\end{aligned} \tag{2.1}$$

where A,B are matrices in $\mathbb{C}^{N\times N}$ satisfying that A is a positive stable and m is positive integer.

By using (2.1) and Taylor series at $t=0$, we can write

$$\begin{aligned}\sum_{n=0}^{\infty} g_n^m(x,y;A,B)\frac{t^n}{n!} &= \left(\sum_{n=0}^{\infty}\frac{\left(xt\sqrt{2A}\right)^n}{n!}\right)\left(\sum_{k=0}^{\infty}\frac{(Byt^m)^k}{k!}\right)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{\left(\sqrt{2A}\right)^{n-mk}(B)^k}{(n-mk)!\,k!}x^{n-mk}y^k t^n.\end{aligned}$$

Then comparing coefficients of t^n, we have explicit representation for matrix version of Gould-Hopper polynomials as:

$$g_n^m(x,y;A,B) = \sum_{k=0}^{\left[\frac{n}{m}\right]}\frac{n!\left(\sqrt{2A}\right)^{n-mk}(B)^k}{(n-mk)!\,k!}x^{n-mk}y^k. \tag{2.2}$$

Now, we consider some special cases as follows.

Remark 1. The case $B = I \in \mathbb{C}^{N\times N}$ in (2.1) gives the matrix polynomials defined by [16].

Remark 2. If we take $m=2$ in (2.1), we have matrix version of the two-variable Hermite–Kampé de Fériet polynomials $H_n(x,y)$ which are generated by (1.2).

Remark 3. Setting νx instead of x and $y=-1$ in (2.1), we get matrix extension of the generalized Hermite polynomials $H_{n,m,\nu}(x)$ specified by (1.3).

Remark 4. If we take $m=2, y=-1$ and $B=I\in\mathbb{C}^{N\times N}$ in (2.1), we have

$$\begin{aligned} g_n^2(x,-1;A,I) &= \sum_{k=0}^{[n/2]} \frac{n!\,(-1)^k\left(x\sqrt{2A}\right)^{n-2k}}{(n-2k)!\,k!} \\ &= H_n(x,A) \end{aligned}$$

where $H_n(x,A)$ is Hermite matrix polynomial given by [9].

Remark 5. For $x\to(-x)$ in (2.2), we get

$$g_n^m(-x,y;A,B)=(-1)^n g_n^m(x,(-1)^m y;A,B).$$

If we write $x=0$ in (2.1), we have

$$\sum_{n=0}^{\infty} g_n^m(0,y;A,B)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{(Byt^m)^n}{n!}. \tag{2.3}$$

On the other hand, we can write

$$\begin{aligned} \sum_{n=0}^{\infty} g_n^m(0,y;A,B)\frac{t^n}{n!} &= \sum_{n=0}^{\infty} g_{nm}^m(0,y;A,B)\frac{t^{nm}}{(nm)!} \\ &+\sum_{n=0}^{\infty} g_{nm+1}^m(0,y;A,B)\frac{t^{nm+1}}{(nm+1)!} \\ &+\ldots+\sum_{n=0}^{\infty} g_{nm+m-1}^m(0,y;A,B)\frac{t^{nm+m-1}}{(nm+m-1)!}. \end{aligned} \tag{2.4}$$

Using (2.3) and (2.4), we have

$$\begin{aligned} g_{nm}^m(0,y;A,B) &= \frac{(By)^n\,(mn)!}{n!} \\ g_{nm+k}^m(0,y;A,B) &= \theta;\ \ k=1,2,...,m-1. \end{aligned} \tag{2.5}$$

Remark 6. For $m=2, y=-1$ and $B=I$ in (2.5), we have

$$g_{2n}^2(0,-1;A,I)=H_{2n}(0,A)=\frac{(-1)^n\,(2n)!I}{n!}$$

and

$$g_{2n+1}^2(0,-1;A,I)=H_{2n+1}(0,A)=\theta.$$

If we differentiate (2.1) with respect to x, matrix polynomials $g_n^m(x,y;A,B)$ satisfy recurrence relations as follows:

$$\frac{\partial}{\partial x}g_n^m(x,y;A,B)=n\sqrt{2A}g_{n-1}^m(x,y;A,B);\ n\geq 1$$

$$\frac{\partial^k}{\partial x^k} g_n^m(x,y;A,B) = n(n-1)\dots(n-k+1)\left(\sqrt{2A}\right)^k \times g_{n-k}^m(x,y;A,B);\ n \geq k. \tag{2.6}$$

Similarly, differentiating (2.1) with respect to y, the following holds for matrix polynomials $g_n^m(x,y;A,B)$

$$\frac{\partial}{\partial y} g_n^m(x,y;A,B) = \frac{n!}{(n-m)!} g_{n-m}^m(x,y;A,B)B \ \ ; n \geq m. \tag{2.7}$$

By using the derivative of the generating function (2.1) with respect to t, we can give next relation

$$\begin{aligned} &g_{n+1}^m(x,y;A,B) - \sqrt{2A}x g_n^m(x,y;A,B) \\ &\qquad = \frac{my\, n!}{(n-m+1)!} g_{n-m+1}^m(x,y;A,B)B;\ \ n \geq m-1. \end{aligned}$$

On the other hand, from (2.1) we get

$$\left.\begin{aligned} &\frac{\partial}{\partial x} G(x,y,t) = t\sqrt{2A}G(x,y,t) \\ &\frac{\partial}{\partial t} G(x,y,t) = x\sqrt{2A}G(x,y,t) + G(x,y,t)mByt^{m-1} \end{aligned}\right\} \tag{2.8}$$

from which, we obtain that

$$t\sqrt{2A}\frac{\partial}{\partial t} G(x,y,t) = x\sqrt{2A}\frac{\partial}{\partial x} G(x,y,t) + \frac{\partial}{\partial x} G(x,y,t)mByt^{m-1}.$$

In view of (2.1) and the last relation, we derive

$$\begin{aligned} &n\sqrt{2A} g_n^m(x,y;A,B) - x\sqrt{2A}\frac{\partial}{\partial x} g_n^m(x,y;A,B) \\ &\qquad = my\frac{\partial}{\partial x} g_{n-m+1}^m(x,y;A,B)\frac{n!}{(n-m+1)!}B;\ \ n \geq m-1. \end{aligned} \tag{2.9}$$

Now, let's find matrix differential equation for the Gould-Hopper matrix polynomials. We start with writing $n+1$ instead of n in (2.7). Then we have

$$\frac{\partial}{\partial y} g_{n+1}^m(x,y;A,B) = \frac{(n+1)!}{(n+1-m)!} g_{n+1-m}^m(x,y;A,B)B. \tag{2.10}$$

Then differentiating (2.10) with respect to x and then using (2.6), we get

$$(n+1)\sqrt{2A}\frac{\partial}{\partial y} g_n^m(x,y;A,B) = \frac{(n+1)!}{(n+1-m)!}\frac{\partial}{\partial x} g_{n+1-m}^m(x,y;A,B)B. \tag{2.11}$$

If we write (2.11) in (2.9), we arrive at the following matrix differential equation

$$n g_n^m(x,y;A,B) = x\frac{\partial}{\partial x} g_n^m(x,y;A,B) + my\frac{\partial}{\partial y} g_n^m(x,y;A,B).$$

3. Multivariable extension of Gould-Hopper matrix polynomials

The multivariable matrix extension of the Gould-Hopper polynomials given by (2.1) is defined by the following generating function

$$\begin{aligned} & G_n^m(\mathrm{X,Y;T,A,B}) \qquad (3.1)\\ &= G_{n_1}^{m_1}(x_1,y_1;t_1,A_1,B_1)\, G_{n_2}^{m_2}(x_2,y_2;t_2,A_2,B_2)\ldots G_{n_r}^{m_r}(x_r,y_r;t_r,A_r,B_r)\\ &= \prod_{i=1}^{r}\left\{\exp(x_i t_i\sqrt{2A_i})\exp(B_i y_i t_i^{m_i})\right\}\\ &= \sum_{n_1,\ldots,n_r=0}^{\infty} g_n^m(\mathrm{X,Y;A,B})\frac{t_1^{n_1}\ldots t_r^{n_r}}{n_1!\ldots n_r!} \end{aligned}$$

where $\mathrm{X}=(x_1,\ldots,x_r)$, $\mathrm{Y}=(y_1,\ldots,y_r)$, $m=(m_1,\ldots,m_r)$, $n=(n_1,\ldots,n_r)$, $|n|=n_1+\ldots+n_r$ $(n_1,\ldots,n_r\in\mathbb{N}_0)$, $m_1,\ldots,m_r\in\mathbb{N}$, $\mathrm{T}=(t_1,\ldots,t_r)$, $\mathrm{A}=(A_1,\ldots,A_r)$, $\mathrm{B}=(B_1,\ldots,B_r)$ and A_i, B_i are matrices in $\mathbb{C}^{N\times N}$ satisfying that A_i is positive stable for $1\le i\le r$.

For the multivariable Gould-Hopper matrix polynomials $g_n^m(\mathrm{X,Y;T,A,B})$, explicit form is

$$\begin{aligned} g_n^m(\mathrm{X,Y;A,B}) = \sum_{k_1,\ldots,k_r=0}^{[n_1/m_1,\ldots,n_r/m_r]} & \frac{n_1!(\sqrt{2A_1})^{n_1-m_1k_1}B_1^{k_1}\ldots n_r!(\sqrt{2A_r})^{n_r-m_rk_r}B_r^{k_r}}{(n_1-m_1k_1)!k_1!\ldots(n_r-m_rk_r)!k_r!}\\ & x_1^{n_1-m_1k_1}y_1^{k_1}\ldots x_r^{n_r-m_rk_r}y_r^{k_r}. \end{aligned}$$

Similar to the relations (2.6) and (2.7), if we differentiate (3.1) with respect to the variables x_i and y_i $(i=1,2,\ldots,r)$, we can easily obtain

$$\begin{aligned} &\frac{\partial}{\partial x_i} g_n^m(\mathrm{X,Y;A,B})\\ &\qquad = n_i\sqrt{2A_i}\, g^m_{n_1,n_2,\ldots n_{i-1},n_i-1,n_{i+1},\ldots,n_r}(\mathrm{X,Y;A,B});\ \ n_i\ge 1, \qquad (3.2) \end{aligned}$$

$$\begin{aligned} &\frac{\partial^k}{\partial x_i^k} g_n^m(\mathrm{X,Y;A,B})\\ &\quad = (n_i-k+1)_k\left(\sqrt{2A_i}\right)^k g^m_{n_1,n_2,\ldots n_{i-1},n_i-k,n_{i+1},\ldots,n_r}(\mathrm{X,Y;A,B});\ \ n_i\ge k, \end{aligned}$$

$$\begin{aligned} &\frac{\partial}{\partial y_i} g_n^m(\mathrm{X,Y;A,B})\\ &\qquad = \frac{n_i!}{(n_i-m_i)!} g^m_{n_1,n_2,\ldots n_{i-1},n_i-m_i,n_{i+1},\ldots,n_r}(\mathrm{X,Y;A,B});\ \ n_i\ge m_i, \qquad (3.3) \end{aligned}$$

$$\frac{\partial^r}{\partial x_1 \ldots \partial x_r} g_n^m(\mathrm{X,Y;A,B}) = n_1\sqrt{2A_1}\ldots n_r\sqrt{2A_r}\, g^m_{n_1-1,n_2-1,\ldots,n_r-1}(\mathrm{X,Y;A,B}),$$

for all $n_i \geq 1$, $i = 1,2,\ldots,r$ and

$$\frac{\partial^r}{\partial y_1 \ldots \partial y_r} g_n^m(\mathrm{X,Y;A,B}) = \frac{n_1!\ldots n_r!}{(n_1-m_1)!\ldots(n_r-m_r)!} g^m_{n_1-m_1,n_2-m_2,\ldots,n_r-m_r}(\mathrm{X,Y;A,B}),$$

for all $n_i \geq m_i$, $i = 1,2,\ldots,r$ where all of the matrices are commutative with each other.

Using the same method in (2.8), we get

$$n_i\sqrt{2A_i}\, g_n^m(\mathrm{X,Y;A,B}) - \sqrt{2A_i}\, x_i \frac{\partial}{\partial x_i} g_n^m(\mathrm{X,Y;A,B}) = m_i y_i \frac{n_i!}{(n_i-m_i+1)!} \frac{\partial}{\partial x_i} g^m_{n_1,n_2,\ldots n_{i-1},n_i-m_i+1,n_{i+1},\ldots,n_r}(\mathrm{X,Y;A,B})\, B_i \quad (3.4)$$

for $n_i \geq m_i - 1$ where all of the matrices are commutative with each other.

In view of the equations (3.2), (3.3) and (3.4), we obtain the following matrix differential equation for multivariable Gould-Hopper matrix polynomials

$$\sum_{i=1}^{r} n_i\, g_n^m(\mathrm{X,Y;A,B}) = \sum_{i=1}^{r} x_i \frac{\partial}{\partial x_i} g_n^m(\mathrm{X,Y;A,B}) + \sum_{i=1}^{r} m_i y_i \frac{\partial}{\partial y_i} g_n^m(\mathrm{X,Y;A,B})$$

where all of the matrices are commutative with each other.

4. Bilinear and bilateral generating matrix functions for multivariable Gould-Hopper matrix polynomials

In order to obtain several families of bilinear and bilateral generating matrix functions for multivariable Gould-Hopper matrix polynomials, we first state our result as the following.

Theorem 1. *For a non-vanishing function $\Omega_\mu(z_1,\ldots,z_s)$ of s complex variables $z_1,\ldots,z_s$ $(s \in \mathbb{N})$ and of complex order μ, let*

$$\Lambda_{\mu,\nu}(z_1,\ldots,z_s;\eta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\nu k}(z_1,\ldots,z_s)\eta^k \ ; \ (a_k \neq 0,\ \mu,\nu \in \mathbb{C})$$

and

$$\Theta^m_{n,p,\mu,\nu}(\mathrm{X,Y};z_1,\ldots,z_s;\zeta) := \sum_{k=0}^{[n_1/p]} \frac{a_k}{(n_1-pk)!n_2!\ldots n_r!} g^m_{n_1-pk,n_2,\ldots,n_r}(\mathrm{X,Y;A,B}) \times \Omega_{\mu+\nu k}(z_1,\ldots,z_s)\zeta^k \quad (4.1)$$

where $m=(m_1,...,m_r)$, $n=(n_1,...,n_r)$, $\mathrm{X}=(x_1,...,x_r)$, $\mathrm{Y}=(y_1,...,y_r)$, $p\in\mathbb{N}$ *and (as usual)* $[\alpha]$ *represents the greatest integer in* $\alpha\in\mathbb{R}$. *Then we have*

$$\sum_{n_1,...,n_r=0}^{\infty}\Theta^{m}_{n,p,\mu,\nu}\left(\mathrm{X},\mathrm{Y};z_1,...,z_s;\frac{\eta}{t_1^p}\right)t_1^{n_1}t_2^{n_2}...t_r^{n_r}$$
$$=\prod_{i=1}^{r}\left\{\exp(x_it_i\sqrt{2A_i})\exp(B_iy_it_i^{m_i})\right\}\Lambda_{\mu,\nu}(z_1,...,z_s;\eta) \quad (4.2)$$

provided that each member of (4.2) *exists.*

Proof. For convenience, let S denote the left hand-side of the assertion (4.2) of Theorem 1. Then, upon substituting for the polynomials $\Theta^{m}_{n,p,\mu,\nu}\left(\mathrm{X},\mathrm{Y};z_1,...,z_s;\frac{\eta}{t_1^p}\right)$ from the definition (4.1) into the left-hand side of (4.2), we obtain

$$S=\sum_{n_1,...,n_r=0}^{\infty}\sum_{k=0}^{[n_1/p]}a_k\frac{g^{m}_{n_1-pk,n_2,...,n_r}(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})}{(n_1-pk)!n_2!...n_r!}$$
$$\cdot\Omega_{\mu+\nu k}(z_1,...,z_s)\eta^k t_1^{n_1-pk}t_2^{n_2}...t_r^{n_r}. \quad (4.3)$$

Upon inverting the order of summation in (4.3), if we replace n_1 by n_1+pk, we can write

$$S=\sum_{n_1,...,n_r=0}^{\infty}\sum_{k=0}^{\infty}a_k\frac{g^{m}_{n}(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})}{n_1!n_2!...n_r!}\Omega_{\mu+\nu k}(z_1,...,z_s)\eta^k t_1^{n_1}...t_r^{n_r}$$
$$=\left(\sum_{n_1,...,n_r=0}^{\infty}\frac{g^{m}_{n}(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})}{n_1!n_2!...n_r!}t_1^{n_1}...t_r^{n_r}\right)\left(\sum_{k=0}^{\infty}a_k\Omega_{\mu+\nu k}(z_1,...,z_s)\eta^k\right)$$
$$=\prod_{i=1}^{r}\left\{\exp(x_it_i\sqrt{2A_i})\exp(B_iy_it_i^{m_i})\right\}\Lambda_{\mu,\nu}(z_1,...,z_s;\eta).$$

The proof is completed. □

In order to give some applications of Theorem 1, we consider the expresses of the multivariable function $\Omega_{\mu+\nu k}(y_1,...,y_s)$ $(k\in\mathbb{N}_0,\ s\in\mathbb{N})$ in terms of simpler function of one and more variables.
First of all, let's get $s=2r$ and $\Omega_{\mu+\nu k}(u_1,...,u_r;v_1,...,v_r)=P^{(C_1,...,C_r)}_{\mu+\nu k}(\mathrm{L},\mathrm{U},\mathrm{V},\mathrm{D})$ in Theorem 1, where $P^{(C_1,...,C_r)}_{\mu+\nu k}(\mathrm{L},\mathrm{U},\mathrm{V},\mathrm{D})$ denotes the multivariable matrix Humbert polynomials generated by [1]

$$\sum_{k=0}^{\infty}P^{(C_1,...,C_r)}_{k}(\mathrm{L},\mathrm{U},\mathrm{V},\mathrm{D})t^k$$

$$= \prod_{i=1}^{r} (D_i - l_i u_i t + v_i t^{l_i})^{-C_i} \left(\left| l_i u_i t - v_i t^{l_i} \right| < |D_i| \;\; ; \, i = 1,2,...,r \right) \quad (4.4)$$

where $C_i \in \mathbb{C}^{N\times N}$, U$= (u_1,...,u_r)$, V$= (v_1,...,v_r)$, D$= (D_1,...,D_r)$; $D_i \neq 0$, $i = 1,2,...,r$, L$= (l_1,...,l_r)$ and l_i $(i = 1,2,...,r)$ is positive integer.

Then we obtain the following class of bilateral generating functions for the g_n^m (X,Y;A,B) and $P_{\mu+\nu k}^{(C_1,...,C_r)}$(L,U,V,D).

Corollary 1. *If* $\Lambda^{\mathrm{L}}_{\mu,\nu}(\mathrm{U,V};\eta) := \sum_{k=0}^{\infty} a_k P_{\mu+\nu k}^{(C_1,...,C_r)}(\mathrm{L,U,V,D})\eta^k$ *where* $(a_k \neq 0,\, \mu,\nu \in \mathbb{N}_0)$; *and*

$$\Theta^{m,\mathrm{L}}_{n,p,\mu,\nu}(\mathrm{X,Y;U,V};\zeta) := \sum_{k=0}^{[n_1/p]} a_k \frac{g^m_{n_1-pk,n_2,...,n_r}(\mathrm{X,Y;A,B})}{(n_1-pk)!n_2!...n_r!} P_{\mu+\nu k}^{(C_1,...,C_r)}(\mathrm{L,U,V,D})\zeta^k$$

where $n, p \in \mathbb{N}$. *Then we have*

$$\sum_{n_1,...,n_r=0}^{\infty} \Theta^{m,\mathrm{L}}_{n,p,\mu,\nu}\left(\mathrm{X,Y;U,V};\frac{\eta}{t_1^p}\right) t_1^{n_1}...t_r^{n_r}$$
$$= \prod_{i=1}^{r} \left\{\exp(x_i t_i \sqrt{2A_i}) \exp(B_i y_i t_i^{m_i})\right\} \Lambda^{\mathrm{L}}_{\mu,\nu}(\mathrm{U,V};\eta) \quad (4.5)$$

provided that each member of (4.5) *exists.*

Remark 7. Using the generating relation (4.4) and taking $a_k = 1$, $\mu = 0$, $\nu = 1$, we have

$$\sum_{n_1,...,n_r=0}^{\infty} \sum_{k=0}^{[n_1/p]} \frac{g^m_{n_1-pk,n_2,...,n_r}(\mathrm{X,Y;A,B})}{(n_1-pk)!n_2!...n_r!} P_k^{(C_1,...,C_r)}(\mathrm{L,U,V,D})\eta^k t_1^{n_1-pk} t_2^{n_2}...t_r^{n_r}$$
$$= \prod_{i=1}^{r} \left\{\exp(x_i t_i \sqrt{2A_i}) \exp(B_i y_i t_i^{m_i})\right\} \prod_{i=1}^{r} \left\{(D_i - l_i u_i \eta + v_i \eta^{l_i})^{-C_i}\right\}.$$

Now, setting $s = 2$ and $\Omega_{\mu+\nu k}(z_1,z_2) = g^l_{\mu+\nu k}(z_1,z_2;C,D)$ in Theorem 1, we obtain the following class of bilinear generating function for the matrix version of the Gould-Hopper polynomials.

Corollary 2. *If* $\Lambda^{l}_{\mu,\nu}(z_1,z_2;\eta) := \sum_{k=0}^{\infty} a_k g^l_{\mu+\nu k}(z_1,z_2;C,D)\eta^k$ *where* $(a_k \neq 0,\, \mu,\nu \in \mathbb{N}_0)$; *and*

$$\Theta^{m,l}_{n,p,\mu,\nu}(\mathrm{X,Y},z_1,z_2;\zeta) := \sum_{k=0}^{[n_1/p]} a_k \frac{g^m_{n_1-pk,n_2,...,n_r}(\mathrm{X,Y;A,B})}{(n_1-pk)!n_2!...n_r!} g^l_{\mu+\nu k}(z_1,z_2;C,D)\zeta^k$$

where $n_1, p \in \mathbb{N}$. Then we have

$$\sum_{n_1,\dots,n_r=0}^{\infty} \Theta^{m,l}_{n,p,\mu,\nu}\left(\mathrm{X},\mathrm{Y},z_1,z_2;\frac{\eta}{t_1^p}\right) t_1^{n_1}\dots t_r^{n_r}$$
$$= \prod_{i=1}^{r}\left\{\exp(x_i t_i\sqrt{2A_i})\exp(B_i y_i t_i^{m_i})\right\} \Lambda^l_{\mu,\nu}(z_1,z_2;\eta) \quad (4.6)$$

provided that each member of (4.6) *exists.*

Remark 8. Using Corollary 2 and taking $a_k = 1$, $\mu = 0$, $\nu = 1$, we have

$$\sum_{n_1,\dots,n_r=0}^{\infty}\sum_{k=0}^{[n_1/p]} \frac{g^m_{n_1-pk,n_2,\dots,n_r}(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})}{(n_1-pk)!n_2!\dots n_r!} g^l_k(z_1,z_2;C,D)\eta^k t_1^{n_1-pk} t_2^{n_2}\dots t_r^{n_r}$$
$$= \prod_{i=1}^{r}\left\{\exp(x_i t_i\sqrt{2A_i})\exp(B_i y_i t_i^{m_i})\right\} \exp(z_1\eta\sqrt{2C})\exp(Dz_2\eta^l).$$

We remark that for every suitable choice of the coefficients a_k $(k \in \mathbb{N}_0)$, if the multivariable function $\Omega_{\mu+\nu k}(z_1,\dots,z_s)$, $(s \in \mathbb{N})$, is expressed as an appropriate product of several simpler functions, the assertions of Theorem 1 can be applied in order to derive various families of multilinear and multilateral generating functions for the $g^m_n(\mathrm{X},\mathrm{Y};\mathrm{A},\mathrm{B})$.

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