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CONVERGENCE THEOREMS FOR ADMISSIBLE PERTURBATIONS OF φ-PSEUDOCONTRACTIVE OPERATORS

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Abstract. We prove some convergence theorems for a Krasnoselskij type fixed point iterative method constructed as the admissible perturbation of a nonlinear φ-pseudocontractive operator defined on a convex and closed subset of a Hilbert space. These new results extend and unify several related results in the current literature established for contractions, strongly pseudocontractive operators and generalized pseudocontractions.

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Keywords: Hilbert space, contraction, strong pseudocontraction, generalized pseudocontraction, fixed point iteration, admissible perturbation, convergence theorem

1. INTRODUCTION AND PRELIMINARIES

There exists a vast literature on the iterative approximation of fixed points of self and nonself single-valued and multi-valued mappings; see for example the recent monographs [3, 9] and references therein.

Let X be a space. Consider the generic fixed point problem

$$x = Tx, \tag{1.1}$$

where $T : K \to X$ is a given mapping and $K \subset X$.

A method that was intensively used to solve (1.1) is the well known *Picard iteration* or *sequence of successive approximations* $\{x_n\}_{n=0}^{\infty}$, defined by $x_0 \in K$ and

$$x_{n+1} = Tx_n, n \geq 0. \tag{1.2}$$

Picard iterative scheme (1.2) is known to converge to a solution of (1.1) under rather strong conditions on K, X and T, see the sample convergence theorems in [3, 9, 14].

In order to solve (1.1) under weaker assumptions and thus to broaden the scope of applications of fixed point theory, the researchers have introduced more reliable fixed point iterative methods; see [3] and [9].

We recall some of the most used fixed point iterative methods.

Let E be a real vector space and $T : E \to E$ a given operator. Let $x_0 \in E$ be arbitrary and $\{\alpha_n\} \subset [0, 1]$ a sequence of real numbers.

The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by $x_0 \in E$ and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n\,, \quad n = 0,1,2,\dots \tag{1.3}$$

is called the *Mann iterative scheme*, in view of [13]. The sequence $\{x_n\}_{n=0}^{\infty} \subset E$ defined by

$$\begin{cases} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n\,, \quad n = 0,1,2,\dots \\ y_n &= (1-\beta_n)x_n + \beta_n T x_n\,, \quad n = 0,1,2,\dots, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$, and $x_0 \in E$ is arbitrary, is called the *Ishikawa iterative scheme*, in view of [10].

Remark 1. For $\beta_n = 0$, (1.4) reduces to (1.3), while, for $\alpha_n = \lambda$ (constant), the Mann iteration scheme (1.3) reduces to the so called *Krasnoselskij iteration scheme*, [11].

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n\,, \quad n = 0,1,2,\dots \tag{1.5}$$

Picard iterative scheme (1.2) is obtained from it with $\lambda = 1$.

The above fixed point iterative methods have been extensively used to approximate fixed points of several classes of contractive type mappings; see for example the recent monographs [3, 9, 14] and references therein. One of the most important class of such mappings is that of *pseudocontractive* type mappings. There exist several concepts of pseudocontractive type mappings: pseudocontractive, strictly pseudocontractive, strongly pseudocontractive, generalized pseudocontractive mappings etc. We recall the definition of the most important ones for the purpose of the present paper.

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and norm $\|\cdot\|$. An operator $T : K \subset H \to H$ is said to be:

- *pseudocontractive* if, for all $x, y \in K$,

$$\|Tx - Ty\|^2 \le \|x-y\|^2 + \|Tx - Ty - (x-y)\|^2. \tag{1.6}$$

  Condition (1.6) is equivalent to the following condition

$$\langle Tx - Ty, x-y \rangle \le \|x-y\|^2. \tag{1.7}$$

- *strongly pseudocontractive* if there exists a constant $k \in (0,1)$ such that, for all $x, y \in K$,

$$\langle Tx - Ty, x-y \rangle \le k\,\|x-y\|^2. \tag{1.8}$$

- *strictly pseudocontractive* if there exists a constant $k < 1$ such that, for all $x, y \in K$,

$$\|Tx - Ty\|^2 \le \|x-y\|^2 + k\,\|Tx - Ty - (x-y)\|^2. \tag{1.9}$$

- *generalized pseudocontractive* if there exists a constant $r > 0$ such that, for all $x, y \in K$,

$$\|Tx - Ty\|^2 \leq r^2 \|x - y\|^2 + \|Tx - Ty - r(x - y)\|^2. \tag{1.10}$$

Condition (1.10) is equivalent to

$$\langle Tx - Ty, x - y \rangle \leq r \|x - y\|^2. \tag{1.11}$$

- *of pseudocontractive type* if, for all $x, y \in K$,

$$\|Tx - Ty\|^2 \leq r^2 \|x - y\|^2 + \max\left\{\|Tx - x\|^2 + \|Ty - y\|^2, \|Tx - Ty - (x - y)\|^2\right\}, \tag{1.12}$$

- *generalized successively pseudocontractive* if there exists a constant $k > 0$ such that, for all $x, y \in K$ and for all $n \geq 1$,

$$\|T^n x - T^n y\|^2 \leq k^2 \|x - y\|^2 + \|T^n x - T^n y - k(x - y)\|^2. \tag{1.13}$$

The pseudocontractive type operators have been studied by many authors ([12, 17–21]) in view of their relationship with accretive operators: the operator $T := I - A$ is pseudocontractive if and only if A is accretive. In this context, Browder [8] has shown that the initial value evolution system

$$\frac{du}{dt} + Au = 0, \; u(0) = u_0,$$

is solvable if A is Lipschitzian and accretive.

In the iterative approximation of fixed point [9], in addition to finding zeros of accretive operators, we are also interesyed to solve the equivalent problem of computing fixed point of pseudocontractive type operators. In this context, some authors tried to unify various classes of pseudocontractive type operators.

The first author introduced in [2], the so called class of pseudo φ-contractions and established a general convergence theorem for the Krasnoselskij fixed point iterative method, while Kumar and Sharma [12] introduced and studied iterative methods for solving variational inequalities that involve Lipschitzian and generalized successively pseudocontractive operators.

On the other hand, Rus [15] considered a new approach to fixed point iterative methods, based on the concept of *admissible perturbation* of a self operator. Further contributions in this direction have been made in [4–6].

As the class of φ-pseudocontractions includes important classes of pseudocontractive type mappings, our aim in the present paper is to establish some convergence theorems for Krasnoselskij type fixed point iterations constructed as admissible perturbations of φ-pseudocontractions in Hilbert spaces, thus obtaining general results that unify and extend several results in literature (see [2, 3] and references therein).

The paper is organized as follows. In Section 2, we present concepts related to admissible mappings and admissible perturbations of a nonlinear operator [15]. Section 3 deals with main aspects and results related to φ-contractions. In Section 4, we prove our main results related to the approximation of fixed points of φ-pseudocontractive operators by means of general iterative methods defined as admissible perturbations of a nonlinear mapping.

2. Admissible perturbations of an operator

Definition 1 ([15])**.** Let X be a nonempty set. A mapping $G : X \times X \to X$ is called *admissible* if it satisfies the following two conditions:
(A_1) $G(x,x) = x$, for all $x \in X$;
(A_2) $G(x,y) = x$ implies $y = x$.

Definition 2 ([15])**.** Let X be a nonempty set. If $f : X \to X$ is a given mapping and $G : X \times X \to X$ is an admissible mapping, then the mapping $f_G : X \to X$, defined by

$$f_G(x) = G(x, f(x)),\ \forall x \in X, \tag{2.1}$$

is called the *admissible perturbation* of f.

Remark 2. The following property of admissible perturbations is fundamental in the iterative approximation of fixed points: if $f : X \to X$ is a given mapping and $f_G : X \to X$ denotes its admissible perturbation, then

$$Fix(f_G) = Fix(f) := \{x \in X | x = f(x)\}, \tag{2.2}$$

that is, the admissible perturbation f_G of f has the same set of fixed points as the operator f itself.

Note that, in general,

$$Fix(f_G^n) \neq Fix(f^n),\ n \geq 2. \tag{2.3}$$

Example 1 ([15]). Let $(V, +, \mathbb{R})$ be a real vector space, $X \subset V$ a convex subset, $\lambda \in (0,1)$, $f : X \to X$ and $G : X \times X \to X$ be defined by

$$G(x, f(x)) := (1-\lambda)x + \lambda f(x),\ x \in X.$$

Then f_G is an admissible perturbation of f. We shall denote f_G by f_λ and call it the Krasnoselskij perturbation of f.

Example 2 ([15]). Let $(V, +, \mathbb{R})$ be a real vector space, $X \subset V$ a convex subset, $\chi : X \times X \to (0,1)$, $f : X \to X$ and $G(x,y) := (1-\chi(x,y))x + \chi(x,y)y$.

Then f_G is an admissible perturbation of f which reduces to the Krasnoselskij perturbation in the case $\chi(x,y)$ is a constant function.

For other important examples of admissible mappings and admissible perturbations of nonlinear operators, see [15] (for the case of self mappings) and [7] (for the case of nonself mappings).

Definition 3 ([15]). Let $f : X \to X$ be a nonlinear mapping and $G : X \times X \to X$ an admissible mapping. Then the iterative scheme $\{x_n\}_{n=0}^{\infty}$ given by $x_0 \in X$ and

$$x_{n+1} = G(x_n, f(x_n)), n \geq 0, \tag{2.4}$$

is called the *Krasnoselskij iterative scheme* corresponding to G which we denote by *GK-iterative scheme for simplicity*.

Remark 3. In the particular case

$$G(x, y) := (1-\lambda)x + \lambda y, \, x, y \in X, \tag{2.5}$$

the GK-iterative scheme (2.4) reduces to the classical Krasnoselskij iterative scheme (1.5).

Definition 4. Let $G : E \times E \to E$ be an admissible operator on a normed space E. We say that G is *affine Lipschitzian* if there exists a constant $\mu \in [0, 1]$ such that

$$\|G(x_1, y_1) - G(x_2, y_2)\| \leq \|\mu(x_1 - x_2) + (1-\mu)(y_1 - y_2)\|, \tag{2.6}$$

for all $x_1, x_2, y_1, y_2 \in E$.

Note that the admissible operator $G : E \times E \to E$ given by (2.5) is affine Lipschitzian. Note also that in the very recent paper [4], Definition 3.7, we used a weaker concept of affine Lipschitzianity. It is easy to show that, if G is affine Lipschitzian in the sense of Definition 4 in the present paper, then it is also affine Lipschitzian in the sense of Definition 3.7 in [4], but the reverse implication is not true, in general.

3. Fixed point theorems for φ-contractions

In this section we present a minimal list of concepts and results on φ-contractions that will be needed in the sequel. They are mainly taken from the monograph [3].

Definition 5. Let (X, d) be a complete metric space. $T : X \to X$ is called a (*strict*) *Picard operator* if T has a unique fixed point, i.e, $Fix\,T = \{x^*\}$, and

$$T^n(x_0) \to x^* \text{ (uniformly) for all } x_0 \in X.$$

Example 3. *(Banach Contraction Principle)*

If (X, d) is a complete metric space, then any contraction $T : X \to X$, i.e., any mapping satisfying

$$d(Tx, Ty) \leq \alpha \cdot d(x, y), \quad \forall\, x, y \in X, \tag{3.1}$$

where $0 < \alpha < 1$, is a Picard operator.

Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function. In connection with the function φ we consider the following properties:

(i_φ) φ is monotone increasing, i.e., $t_1 \leq t_2$ implies $\varphi(t_1) \leq \varphi(t_2)$;

(ii_φ) $\{\varphi^n(t)\}$ converges to 0 for all $t \geq 0$;

(iii_φ) $\sum_{n=0}^{\infty} \varphi^n(t)$ converges for all $t > 0$.

Definition 6. A function φ satisfying (i_φ) and (ii_φ) is said to be a *comparison function*;
2) A function φ satisfying (i_φ) and (iii_φ) is said to be a *(c)-comparison function*.

The following assertions are immediate consequences of the previous definition. For other properties see [3], Chapter 2.
1) Any (c)-comparison function is a comparison function ;
2) If φ is a (c)-comparison function, then the function

$$s : \mathbb{R}_+ \to \mathbb{R}_+, \;\; s(t) = \sum_{k=0}^{\infty} \varphi^k(t), \; t \in \mathbb{R}_+ \tag{3.2}$$

is continuous at 0 and satisfies (i_φ).

Example 4 (Example 2.8 in [3]). 1) $\varphi(t) = at$, $t \in \mathbb{R}_+$, $a \in [0,1)$ satisfies all conditions (i_φ)-(iii_φ);
2) $\varphi(t) = \dfrac{t}{1+t}$, $t \in \mathbb{R}_+$ is a comparison function but not a (c)-comparison function;
3) $\varphi(t) = \dfrac{1}{2}t$, if $0 \le t \le 1$ and $\varphi(t) = t - \dfrac{1}{3}$, if $t > 1$ is a discontinuous (c)-comparison function.

Definition 7. Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be a ’*-contraction* if there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d(Tx,Ty) \le \varphi(d(x,y)), \;\; \text{for all } x,y \in X. \tag{3.3}$$

A general fixed point principle for φ-contractions is given by the well-known Matkowski-Rus theorem (see [16], page 31), established for comparison functions (part (i) and (ii)) and strict comparison functions (part (iii)), respectively. Here we consider a particular version of it which additionally provides an posteriori error estimate for Picard iteration.

Theorem 1 (Theorem 2.8 in [3])**.** *Let (X,d) be a complete metric space and $T : X \to X$ a $\varphi-$contraction with φ a (c)-comparison function. Then*
(i) $Fix\,(T) = \{x^*\}$;
(ii) The Picard iterative scheme $\{x_n\} = \{T^n x_0\}_{n \in \mathbb{N}}$ converges to x^* (as $n \to \infty$), for any $x_0 \in X$;
(iii) The following estimate holds

$$d(x_n,x^*) \le s(d(x_n,x_{n+1})), \;\; n = 0,1,2,\ldots, \tag{3.4}$$

where $s(t) = \sum\limits_{k=0}^{\infty} \varphi^k(t)$ is the sum of the comparison series.

Note that if φ is the comparison function given in Example 4, then by Theorem 1, we obtain the well known Banach Contraction Mapping Principle.

It is possible to extend Theorem 1, the so called generalized φ-contractions by replacing (3.3) with the following (see [3], Chapter 2):
there exists a 5-dimensional comparison function $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \leq \varphi(d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)).$$

4. APPROXIMATING FIXED POINTS OF φ-PSEUDOCONTRACTIONS

Definition 8 ([2])**.** Let H be a real Hilbert. An operator $T : K \subset H \to H$ is said to be (strictly) φ-pseudocontractive if, for given $a, b, c \in \mathbb{R}_+$ with $a+b+c=1$, there exists a (comparison) function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$a \cdot \|x-y\|^2 + b\,\langle Tx - Ty, x-y\rangle + c\|Tx - Ty\|^2 \leq \varphi^2\big(\|x-y\|\big), \tag{4.1}$$

holds, for all $x, y \in K$.

The class of φ-pseudocontractions includes various other important classes of contractive and pseudocontractive type operators, as partially illustrated by the following examples.

Example 5. Any pseudocontractive operator T is also $\varphi-$pseudocontractive with $a=0, b=1, c=0$ and $\varphi(t)=t$, by virtue of the equivalent condition (1.7).

Example 6. Any generalized pseudocontractive operator T is a φ-pseudocontractive operator, with $a=0, b=1,\ c=0$ and $\varphi(t) = r \cdot t,\ \ r>0$.

Example 7. Any strongly pseudocontractive operator T is a strictly φ-pseudocontractive operator, with $a=0, b=1,\ c=0$ and $\varphi(t) = k \cdot t,\ \ k \in (0,1)$.

Example 8. Any strictly pseudocontractive operator T is a φ-pseudocontractive operator, with $a = \dfrac{k-1}{2k},\ b=1,\ c=\dfrac{1-k}{2k}$ and $\varphi(t)=t$.

Example 9. Any Lipschitzian operator T (and hence any contraction and any nonexpansive operator) is φ-pseudocontractive with $a=0, b=0, c=1$ and $\varphi(t) = L \cdot t,\ L>0$.

The main result of this paper is the following convergence theorem, which extends Theorem 1 from the case of Krasnoselskij iterative scheme (1.5) to the case of more general GK-iterative scheme (2.4).

Theorem 2. *Let K be a nonempty closed and convex subset of a real Hilbert space H and $T : K \to K$ a strictly φ-pseudocontractive. Then*
(i) T has a unique fixed point p in K;
(ii) If $G : K \times K \to K$ is an affine Lipschitzian admissible operator with constant $\lambda \in (0,1)$, then the GK-iterative scheme $\{x_n\}_{n=0}^{\infty}$ given by x_0 in K and

$$x_{n+1} = G(x_n, f(x_n)), n \geq 0, \tag{4.2}$$

converges (strongly) to p, for any $x_0 \in K$;

(iii) If, additionally φ is a (c)-comparison function, then the following estimate

$$\|x_n - p\| \le s\big(\|x_n - x_{n-1}\|\big), \quad n = 1, 2, \dots$$

holds, with $s(t) = \sum\limits_{k=0}^{\infty} \varphi^k(t)$.

Proof. We consider the admissible perturbation operator $F : K \to K$, associated with T and defined by

$$Fx = G(x, Tx),\, x \in K. \tag{4.3}$$

By the fundamental property of an admissible mapping in (2.2), we have

$$Fix(F) = Fix(T). \tag{4.4}$$

Moreover, as the admissible mapping G is affine Lipschitzian, so there exists a constant $\lambda \in [0,1]$ such that

$$\|G(x, Tx) - G(y, Ty)\| \le \|(1-\lambda)(x-y) + \lambda(Tx - Ty)\|,$$

for all $x, y \in K$. Then

$$\begin{aligned}\|Fx - Fy\|^2 &= \|G(x,Tx) - G(y,Ty)\|^2 \\ &\le \|(1-\lambda)(x-y) + \lambda(Tx-Ty)\|^2 = (1-\lambda)^2 \cdot \|x-y\|^2 + 2\lambda(1-\lambda)\cdot \\ &\qquad \cdot \langle Tx - Ty, x - y\rangle + \lambda^2 \cdot \|Tx - Ty\|^2 .\end{aligned}$$

Now put $a = (1-\lambda)^2 \ge 0$, $b = 2\lambda(1-\lambda) \ge 0$ and $c = \lambda^2 \ge 0$. We have $a + b + c = (1-\lambda)^2 + 2\lambda(1-\lambda) + \lambda^2 = 1$ and since T is a strictly φ-pseudocontractive, it results that there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$\|Fx - Fy\|^2 \le \varphi^2\big(\|x-y\|\big), \quad \forall\, x, y \in K,$$

which yields

$$\|Fx - Fy\| \le \varphi\big(\|x-y\|\big), \quad \forall\, x, y \in K.$$

This means that F is a φ-contraction, and hence by Theorem 1, the conclusions of our theorem follow. □

Our result unifies and generalizes many convergence theorems in the existing literature.

Remark 4. 1) If T is strongly pseudocontractive, that is, T satisfies (1.8) with $k \in (0,1)$, then T is a strictly φ-pseudocontraction and hence, by Theorem 2, T has a unique fixed point p in K and the GK-iterative scheme converges strongly to p.

2) If T is a generalized pseudocontractive mapping with constant $r < 1$ and Lipschitzian with constant $s \ge 1$, then by Theorem 2, we obtain a generalization of the main result in [6].

3) In case $G(x, y)$ is the admissible mapping given by (2.5), then by Theorem 2, we obtain the main results in [2] (see also Theorem 3.8 in [3]).

Corollary 1. *Let K be a nonempty closed and convex subset of a Hilbert space H and let $T : K \to K$ be a generalized pseudocontractive and Lipschitzian operator with the corresponding constants $r < 1$ and $s \geq 1$, respectively. Then*

(i) T has an unique fixed point p in K;

(ii) If $G : K \times K \to K$ is an affine Lipschitzian admissible operator with constant $\lambda \in (0,1)$, then the GK-iterative scheme $\{x_n\}_{n=0}^{\infty}$ given by x_0 in K and

$$x_{n+1} = G(x_n, f(x_n)), n \geq 0,$$

converges (strongly) to p for all $\lambda \in (0,1)$ satisfying

$$0 < \lambda < 2(1-r)/(1-2r+s^2).$$

(iii) The priori

$$\| x_n - p\| \leq \frac{\theta^n}{1-\theta} \cdot \| x_1 - x_0\| \, , \; n = 1,2,\dots$$

and the posteriori

$$\| x_n - p\| \leq \frac{\theta}{1-\theta} \cdot \| x_n - x_{n-1}\| \, , \; n = 1,2,\dots$$

estimates hold, with

$$\theta = \left((1-\lambda)^2 + 2\lambda(1-\lambda)r + \lambda^2 s^2\right)^{1/2}.$$

In case $G(x,y)$ is the admissible mapping given by (2.5), then by Corollary 1, we obtain the main results in [1] and [20] (see also [3], Theorem 3.6).

A more general result than the one given in Theorem 2, can be similarly proven, if we consider generalized φ-pseudocontractive operators instead of φ-pseudocontractions, as shown by the next theorem.

Theorem 3. *Let K be a nonempty closed and convex subset of a real Hilbert space H and $T : K \to K$ a strictly generalized φ-pseudocontraction with $\varphi : \mathbb{R}_+^5 \to \mathbb{R}_+$ satisfying:*

(i) *$\psi(t) = \varphi(t,t,t,t,t)$ is a continuous comparison function;*

(ii) *$h(t) = t - \psi(t)$ is increasing and bijective.*

Then

(i) *T has a unique fixed point p in K;*

(ii) *If $G : K \times K \to K$ is an affine Lipschitzian admissible operator with constant $\lambda \in (0,1)$, then the GK-iterative scheme $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_{n+1} = G(x_n, Tx_n), \quad n = 0,1,2,\dots$$

converges strongly to p, for all $\lambda \in (0,1)$.

(iii) $\quad \|x_n - p\| \leq \psi^n\left(h^{-1}(\|x_0 - x_1\|)\right), \quad n \geq 1.$

Proof. The proof is similar to that of Theorem 2 with the only difference that, we apply the generalized φ-contraction principle ([3],Theorem 2.10) instead of Theorem 2. $\square$

Remark 5. If T is not a strictly φ-pseudocontraction, then the conclusion of Theorem 2 (and 1) is no longer valid. If for example, $\varphi(t) = t$, then the admissible perturbation of T, $Fx = G(x, Tx)$, is nonexpansive but in general it is not a contraction (or a φ-contraction). This problem will be considered in a forthcoming paper (cf. [4]).

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REFERENCES

[1] V. Berinde, "Approximating fixed points of Lipschitzian generalized pseudo-contractions," in *Proceedings of the 3rd international Palestinian conference on mathematics and mathematics education, Bethlehem, Palestine, August 9–12, 2000*, S. Elaydi, Ed. Singapore: World Scientific, 2002, pp. 73–81.

[2] V. Berinde, "Approximating fixed points of weak φ-contractions using the Picard iteration," *Fixed Point Theory*, vol. 4, no. 2, pp. 131–142, 2003.

[3] V. Berinde, *Iterative approximation of fixed points*, 2nd ed. Berlin: Springer, 2007.

[4] V. Berinde, "Convergence theorems for fixed point iterative methods defined as admissible perturbations of a nonlinear operator," *Carpathian J. Math.*, vol. 29, no. 1, pp. 9–18, 2013.

[5] V. Berinde, A. R. Khan, and H. Fukhar-ud-din, "Fixed point iterative methods defined as admissible perturbations of generalized pseudocontractive operators," *J. Nonlinear Convex Anal.*, accepted.

[6] V. Berinde and G. Kovács, "Stabilizing discrete dynamical systems by monotone Krasnoselskij type iterative schemes," *Creat. Math. Inform.*, vol. 17, no. 3, pp. 298–307, 2008.

[7] V. Berinde, Şt. Măruşter, and I. A. Rus, "An abstract point of view on iterative approximation of fixed points of nonself operators," *J. Nonlinear Convex Anal.*, accepted.

[8] F. E. Browder, "Nonlinear monotone and accretive operators in Banach spaces," *Proc. Natl. Acad. Sci. USA*, vol. 61, pp. 388–393, 1968.

[9] C. Chidume, *Geometric properties of Banach spaces and nonlinear iterations*, ser. Lecture Notes in Mathematics. Berlin: Springer, 2009, vol. 1965.

[10] S. Ishikawa, "Fixed points by a new iteration method," *Proc. Am. Math. Soc.*, vol. 44, pp. 147–150, 1974.

[11] M. A. Krasnosel'skij, "Zwei Bemerkungen über die Methode der sukzessiven Approximationen," *Usp. Mat. Nauk*, vol. 10, no. 1(63), pp. 123–127, 1955.

[12] K. Kumar and B. K. Sharma, "A generalized iterative algorithm for generalized successively pseudocontractions," *Appl. Math. E-Notes*, vol. 6, pp. 202–210, 2006.

[13] W. R. Mann, "Mean value methods in iteration," *Proc. Am. Math. Soc.*, vol. 4, pp. 506–510, 1953.

[14] M. Păcurar, *Iterative methods for fixed point approximation*. Cluj-Napoca: Editura Risoprint, 2009.

[15] I. A. Rus, "An abstract point of view on iterative approximation of fixed points: impact on the theory of fixed point equations," *Fixed Point Theory*, vol. 13, no. 1, pp. 179–192, 2012.

[16] I. A. Rus, A. Petruşel, and G. Petruşel, *Fixed point theory*. Cluj-Napoca: Cluj University Press, 2008.
[17] D. R. Sahu, N. C. Wong, and J. C. Yao, "A unified hybrid iterative method for solving variational inequalities involving generalized pseudocontractive mappings," *SIAM J. Control Optim.*, vol. 50, no. 4, pp. 2335–2354, 2012.
[18] Ş. M. Şoltuz, "Mann iteration for generalized pseudocontractive maps in Hilbert spaces," *Math. Commun.*, vol. 6, no. 1, pp. 97–100, 2001.
[19] Ş. M. Şoltuz, "Two Mann iteration types for generalized pseudocontractive maps," in *Fixed point theory and applications. Vol. 2. Proceedings of the 6th international conference on nonlinear functional analysis and applications, Masan, Korea, September 1–5, 2000.*, Y. J. Cho, Ed. Huntington: Nova Science Publishers, 2001, pp. 105–110.
[20] R. U. Verma, "A fixed-point theorem involving Lipschitzian generalised pseudo-contractions," *Proc. R. Ir. Acad., Sect. A*, vol. 97, no. 1, pp. 83–86, 1997.
[21] R. U. Verma, "Generalized pseudo-contractions and nonlinear variational inequalities," *Publ. Math.*, vol. 53, no. 1-2, pp. 23–28, 1998.

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