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THE CAUCHY-SCHWARZ INEQUALITY IN CAYLEY GRAPH AND TOURNAMENT STRUCTURES ON FINITE FIELDS

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Abstract. The Cayley graph construction provides a natural grid structure on a finite vector space over a field of prime or prime square cardinality, where the characteristic is congruent to 3 modulo 4, in addition to the quadratic residue tournament structure on the prime subfield. Distance from the null vector in the grid graph defines a Manhattan norm. The Hermitian inner product on these spaces over finite fields behaves in some respects similarly to the real and complex case. An analogue of the Cauchy-Schwarz inequality is valid with respect to the Manhattan norm. With respect to the non-transitive order provided by the quadratic residue tournament, an analogue of the Cauchy-Schwarz inequality holds in arbitrarily large neighborhoods of the null vector, when the characteristic is an appropriate large prime.

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1. MANHATTAN NORMS AND GRID GRAPHS

We consider the finite fields $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ of prime and prime square cardinality, where $p \equiv 3 \mod 4$. The field $\mathbb{F}_{p^2}$ has a natural graph structure with the field elements as vertices, two distinct vertices u, z being adjacent if $(z-u)^4 = 1$. The subfield $\mathbb{F}_p$ of $\mathbb{F}_{p^2}$ then induces a subgraph in which x and y are adjacent if and only if $(y-x)^2 = 1$. The graph $\mathbb{F}_{p^2}$ is isomorphic to the Cartesian square $C_p^2 = C_p \square C_p$, where C_p is a p-cycle and within $\mathbb{F}_{p^2}$ the induced subgraph $\mathbb{F}_p$ is itself a p-cycle. Clearly the graph $\mathbb{F}_{p^2}$ is not planar, but can be drawn as a grid on the torus.

For any connected graph whose vertex set is a group, the distance of any vertex z from the identity element of the group is called the *norm* of z, denoted $N(z)$. In general, distances and norms measured in connected subgraphs induced by subgroups can be larger than distances and norms measured with reference to the whole graph. However, with respect to the distance-preserving subgraph induced by $\mathbb{F}_p$ in $\mathbb{F}_{p^2}$, the norm of any $z \in \mathbb{F}_p$ is the same as its norm with respect to the whole graph $\mathbb{F}_{p^2}$: this is simply the length of the shortest path from 0 to z in the cycle induced by $\mathbb{F}_p$.

For $q = p$ or $q = p^2$, the n-dimensional vector space $\mathbb{F}_q^n$ is also endowed with the Cartesian product graph structure $\mathbb{F}_q \square \cdots \square \mathbb{F}_q$ isomorphic to C_p^n or C_p^{2n}. The norm of a vector $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{F}_q^n$ is then equal to the sum $N(v_1) + \cdots + N(v_n)$ and we also write $N(\mathbf{v})$ for this vector norm.

The Gaussian integers $\mathbb{Z}[i]$ also constitute a graph in which u and z are adjacent if and only if $(z-u)^4 = 1$.

It is easy to see that the norm in this *infinite Manhattan grid* satisfies the triangle and submultiplicative inequalities

$$N(u+z) \leq N(u) + N(z)$$
$$N(uz) \leq N(u)N(z)$$

To emphasize that the norms on $\mathbb{F}_{p^2}$, $\mathbb{F}_{p^2}^n$ and $\mathbb{Z}[i]$ are understood with reference to the specific grid graphs defined above, we call these norms *Manhattan norms*. Throughout this paper we think of $\mathbb{F}_{p^2}$ as the ring quotient $\mathbb{Z}[i]/(p)$.

2. Graph quotients and Cayley graphs

Given a graph G (undirected, with possible loops) on vertex set V and an equivalence relation $\equiv$ on V, the *quotient graph* $G/\equiv$ is defined as follows: the vertices of $G/\equiv$ are the equivalence classes of $\equiv$, and classes A, B are adjacent if for some $a \in A$, $b \in B$, the elements a, b are adjacent in G. Note that the distance of A to B in the quotient graph is at most equal to, but possibly less than the minimum of the distances a to b for all $a \in A$, $b \in B$. Note also that $G/\equiv$ can have loops even if G has not.

Given a group G with identity element e and a set Γ of group elements that generates G, the *(left) Cayley graph* $\mathcal{C}(G, \Gamma)$ of G with respect to Γ has vertex set G, elements $a, b \in G$ being considered adjacent if ab^{-1} or ba^{-1} belongs to Γ. For each congruence $\equiv$ of the group G, corresponding to some normal subgroup H, Γ yields a generating set $\Gamma_\equiv$ of $G/\equiv$ consisting with those classes of $\equiv$ that intersect Γ. The graph quotient of $\mathcal{C}(G, \Gamma)$ by the equivalence $\equiv$ coincides with the Cayley graph of the quotient graph $G/\equiv$ with respect to $\Gamma_\equiv$. For $R \subseteq G$ inducing a connected subgraph $[R]$ in $\mathcal{C}(G, \Gamma)$, denote by $d_R(x, y)$ the distance function of the subgraph $[R]$. Denoting by xH the H-coset of any $x \in G$, this relates to norms in $\mathcal{C}(G, \Gamma)$ and $\mathcal{C}(G, \Gamma)/\equiv$ as follows: for all $x \in R$,

$$d_R(x, e) \geq N(x) \geq N(xH)$$

Both inequalities can be strict. However, we have:

Cayley Graph Quotient Lemma. *Let a group G with identity e be generated by $\Gamma \subseteq G$, and consider any normal subgroup H with corresponding congruence $\equiv$. There is a set $R \subseteq G$ having exactly one element in common with each congruence* *class modulo H, and such that for every $x \in R$*

$$d_R(x,e) = N(x) = N(xH)$$

Proof. We can define the unique (representative) element $r(A) \in R \cap A$ for each coset A by induction on the distance $d(H,A)$ of A from H in $\mathcal{C}(G,\Gamma)/\equiv$. Let $r(H) = e$. Assuming $r(A)$ defined for all A with $d(H,A) \le m$, let a coset B have distance $m+1$ from H. Choose any coset A adjacent to B with $d(H,A) = m$ and elements $a \in A$, $b \in B$ that are adjacent in $\mathcal{C}(G,\Gamma)$. Let $r(B) = ba^{-1}r(A)$. □

We can apply the above lemma in the case where $G = \mathbb{Z}[i]$, $\Gamma = \{1,i\}$ and $H = p\mathbb{Z}[i] = \{pa + pbi : a,b \in \mathbb{Z}\}$ for a prime integer $p \equiv 3 \mod 4$. Now $\mathcal{C}(G,\Gamma)$ and $\mathcal{C}(G,\Gamma)/\equiv$ are the Manhattan grid graphs on $\mathbb{Z}[i]$ and $\mathbb{Z}[i]/H = \mathbb{F}_{p^2}$, respectively. Referring to the set R of representatives in the lemma, for any H-cosets X, Y let x, y be the unique elements in $X \cap R$, $Y \cap R$. As $xy \in XY$, we have $N(XY) \le N(xy)$. By the submultiplicative inequality in $\mathbb{Z}[i]$ we have $N(xy) \le N(x)N(y)$. Using the lemma we have $N(x)N(y) = N(X)N(Y)$. This yields a submultiplicative inequality in $\mathbb{F}_{p^2}$ and a similar reasoning on the coset $X+Y$ yields a triangle inequality:

Triangle and Submultiplicative Inequalities in $\mathbb{F}_{p^2}$. *For all u,z in $\mathbb{F}_{p^2}$*

$$N(u+z) \le N(u) + N(z)$$
$$N(uz) \le N(u)N(z)$$

This indicates that Manhattan distance provides a well-behaved notion of neighborhood of 0 in the finite fields $\mathbb{F}_{p^2}$.

3. Squares in $\mathbb{F}_p$ and non-transitive order

For each prime $p \equiv 3 \mod 4$ the *quadratic residue tournament* on $\mathbb{F}_p$ is the directed graph with vertex set $\mathbb{F}_p$ in which there is an arrow from vertex x to vertex y if $y - x$ is a non-zero square in $\mathbb{F}_p$, in which case we write $x <_p y$. We write $x \le_p y$ if $x <_p y$ or $x = y$. The relation $\le_p$ is reflexive, anti-symmetric but not transitive, and for every $x \ne y$ exactly one of $x \le_p y$ or $y \le_p x$ holds. Using Dirichlet's theorem on primes in arithmetic progressions, Kustaanheimo showed [4] that for every positive integer k, there is a prime $p \equiv 3 \mod 4$, such that $\le_p$ is a transitive (and linear) order relation on $\{0,1,\dots,k\} \subseteq \mathbb{F}_p$, that is, all positive integers up to k are quadratic residues $\mod p$. Obviously k cannot exceed $(p-1)/2$. Implications of [4] and related questions were investigated by Järnefelt, Kustaanheimo, Quist [3, 5], in particular with a view to discrete models in physics, also in subsequent application-oriented work between the 1950's (Coish [1]) and the 1980's (Nambu [6]). For further references see [2]. In particular [4] implies that for every positive integer k, there is a prime $p \equiv 3 \mod 4$, such that all $z \in \mathbb{F}_{p^2}$ with $N(z) \le k$ are squares in $\mathbb{F}_{p^2}$. (Note that all elements of the prime subfield $\mathbb{F}_p$ are squares in $\mathbb{F}_{p^2}$.) To emphasise the analogy of the relation $\le_p$ with the ordinary inequality relation $\le$ among numbers, we say that a non-zero $z \in \mathbb{F}_{p^2}$ is *positive* if $z \in \mathbb{F}_p$ and $0 \le_p z$.

4. Inner products compared in non-transitive order

The only non-trivial automorphism of the field $\mathbb{F}_{p^2}$ associates to each $z \in \mathbb{F}_{p^2}$ its *conjugate* $\overline{z}$. The *inner product* $\mathbf{v}\cdot\mathbf{w}$ of vectors $\mathbf{v} = (v_1,\dots,v_n)$ and $\mathbf{w} = (w_1,\dots,w_n)$ in $\mathbb{F}_{p^2}^n$ is defined as the scalar $v_1\overline{w_1} + \cdots + v_n\overline{w_n} \in \mathbb{F}_{p^2}$. This inner product is left and right distributive over vector addition, satisfies $\mathbf{v}\cdot\mathbf{w} = \overline{\mathbf{w}\cdot\mathbf{v}}$, $c(\mathbf{v}\cdot\mathbf{w}) = (c\mathbf{v})\cdot\mathbf{w} = \mathbf{v}\cdot(\overline{c}\mathbf{w})$ for all $c \in \mathbb{F}_{p^2}$. However, while $\mathbf{v}\cdot\mathbf{v}$ belongs to the prime subfield $\mathbb{F}_p$, $\mathbf{v}\cdot\mathbf{v}$ is not necessarily positive, and can be 0 even if $\mathbf{v} \neq \mathbf{0}$. Still, a conditional version of positive definiteness holds locally:

Theorem 1. *For every $k \geq 1$ there is a prime $p \equiv 3 \mod 4$, such that for all $n \geq 1$ and for all vectors $\boldsymbol{v} \in \mathbb{F}_{p^2}^n$ of Manhattan norm $N(\boldsymbol{v}) \leq k$, we have $0 \leq_p \boldsymbol{v}\cdot\boldsymbol{v}$ with equality if and only if $\boldsymbol{v} = \boldsymbol{0}$.*

Proof. By Kustaanheimo's result in [4] there is a prime integer $p \equiv 3 \mod 4$ such that $0, 1, \dots, 2k^3$ are all quadratic residues mod p. For $\mathbf{v} = (v_1,\dots,v_n)$ in $\mathbb{F}_{p^2}^n$, let $v_j = a_j + b_j i$, where $i^2 = -1$. If $N(\mathbf{v}) \leq k$ then for all j, $N(a_j) \leq k$ and $N(b_j) \leq k$, $v_j\overline{v_j} = a_j^2 + b_j^2$ belongs to the set of squares $\{0,\dots,2k^2\}$. Since v_j can be non-zero for at most k indices $1 \leq j \leq n$ only, the sum of the corresponding terms $a_j^2 + b_j^2$ belongs to the set of squares $\{0, 1, \dots, 2k^3\}$. $\square$

Note that for all vectors $\mathbf{v}, \mathbf{w} \in \mathbb{F}_{p^2}^n$

$$(\mathbf{v}\cdot\mathbf{w})(\mathbf{w}\cdot\mathbf{v}) = (\mathbf{v}\cdot\mathbf{w})(\overline{\mathbf{v}\cdot\mathbf{w}}) \in \mathbb{F}_p \quad \text{and}$$
$$(\mathbf{v}\cdot\mathbf{v})(\mathbf{w}\cdot\mathbf{w}) \in \mathbb{F}_p.$$

If $\mathbf{v}$ and $\mathbf{w}$ are *proportional*, i.e. if there exists a scalar c in $\mathbb{F}_{p^2}$ such that $\mathbf{v} = c\mathbf{w}$ or $\mathbf{w} = c\mathbf{v}$, then the above two products are equal. Generally, they are related in the quadratic residue tournament of $\mathbb{F}_p$ as follows.

Theorem 2 (**Cauchy-Schwarz Inequality**). *For every $k \geq 1$ there is a prime $p \equiv 3 \mod 4$, such that for all $n \geq 1$ and for all vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{F}_{p^2}^n$ of Manhattan norm at most k,*

$$(\boldsymbol{v}\cdot\boldsymbol{w})(\boldsymbol{w}\cdot\boldsymbol{v}) \leq_p (\boldsymbol{v}\cdot\boldsymbol{v})(\boldsymbol{w}\cdot\boldsymbol{w}).$$

Proof. For $n = 1$ the inequality holds trivially as the two sides are equal. Assume $n \geq 2$, $\mathbf{v} = (v_1,\dots,v_n)$, $\mathbf{w} = (w_1,\dots,w_n)$. For all $1 \leq i \leq n$, $N(v_i) \leq k$, $N(w_i) \leq k$. By Kustaanheimo's result [4] there is a prime $p \equiv 3 \mod 4$ such that all positive integers up to $4k^6$ are quadratic residues modulo p. For each of the $\binom{n}{2}$ pairs $\{i, j\} \subseteq \{1,\dots,n\}$, $i \neq j$, by the triangle and submultiplicative inequalities in $\mathbb{F}_{p^2}$

$$N[(v_i w_j - v_j w_i)(\overline{v}_i\overline{w}_j - \overline{v}_j\overline{w}_j)] \leq (k^2 + k^2)^2 = 4k^4$$

Thus the element

$$(v_i w_j \overline{v}_i \overline{w}_j + v_j w_i \overline{v}_j \overline{w}_i) - (v_i w_j \overline{v}_j \overline{w}_i + v_j w_i \overline{v}_i \overline{w}_j) = (v_i w_j - v_j w_i)(\overline{v}_i \overline{w}_j - \overline{v}_j \overline{w}_j)$$

is a square of Manhattan norm at most $4k^4$ in $\mathbb{F}_p$, and it is non-zero for at most $\binom{k}{2} \leq k^2$ pairs $\{i, j\}$. Summing over all pairs $\{i, j\}$, all but at most $\binom{k}{2} \leq k^2$ terms vanish in the sum

$$\sum [(v_i w_j \overline{v}_i \overline{w}_j + v_j w_i \overline{v}_j \overline{w}_i) - (v_i w_j \overline{v}_j \overline{w}_i + v_j w_i \overline{v}_i \overline{w}_j)]$$

which therefore has Manhattan norm at most $4k^6$ and it must also be a square in $\mathbb{F}_p$. But this sum is equal to the difference of products

$$\sum_{i=1}^{n} v_i \overline{v}_i \sum_{j=1}^{n} w_j \overline{w}_j - \sum_{i=1}^{n} v_i \overline{w}_i \sum_{j=1}^{n} \overline{v}_j w_j = (\mathbf{v} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{w}) - (\mathbf{v} \cdot \mathbf{w})(\mathbf{w} \cdot \mathbf{v})$$

which is consequently a square in $\mathbb{F}_p$. □

Remark. From the proof it is clear that, in analogy with the classical Cauchy-Schwarz inequality, for vectors $\mathbf{v}, \mathbf{w}$ of norm not exceeding k in $\mathbb{F}_{p^2}^n$, where p is related to k as stipulated above, the Cauchy-Schwarz inequality with respect to $\leq_p$ holds with equality if and only if $v_i w_j - v_j w_i = 0$ for all i, j, i.e. if and only if $\mathbf{v}, \mathbf{w}$ are proportional.

We note that the inequality established above is conditional, it holds only in a specified Manhattan neighborhood of the null vector. Every non-zero element of $\mathbb{F}_p$ can be written as a sum of two squares, in particular there are $a, b \in \mathbb{F}_p$, such that $a^2 + b^2 = -1$. For $z = a + bi$ we have $z\overline{z} = -1$. As soon as $n \geq 2$, in $\mathbb{F}_{p^2}^n$ let

$$\mathbf{v} = (a, b, 0, \ldots, 0) \quad \text{and} \quad \mathbf{w} = (bz, -az, 0, \ldots, 0)$$

The inequality $(\mathbf{v} \cdot \mathbf{w})(\mathbf{w} \cdot \mathbf{v}) \leq_p (\mathbf{v} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{w})$ fails because the left-hand side is 0 and the right-hand side is -1. In fact if $n \geq 3$, the inequality can be invalidated with vectors $\mathbf{v}, \mathbf{w}$ in $\mathbb{F}_p^n$ as follows. Taking again $a, b \in \mathbb{F}_p$ with $a^2 + b^2 = -1$, let

$$\mathbf{v} = (1, a, b, 0, \ldots, 0) \quad \text{and} \quad \mathbf{w} = (1, 0, 0, 0, \ldots, 0)$$

However, the Cauchy-Schwarz inequality holds unconditionally in the 2-dimensional case for vectors with components in $\mathbb{F}_p$:

Special case of $\mathbb{F}_p^2$. *Let p be a prime congruent* 3 *modulo* 4. *For all vectors $\boldsymbol{v}, \boldsymbol{w}$ in $\mathbb{F}_p^2$*

$$(\boldsymbol{v} \cdot \boldsymbol{w})(\boldsymbol{w} \cdot \boldsymbol{v}) \leq_p (\boldsymbol{v} \cdot \boldsymbol{v})(\boldsymbol{w} \cdot \boldsymbol{w}).$$

Proof. Now the conjugation appearing in the inner products is the identity. Written in components,

$$(\mathbf{v} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{w}) - (\mathbf{v} \cdot \mathbf{w})(\mathbf{w} \cdot \mathbf{v}) = (v_1^2 + v_2^2)(w_1^2 + w_2^2) - (v_1 w_1 + v_2 w_2)^2 =$$
$$= v_1^2 w_2^2 + v_2^2 w_1^2 - 2 v_1 w_1 v_2 w_2 = (v_1 w_2 - v_2 w_1)^2$$

□

5. Manhattan norm of inner product

The Manhattan norm can be seen to be submultiplicative not only on the ring $\mathbb{Z}[i]$ and its quotient field $\mathbb{F}_{p^2}$, but on all vector spaces $\mathbb{F}^n_{p^2}$, with respect to the inner product:

Cauchy-Schwarz Inequality for Manhattan Norm on $\mathbb{F}^n_{p^2}$. *Consider any prime* $p \equiv 3 \mod 4$ *and let* $n \geq 1$. *For all* $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{F}^n_{p^2}$

$$N(\boldsymbol{v} \cdot \boldsymbol{w}) \leq N(\boldsymbol{v}) N(\boldsymbol{w}).$$

Proof. Let $\mathbf{v} = (v_1, \ldots, v_n), \mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{F}^n_{p^2}$. Then $\mathbf{v} \cdot \mathbf{w} = \sum v_j \overline{w_j}$. Clearly $N(z) = N(\overline{z})$ for any $z \in \mathbb{F}_{p^2}$. By the triangle and submultiplicative inequalities in $\mathbb{F}_{p^2}$ we have

$$\begin{aligned} N(\mathbf{v} \cdot \mathbf{w}) = N\left(\sum v_j \overline{w_j}\right) \leq \sum N\left(v_j \overline{w_j}\right) \leq \sum N(v_j) N(w_j) \leq \\ \leq \sum N(v_j) \sum N(w_j) = N(\mathbf{v}) N(\mathbf{w}) \end{aligned}$$

□

Remark. The inequality $N(\mathbf{v} \cdot \mathbf{w}) \leq N(\mathbf{v}) N(\mathbf{w})$ is easily interpreted and continues to hold for $\mathbf{v}, \mathbf{w}$ in the module $(\mathbb{Z}[i]/m\mathbb{Z}[i])^n$ for any positive integer m. As soon as m is composite, or a prime not congruent to 3 modulo 4, the ring $\mathbb{Z}[i]/m\mathbb{Z}[i]$ fails to be an integral domain.

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