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STUDIES ON CONCAVE YOUNG-FUNCTIONS

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ABSTRACT. We succeeded in isolating a special class of concave Young-functions enjoying the so-called *density-level property*. In this class there is a proper subset whose members have each the so-called degree of contraction denoted by c^*, and map bijectively the interval $[c^*, \infty)$ onto itself. We constructed the fixed point of each of these functions. Later we proved that every positive number b is the fixed point of a concave Young-function having b as degree of contraction. We showed that every concave Young-function is square integrable with respect to a specific Lebesgue measure. We also proved that the distance generated by the L^2-norm is a metric in the set of concave Young-functions and then derived that the concave Young-functions possessing the density-level property constitute a dense set in the space of concave Young-functions.

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Keywords: concave Young-functions, degree of contraction, fixed points, dense set

1. INTRODUCTION

Let $\varphi : (0, \infty) \to (0, \infty)$ be a right-continuous and decreasing function such that it is integrable on every finite interval $(0, x)$. It is easily seen that the function $\Phi : [0, \infty) \to [0, \infty)$, defined by

$$\Phi(x) = \int_0^x \varphi(t)\, dt, \tag{1.1}$$

is a nonnegative, increasing and concave function with $\Phi(0) = 0$. We further assume that $\Phi(\infty) = \infty$ (Φ is referred to as *concave Young-function* in the literature [4].) We note that if Φ is a concave Young-function, then so is $b\Phi$ for all positive constants b. We shall recall the following definition and result in [1].

Definition A. We say that for the concave Young-function Φ the maximal inequality is valid with some positive constant K_Φ (depending only on Φ) if for an arbitrary nonnegative submartingale $(X_n, \mathcal{F}_n)$, $n \geq 1$, the inequality

$$E\Phi(X_n^*) \leq K_\Phi (1 + EX_n)$$

holds for all $n \geq 1$, where $X_n^* = \max_{1\leq k\leq n} X_k$.

Theorem B. *Let Φ be any concave Young-function. In order that Φ satisfy the maximal inequality, it is necessary and sufficient that*

$$A_\Phi(\infty) := \int_1^\infty \frac{\varphi(t)}{t} dt < \infty.$$

Moreover, if $A_\Phi(\infty) < \infty$, then $K_\Phi = \max(\Phi(1), A_\Phi(\infty))$.

Theorem C ([3, p. 205])**.** *Each non-empty subset B of a metric space X is a metric space, the distance in B being the same as in X.*

We shall say that a concave Young-function Φ satisfies the *density-level property* if $A_\Phi(\infty) < \infty$. The quantity $A_\Phi(\infty)$ will be referred to as *density-level* and the function $A_\Phi : [1, \infty) \to [0, \infty)$, defined by

$$A_\Phi(x) = \int_1^x \frac{\varphi(t)}{t} dt,$$

will be called *density-level function*.

For instance the concave Young-functions $\Phi_1(x) = \sqrt{x}$ and $\Phi_2(x) = \ln(x+1)$, defined for $x \in [0, \infty)$, have finite density-levels. The concave Young-function $\Phi_3(x) = 2x + 1 - e^{-x}$ is of infinite density-level. In fact, if we let $\varphi_3(x)$ stand for the derivative of function $\Phi_3(x)$, then

$$A_{\Phi_3}(\infty) = \int_1^\infty \frac{\varphi_3(t)}{t} dt \geq \int_1^\infty \frac{2}{t} dt = \infty.$$

Theorem B suggests that the set of concave Young-functions that satisfy the density-level property is a rather broad class.

Define function $A_\Phi^* : (0, \infty) \to (0, \infty]$ by

$$A_\Phi^*(b) = \int_b^\infty \frac{\varphi(x)}{x} dx,$$

where $\Phi \in \mathcal{Y}_{\text{conc}}$.

It is not difficult to see that $A_{\Phi_1}^*(b) < \infty$ and $A_{\Phi_3}^*(b) = \infty$, for any number $b \in (0, \infty)$, where functions $\Phi_1(x) = \sqrt{x}$ and $\Phi_3(x) = 2x + 1 - e^{-x}$ are defined for $x \in [0, \infty)$.

Remark 1. The function $x^{-1}\Phi(x)$ is decreasing on the interval $(0, \infty)$ and

$$0 \leq \lim_{x\to\infty} \frac{\Phi(x)}{x} < \infty.$$

Notation. $\mathcal{Y}_{\text{conc}}$ will stand for the collection of all concave Young-functions. $\mathcal{A}$ will denote the set of all functions $\Phi \in \mathcal{Y}_{\text{conc}}$ that satisfy the density-level property.

We note that $\mathcal{A}$ is a proper subset of $\mathcal{Y}_{\text{conc}}$, since the concave Young-function $\Phi_3 : [0, \infty) \to [0, \infty)$, defined above by $\Phi_3(x) = 2x + 1 - e^{-x}$, was shown to be of infinite density-level.

We recall the following fact: *A function T from a metric space (M, ϱ) to itself is called a contraction if there is an α which satisfies $0 \leq \alpha < 1$ so that*

$$\varrho(T(x), T(y)) \leq \alpha \varrho(x, y)$$

for all x, $y \in M$.

We also recall the following well-known principle.

Contraction Mapping Principle ([5])**.** *Let T be a contraction on a complete metric space (M, ϱ). Then there is a unique point $x \in M$ (called* fixed point*) such that $T(x) = x$. Furthermore, if x_0 is any point in M and we define $x_{n+1} = T(x_n)$, then $\lim_{n\to\infty} x_n = x$.*

In this communication we study, among others, the closure of $\mathcal{A}$ under the composition operation. In a sense, Theorems 1 and 2 show that the concave Young-functions with the density-level property behave like left and right ideals with respect to the composition operation. We also realize that not every function $\Phi \in \mathcal{A}$ admits a fixed point. The investigation in this direction leads us to isolate a proper subset $\mathcal{A}_1$ of $\mathcal{A}$ such that every function $\Phi \in \mathcal{A}_1$ possesses the so-called *degree of contraction*, which is closely related to the fixed point of Φ if it exists. We show that every concave Young-function is square integrable with respect to a specific given Lebesgue measure, and we prove that the natural distance defined by the L^2-norm satisfies the metric axioms in $\mathcal{Y}_{\text{conc}}$. We then demonstrate that the subset $\mathcal{A}$ proves to be a dense set in $\mathcal{Y}_{\text{conc}}$.

2. The closure of $\mathcal{A}$ under addition and composition operations

Remark 2. For every number $s \in (0, \infty)$ we have that $s\varphi(s) < \Phi(s)$.

Proof. Fix arbitrarily two numbers $s \in (0, \infty)$ and $b \in (0, s)$. Then by applying twice the fact that φ decreases on $(0, \infty)$, we have that

$$\begin{aligned}\Phi(s) &= \int_0^s \varphi(t)\,dt = \int_0^b \varphi(t)\,dt + \int_b^s \varphi(t)\,dt \geq b\varphi(b) + (s-b)\varphi(s) \\ &> b\varphi(s) + (s-b)\varphi(s) = s\varphi(s),\end{aligned}$$

as required. □

The following remark is an immediate consequence of Theorem B.

Remark 3. Let $\Phi \in \mathcal{Y}_{\text{conc}}$. If $\Phi \in \mathcal{A}$, then $\Phi(x) \leq K_\Phi(1 + x)$ for all $x \in (0, \infty)$, where $K_\Phi = \max(\Phi(1), A_\Phi(\infty))$.

Remark 4. The composition of two concave Young-functions is also a concave Young-function.

The following two lemmas are trivial.

Lemma 1. *For any number $b \in (0, \infty)$ and function $\Phi \in \mathcal{Y}_{\text{conc}}$, we have that $b\Phi \in \mathcal{A}$ if and only if $\Phi \in \mathcal{A}$. Moreover, $A_{b\Phi}(x) = bA_\Phi(x)$, $x \in [1, \infty)$.*

Lemma 2. *Let functions Φ_1 and $\Phi_2 \in \mathcal{Y}_{\text{conc}}$ be arbitrary. Then Φ_1 and $\Phi_2 \in \mathcal{A}$ if and only if $\Phi_1 + \Phi_2 \in \mathcal{A}$. Furthermore,*

$$A_{\Phi_1+\Phi_2}(x) = A_{\Phi_1}(x) + A_{\Phi_2}(x), \qquad x \in [1, \infty).$$

Theorem 1. *Let functions Φ_1 and $\Phi_2 \in \mathcal{Y}_{\text{conc}}$ be arbitrary. If $\Phi_2 \in \mathcal{A}$, then $\Phi_1 \circ \Phi_2 \in \mathcal{A}$.*

Proof. Write φ_i for the derivative of Φ_i ($i \in \{1, 2\}$). Compute the density-level of the composition $\Phi_1 \circ \Phi_2$.

$$\begin{aligned} A_{\Phi_1\circ\Phi_2}(\infty) &= \int_1^\infty \frac{\varphi_2(x)\,\varphi_1(\Phi_2(x))}{x}dx \\ &\le \varphi_1(\Phi_2(1)) \int_1^\infty \frac{\varphi_2(x)}{x}dx = \varphi_1(\Phi_2(1))\,A_{\Phi_2}(\infty) < \infty, \end{aligned}$$

via the monotonicity of function φ_1. □

Remark 5. Let $\Phi \in \mathcal{Y}_{\text{conc}}$. Then for Φ to belong to $\mathcal{A}$ it is necessary that

$$\lim_{t\to\infty} \varphi(t) = 0.$$

Proof. Assume that $\Phi \in \mathcal{A}$ but $\lim_{t\to\infty} \varphi(t) = l_0 > 0$. Pick an arbitrarily fixed number $t \in (1, \infty)$. Then

$$\infty > A_\Phi(\infty) \ge \int_1^t \frac{\varphi(x)}{x}dx \ge \varphi(t)\log(t) > l_0 \log(t).$$

Passing to the limit, it will follow that $\infty = A_\Phi(\infty) < \infty$, which is absurd. This completes the proof. □

The following remark suggests that if $\Phi \in \mathcal{Y}_{\text{conc}}$, then either $A^*_\Phi(b) = \infty$ for all $b \in (0, \infty)$, or $A^*_\Phi(b) < \infty$ for all $b \in (0, \infty)$.

Remark 6. Let $\Phi \in \mathcal{Y}_{\text{conc}}$. Then $A^*_\Phi(b) < \infty$ for every constant $b \in (0, \infty) \setminus \{1\}$ if and only if $A_\Phi(\infty) < \infty$.

Proof. A simple computation shows that

$$A^*_\Phi(b) = \int_b^\infty \frac{\varphi(x)}{x}dx = \begin{cases} A_\Phi(\infty) + \int_b^1 \frac{\varphi(x)}{x}dx & \text{if } b < 1 \\ A_\Phi(\infty) - \int_1^b \frac{\varphi(x)}{x}dx & \text{if } b > 1, \end{cases}$$

which yields the result. □

Theorem 2. *Let functions Φ_1 and $\Phi_2 \in \mathcal{Y}_{\text{conc}}$ be arbitrary. If $\Phi_1 \in \mathcal{A}$, then $\Phi_1 \circ \Phi_2 \in \mathcal{A}$.*

Proof. We first show that

$$A_{\Phi_1}(\infty) = \int_{\Phi_2^{-1}(1)}^\infty \frac{\varphi_2(t)\,\varphi_1(\Phi_2(t))}{\Phi_2(t)}dt,$$

where Φ_2^{-1} is the inverse function of Φ_2 (whose existence is guaranteed by the continuity of Φ_2).

In fact, by definition we have that

$$A_{\Phi_1}(\infty) = \int_1^\infty \frac{\varphi_1(x)}{x} dx.$$

Now, setting $x = \Phi_2(t)$ we observe that $dx = \varphi_2(t)\,dt$ and thus

$$A_{\Phi_1}(\infty) = \int_{\Phi_2^{-1}(1)}^\infty \frac{\varphi_2(t)\,\varphi_1(\Phi_2(t))}{\Phi_2(t)} dt.$$

Next, compute the density-level of the composition $\Phi_1 \circ \Phi_2$. Remark 1 implies that

$$\begin{aligned} A_{\Phi_1 \circ \Phi_2}(\infty) &= \int_1^\infty \frac{\varphi_2(t)\,\varphi_1(\Phi_2(t))}{t} dt \\ &= \int_1^\infty \frac{\Phi_2(t)}{t} \frac{\varphi_2(t)\,\varphi_1(\Phi_2(t))}{\Phi_2(t)} dt \\ &\le c \int_{\Phi_2^{-1}(1)}^\infty \frac{\varphi_2(t)\,\varphi_1(\Phi_2(t))}{\Phi_2(t)} dt = c A_{\Phi_1}(\infty), \end{aligned}$$

where $c = 1/\Phi_2^{-1}(1)$ (the second equality holds because of the claim shown above), which was to be proven. □

Corollary 1. *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ and $\alpha \in (0,1)$ be arbitrary. Then $\Phi_\alpha \in \mathcal{A}$, where the function $\Phi_\alpha : [0,\infty) \to [0,\infty)$ is defined by $\Phi_\alpha(x) = \Phi^\alpha(x) = (\Phi(x))^\alpha$.*

Proposition 1. *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ be arbitrary and fix any number $s \in (0,\infty)$. Then*

$$|\Phi(x) - \Phi(y)| \le \varphi(s)\,|x-y|$$

for all numbers $x,\ y \in [s,\infty)$.

Proof. Pick numbers $x,\ y \in [s,\infty)$ arbitrarily. Via the monotonicity of Φ it follows that

$$\begin{aligned} |\Phi(x) - \Phi(y)| &= \max(\Phi(x),\ \Phi(y)) - \min(\Phi(x),\ \Phi(y)) \\ &= \Phi(\max(x,\ y)) - \Phi(\min(x,\ y)). \end{aligned}$$

Hence the monotonicity of φ yields that

$$|\Phi(x) - \Phi(y)| = \int_{\min(x,y)}^{\max(x,y)} \varphi(t)\,dt \le \varphi(s)(\max(x,\ y) - \min(x,\ y)) = \varphi(s)\,|x-y|.$$

This was to be proved. □

We shall similarly show the following proposition.

Proposition 2. *Let $\Phi \in \mathcal{A}$ be arbitrary and fix any number $s \in [1,\infty)$. Then*

$$|A_\Phi(x) - A_\Phi(y)| \le \varphi(s)\,|x-y|$$

for all numbers $x,\ y \in [s,\infty)$.

Proof. Pick numbers $x,\ y \in [s, \infty)$ arbitrarily. Via the monotonicity of A_Φ it follows that

$$\begin{aligned}|A_\Phi(x) - A_\Phi(y)| &= \max(A_\Phi(x), A_\Phi(y)) - \min(A_\Phi(x), A_\Phi(y)) \\ &= A_\Phi(\max(x, y)) - A_\Phi(\min(x, y)) \\ &= \int_{\min(x,y)}^{\max(x,y)} \frac{\varphi(t)}{t} dt \leq \frac{\varphi(s)}{s}|x-y| \leq \varphi(s)|x-y|,\end{aligned}$$

because $t^{-1}\varphi(t)$ is a decreasing function. □

Proposition 3. *Let $x,\ y \in (0, \infty)$ and $\Delta \subset \mathcal{Y}_{\text{conc}}$ (with $\Delta \neq \emptyset$) be arbitrary. Then*

$$\left|\sup_{\Phi\in\Delta} \Phi(x) - \sup_{\Phi\in\Delta} \Phi(y)\right| \leq \sup_{\Phi\in\Delta} |\Phi(x) - \Phi(y)|,$$

provided that $\sup_{\Phi\in\Delta} \Phi(t) < \infty$ *for all* $t \in (0, \infty)$.

Proof. We first note that

$$\Phi(x) \leq |\Phi(x) - \Phi(y)| + \Phi(y) \quad \text{and} \quad \Phi(y) \leq |\Phi(x) - \Phi(y)| + \Phi(x).$$

Taking the supremum we can easily observe that

$$\sup_{\Phi\in\Delta} \Phi(x) \leq \sup_{\Phi\in\Delta} |\Phi(x) - \Phi(y)| + \sup_{\Phi\in\Delta} \Phi(y)$$

and

$$\sup_{\Phi\in\Delta} \Phi(y) \leq \sup_{\Phi\in\Delta} |\Phi(x) - \Phi(y)| + \sup_{\Phi\in\Delta} \Phi(x).$$

Combining these inequalities we have that

$$-\sup_{\Phi\in\Delta} |\Phi(x) - \Phi(y)| \leq \sup_{\Phi\in\Delta} \Phi(x) - \sup_{\Phi\in\Delta} \Phi(y) \leq \sup_{\Phi\in\Delta} |\Phi(x) - \Phi(y)|,$$

which yields the result. □

We know that $k\Phi \in \mathcal{Y}_{\text{conc}}$ for any fixed $\Phi \in \mathcal{Y}_{\text{conc}}$ and all $k \geq 1$. Then

$$\sup_{\Phi\in\mathcal{Y}_{\text{conc}}} \Phi(x) \geq \sup_{k\geq 1} k\Phi(x) = \begin{cases} 0 & \text{if } x = 0, \\ \infty & \text{if } x \in (0, \infty), \end{cases}$$

meaning that there is no real function $g(x)$ such that $\Phi(x) \leq g(x)$ for all $\Phi \in \mathcal{Y}_{\text{conc}}$ and $x \in [0, \infty)$. Nevertheless, this is possible for their suitably normalised forms, as shown in the following lemma.

Lemma 3. *The function $S : [0, \infty) \to [0, \infty)$, defined by*

$$S(x) = \sup_{\Phi\in\mathcal{Y}_{\text{conc}}} (\Phi(1))^{-1} \Phi(x),$$

has the following properties:

(1) $S(0) = 0$ *and* $S(1) = 1$.

(2) *S is a non-decreasing function such that*

$$(\Phi(1))^{-1}\,\Phi(x) \leq S(x)$$

for all $\Phi \in \mathcal{Y}_{\mathrm{conc}}$ and $x \in [0,\infty)$.

(3) *The identity*

$$\sup_{\Phi\in\mathcal{Y}_{\mathrm{conc}}} (1+\Phi(1))^{-1} = 1$$

holds.

(4) *For every number $x \in [0,\infty)$, the chain of inequalities $x \leq S(x) \leq x+1$ holds true.*

(5) *We have that $\lim_{x\to\infty} x^{-1}S(x) = 1$ and $\lim_{x\to\infty} S(x) = \infty$.*

Proof. The first part is obvious. We show that $S(x)$ is a non-decreasing function. In fact, pick arbitrarily two numbers $x_1$and $x_2 \in [0,\infty)$ with $x_1 < x_2$. By the monotonicity we have that $\Phi(x_1) < \Phi(x_2)$. If we normalize this inequality suitably and then take the supremum on both sides over all $\Phi \in \mathcal{Y}_{\mathrm{conc}}$ we can then observe that $S(x_1) \leq S(x_2)$. Thus S is a non-decreasing function. To show the identity in the third part we begin by establishing the inequality $(1+\Phi(1))^{-1} \leq 1$, which holds for every $\Phi \in \mathcal{Y}_{\mathrm{conc}}$. Then $\sup_{\Phi\in\mathcal{Y}_{\mathrm{conc}}} (1+\Phi(1))^{-1} \leq 1$. We also know that $k^{-1}\Phi \in \mathcal{Y}_{\mathrm{conc}}$ for any fixed integer $k \geq 1$. Hence

$$\left(1+k^{-1}\Phi(1)\right)^{-1} \leq \sup_{\Phi\in\mathcal{Y}_{\mathrm{conc}}} (1+\Phi(1))^{-1}.$$

Passing to the limit we observe that $\lim_{k\to\infty}(1+k^{-1}\Phi(1))^{-1} = 1$. Consequently $\sup_{\Phi\in\mathcal{Y}_{\mathrm{conc}}} (1+\Phi(1))^{-1} = 1$. The fourth part will be proved if we show that $S(x) \leq x+1$ and $S(x) \geq x$ for all $x \in [0,\infty)$. In fact, take arbitrarily a function $\Phi \in \mathcal{Y}_{\mathrm{conc}}$. Clearly the equation of the tangent line of Φ at the point $(1, \Phi(1))$ is given by $y = \varphi(1)(x-1) + \Phi(1)$, $x \in [0,\infty)$. Via the concavity of Φ, it is obvious that $\Phi(x) \leq \varphi(1)(x-1) + \Phi(1)$, $x \in [0,\infty)$. Hence by Remark 2 we have $\Phi(x) \leq \varphi(1)\,x + \Phi(1) < \Phi(1)(x+1)$, $x \in [0,\infty)$. This implies that $S(x) < x+1$, for all $x \in [0,\infty)$. Finally fix arbitrarily a function $\Phi \in \mathcal{Y}_{\mathrm{conc}}$. Then the function, defined on $[0,\infty)$ by $x + \Phi(x)$ (for any fixed $\Phi \in \mathcal{Y}_{\mathrm{conc}}$), also belongs to $\mathcal{Y}_{\mathrm{conc}}$. Hence

$$S(x) \geq \frac{x+\Phi(x)}{1+\Phi(1)} \geq \frac{x}{1+\Phi(1)},\ x \in [0,\infty).$$

Now taking the supremum over $\Phi \in \mathcal{Y}_{\mathrm{conc}}$, the third part leads to the desired inequality $S(x) \geq x$. To complete the proof we just point out that the fifth part becomes obvious because of the fourth part. □

Lemma 4. *The function $H : [1,\infty) \to [0,\infty)$ defined by*

$$H(x) = \sup_{\Phi\in\mathcal{A}} (\varphi(1))^{-1}\,\Phi(x)$$

is increasing and has the property that

$$|H(x) - H(y)| \leq |x - y|$$

for all $x,\ y \in [1, \infty)$.

3. The fixed points of a class of concave Young-functions

We shall introduce the following notion (probably new, at least in the author's opinion). Some series of examples will justify that it is well founded.

Definition 1. Let $\Phi \in \mathcal{A}$ be arbitrary. The number $c^* \in (0, \infty)$ will be called the degree of contraction of function Φ if there is some constant $\alpha > 1$ such that both the identities

$$\int_{c^*}^{\infty} \frac{\varphi(t)}{t} dt = 1 \text{ and } \int_{c^*}^{\alpha c^*} \frac{\varphi(t)}{t} dt = \varphi(c^*)$$

hold simultaneously. (In this case we shall say that Φ admits the number c^* as its degree of contraction.)

Example 1. For $\Phi(x) = \sqrt{x+1} - 1$, $x \in [0, \infty)$, the degree of contraction of Φ is equal to $\frac{4e^2}{e^4 - 2e^2 + 1} \approx 0.7240616609$ with

$$\alpha = \frac{\left(e^2 - 1\right)^2 e^{2/(e^2+1)-1}}{e^{4/(e^2+1)+2} - 2e^{2/(e^2+1)+1} + 1} \approx 3.175019732.$$

Example 2. For any fixed number $p \in (0, 1)$, the degree of contraction of the function $\Phi_p(x) = x^p$, $x \in [0, \infty)$, is equal to $\left(\frac{p}{1-p}\right)^{\frac{1}{1-p}}$ with $\alpha = \frac{1}{p^{1/(1-p)}}$.

Example 3. The degree of contraction of the function $\log(x + 1)$, $x \in [0, \infty)$, equals $(e - 1)^{-1}$ with $\alpha = \frac{e-1}{e^{1/e}-1} \approx 3.864191634$.

Proposition 4. *Let $\Phi \in \mathcal{A}$ admit the number c^* as its degree of contraction. Then $\varphi(c^*) < 1$.*

Proof. By Definition 1, there is a number $\alpha > 1$ such that both the identities

$$\int_{c^*}^{\infty} \frac{\varphi(t)}{t} dt = 1 \text{ and } \int_{c^*}^{\alpha c^*} \frac{\varphi(t)}{t} dt = \varphi(c^*)$$

hold simultaneously. Consequently we have that

$$1 = \int_{c^*}^{\infty} \frac{\varphi(t)}{t} dt = \int_{c^*}^{\alpha c^*} \frac{\varphi(t)}{t} dt + \int_{\alpha c^*}^{\infty} \frac{\varphi(t)}{t} dt > \int_{c^*}^{\alpha c^*} \frac{\varphi(t)}{t} dt = \varphi(c^*)$$

because

$$0 < \int_{\alpha c^*}^{\infty} \frac{\varphi(t)}{t} dt < \infty$$

by the assumption and the monotonicity of the function $t^{-1}\varphi(t)$ on the interval $(0, \infty)$. This was to be proved. □

On the one hand it is not difficult to verify that 1 is the degree of contraction of function $\Phi(x) = \sqrt{x}$, $x \in [0, \infty)$, with $\alpha = 4$, and 1 is the unique solution of equation $\Phi(x) = x$ on interval $[1, \infty)$. On the other hand, we know, for instance, in Example 3, that $(e-1)^{-1}$ is the degree of contraction of function $\log(x+1)$. Nevertheless,

$$\log\left(\frac{1}{e-1} + 1\right) = \log\left(\frac{e}{e-1}\right) \neq \frac{1}{e-1}.$$

The question thus arises which are those functions $\Phi \in \mathcal{A}$ that are contractions. We shall provide a proper subset of $\mathcal{A}$ enjoying this property.

Theorem 3. *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ and c^* be any positive number. In order that the equality $\Phi(c^*) = c^*$ hold, it is necessary and sufficient that the range of the function $\Phi|_{[c^*,\infty)} : [c^*, \infty) \to [0, \infty)$, defined by the fotmula*

$$\Phi|_{[c^*,\infty)}(x) = \Phi(x),$$

should equal the interval $[c^*, \infty)$.

Proof. Suppose that $\Phi|_{[c^*,\infty)}(c^*) = \Phi(c^*) = c^*$. Obviously Φ is a bijection on $[0, \infty)$. Hence it follows that $\Phi|_{[c^*,\infty)}$ is an injection on $[c^*, \infty)$. Since $\Phi|_{[c^*,\infty)}$ is continuous on $[c^*, \infty)$and tends increasingly to ∞, we have that the range of function $\Phi|_{[c^*,\infty)}$ equals $[\Phi(c^*), \infty) = [c^*, \infty)$, by assumption. Conversely, assume that the range of $\Phi|_{[c^*,\infty)}$ equals interval $[c^*, \infty)$, but, on the contrary, there is some number $y \in (c^*, \infty)$ such that $\Phi(y) = \Phi|_{[c^*,\infty)}(y) = c^*$. By the assumption it is obvious that function $\Phi|_{[c^*,\infty)}$ is surjective on $[c^*, \infty)$. Moreover, $\Phi|_{[c^*,\infty)}$ maps bijectively the interval $[c^*, \infty)$ onto itself because it is also an injection. The monotonicity of Φ yields that $\Phi(c^*) = \Phi|_{[c^*,\infty)}(c^*) < \Phi(y) = c^*$. However, by the bijective property of $\Phi|_{[c^*,\infty)}$, we have that $\Phi(c^*) \geq c^*$. Consequently the inequality $c^* < c^*$ follows. This, however, is absurd. Therefore, we can conclude on the validity of the argument. □

Notation. *$\mathcal{A}_1$ will stand for the collection of all functions $\Phi \in \mathcal{A}$ mapping bijectively the interval $[c^*, \infty)$ onto itself, where c^* is the degree of contraction of Φ.*

Theorem 4. *Let $\Phi \in \mathcal{A}_1$ with c^* its degree of contraction. Endow the interval $[c^*, \infty)$ with the metric $\varrho(\cdot, \cdot) : [c^*, \infty) \times [c^*, \infty) \to [0, \infty)$ defined by $\varrho(x, y) = |x - y|$. Then Φ is a contraction over the metric space $([c^*, \infty), \varrho)$.*

Moreover, the number c^* is the unique solution of the equation $\Phi(x) = x$ on $[c^*, \infty)$.

Proof. We first note that the pair $([c^*, \infty), \varrho)$ is a complete metric space. Combining Propositions 1 and 4 we can easily derive that Φ is a contraction on the metric space $([c^*, \infty), \varrho)$. But since $\Phi(c^*) = c^*$ (via Theorem 3), we deduce, referring to the Contraction Mapping Principle, that the degree of contraction c^* is the unique solution to the equation $\Phi(x) = x$ on the interval $[c^*, \infty)$. This completes the proof. □

Theorem 5. *For every number $b \in (0,\infty)$ there can be found some function $\Phi_b \in \mathcal{A}_1$ with degree of contraction b. Furthermore, number b is the unique solution of equation $\Phi_b(x) = x$ on the interval $[b,\infty)$.*

Proof. Let $b \in (0,\infty)$ be any number and define the function $\Phi_b(t) = \sqrt{bt}$, $t \in [0,\infty)$. Clearly, the derivative of $\Phi_b(t)$ is expressed by $\varphi(t) = \frac{\sqrt{b}}{2\sqrt{t}}$, $t \in (0,\infty)$. On the one hand, an easy calculation shows that

$$\int_b^\infty \frac{\varphi(t)}{t}dt = \sqrt{b}\int_b^\infty \frac{1}{2t\sqrt{t}}dt = \sqrt{b}\lim_{x\to\infty}\left(\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{x}}\right) = 1,$$

and

$$\int_b^{4b} \frac{\varphi(t)}{t}dt = \sqrt{b}\int_b^{4b} \frac{1}{2t\sqrt{t}}dt = \frac{1}{2} = \varphi(b)$$

i. e., number b is the degree of contraction of Φ_b. On the other hand, an easy substitution leads to

$$\Phi_b(b) = \sqrt{b^2} = b.$$

Then Theorem 3 implies that $\Phi_b \in \mathcal{A}_1$. Consequently Theorem 4 entails that Φ_b is a contraction over the metric space $([b,\infty), \varrho)$ and moreover, number b is the unique solution of equation $\Phi_b(x) = x$ on $[b,\infty)$, with ϱ being the metric induced by the absolute value function. This concludes the proof. □

To end this section we should like to point out that the set

$$\{\Phi \in \mathcal{Y}_{\text{conc}} \setminus \mathcal{A} : \Phi \text{ admits a positive fix point}\}$$

is a non-empty set. In fact, it is not hard to check that the function Φ, defined by $\Phi(x) = \frac{x}{2} + \sqrt{x}$ whenever $x \in [0,\infty)$, belongs to $\mathcal{Y}_{\text{conc}} \setminus \mathcal{A}$ and $\Phi(4) = 4$.

4. Is the set $\mathcal{A}$ dense in $\mathcal{Y}_{\text{conc}}$?

We shall answer this question in the affirmative.

Theorem 6. *For any concave Young-function Φ, there exists a sequence $(\Phi_n) \subset \mathcal{A}$ such that (Φ_n) converges pointwise to Φ, i. e., $\lim_{n\to\infty}\Phi_n(x) = \Phi(x)$ whenever $x \in [0,\infty)$.*

Proof. Fix arbitrarily an index $n \geq 1$ and define $\Phi_n(x) = \Phi^{n/(n+1)}(x)$, $x \in [0,\infty)$. Obviously, $(\Phi_n) \subset \mathcal{Y}_{\text{conc}}$ because of Remark 4. So, on the one hand, Corollary 1 yields that $(\Phi_n) \subset \mathcal{A}$. On the other hand, we can easily see in the limit that

$$\lim_{n\to\infty}\Phi_n(x) = \lim_{n\to\infty}\Phi^{n/(n+1)}(x) = \Phi(x)$$

for every $x \in [0,\infty)$. Therefore, we conclude on the validity of the theorem. □

Lemma 5. *Let $\Phi \in \mathcal{Y}_{\text{conc}}$. Then there are constants $C_\Phi > 0$ and $B_\Phi \geq 0$ such that*

$$A_\Phi(\infty) - B_\Phi \leq \int_0^\infty \frac{\Phi(t)}{(t+1)^2}dt \leq C_\Phi + A_\Phi(\infty).$$

Proof. An integration by parts leads to

$$\int_0^\infty \frac{\Phi(t)}{(t+1)^2}dt = \left[\frac{-\Phi(t)}{t+1}\right]_0^\infty + \int_0^\infty \frac{\varphi(t)}{t+1}dt = \int_0^\infty \frac{\varphi(t)}{t+1}dt - B_\Phi, \tag{4.1}$$

where $0 \le B_\Phi := \lim_{t\to\infty} \frac{\Phi(t)}{t+1} < \infty$, as $\frac{\Phi(t)}{t+1} < \frac{\Phi(t)}{t}$ for all $t \in (0,\infty)$. On the one hand,

$$\int_0^\infty \frac{\varphi(t)}{t+1}dt = \int_0^1 \frac{\varphi(t)}{t+1}dt + \int_1^\infty \frac{\varphi(t)}{t+1}dt \le \int_0^1 \frac{\varphi(t)}{t+1}dt + A_\Phi(\infty). \tag{4.2}$$

On the other hand, by the monotonicity of function $\varphi(t)$ and by the change of variables, we have that

$$\int_0^\infty \frac{\varphi(t)}{t+1}dt \ge \int_0^\infty \frac{\varphi(t+1)}{t+1}dt = \int_1^\infty \frac{\varphi(x)}{x}dx = A_\Phi(\infty). \tag{4.3}$$

Consequently if we combine (4.1)–(4.3), one can observe that

$$A_\Phi(\infty) - B_\Phi \le \int_0^\infty \frac{\Phi(t)}{(t+1)^2}dt \le \int_0^1 \frac{\varphi(t)}{t+1}dt + B_\Phi + A_\Phi(\infty).$$

This leads to the desired result. □

Lemma 5 suggests that the quantity $\int_0^\infty \frac{\Phi(t)}{(t+1)^2}dt$ and the density-level $A_\Phi(\infty)$ are equivalent, in the sense that they are both either finite or infinite. This gives rise to the following essential result.

Lemma 6. *Let $\Phi \in \mathcal{Y}_{\mathrm{conc}}$ be arbitrary. Then*

$$\int_0^\infty \frac{(\Phi(x))^2}{(x+1)^4}dx < \infty.$$

Proof. Clearly,

$$\begin{aligned}\int_0^\infty \frac{(\Phi(x))^2}{(x+1)^4}dx &= \int_0^1 \frac{(\Phi(x))^2}{(x+1)^4}dx + \int_1^\infty \frac{(\Phi(x))^2}{(x+1)^4}dx \\ &\le \int_0^1 \frac{(\Phi(x))^2}{(x+1)^4}dx + \int_1^\infty \frac{(\Phi(x))^2}{x^4}dx.\end{aligned}$$

Integration by parts yields

$$\int_1^\infty \frac{(\Phi(x))^2}{x^4}dx = \frac{\Phi(1)}{3} + \frac{2}{3}\int_1^\infty \frac{\varphi(x)\Phi(x)}{x^3}dx \le \frac{\Phi(1)}{3} + \frac{2\varphi(1)\Phi(1)}{3},$$

because $\varphi(x)$ and $\frac{\Phi(x)}{x}$ are decreasing functions. □

Now endow the half line $[0,\infty)$ with a σ-algebra $\mathcal{M}$ containing the Borel sets. Define a Lebesgue measure $\mu : \mathcal{M} \to [0,\infty)$ by setting

$$\mu([0,\ x)) = \frac{1}{3}\left(1 - \frac{1}{(x+1)^3}\right)$$

for all $x \in [0, \infty)$. Let $L^2 := L^2([0, \infty), \mathcal{M}, \mu)$ be the collection of all (measurable) square integrable functions. We know (see [6, p. 326], Remark 11.37) that the pair (L^2, d) is not a metric space unless we identify functions which differ only on a set of measure zero, where the mapping $d : L^2 \times L^2 \to [0, \infty)$ is defined by

$$d(f, g) = \sqrt{\int_{[0,\infty)} (f - g)^2 \, d\mu} = \sqrt{\int_0^\infty \frac{(f(x) - g(x))^2}{(x+1)^4} dx}.$$

By Lemma 6, we observe that $\mathcal{Y}_{\text{conc}} \subset L^2$. Unfortunately, we note that this does not guarantee that the pair $(\mathcal{Y}_{\text{conc}}, d)$ is a metric space, for the reason mentioned above. Nevertheless, we shall prepare the ground for showing that $(\mathcal{Y}_{\text{conc}}, d)$ is actually a metric space.

Whenever $\Phi \in \mathcal{Y}_{\text{conc}}$ write G_Φ for the graph of Φ on $[0, \infty)$, i. e.,

$$G_\Phi = \{(x, \Phi(x)) : x \in [0, \infty)\}$$

and write $G_\Phi^{a\|b}$ for the graph of Φ on the interval $[a, b)$, i. e.,

$$G_\Phi^{a\|b} = \{(x, \Phi(x)) : x \in (a, b)\},$$

where $a < b$ are any non-negative numbers.

Lemma 7. *Let Φ and $\Psi \in \mathcal{Y}_{\text{conc}}$ be arbitrary with distinct graphs. Then*

$$|\{x \in (0, \infty) : \Phi(x) = \Psi(x)\}| \leq 1,$$

where $|B|$ stands for the cardinality of B whenever B is a set.

Proof. Suppose on the contrary that

$$|\{x \in (0, \infty) : \Phi(x) = \Psi(x)\}| \geq 2.$$

Write

$$x_1 = \inf\{x \in (0, \infty) : \Phi(x) = \Psi(x)\}$$

and

$$x_2 = \inf\{x \in (0, \infty) \setminus \{x_1\} : \Phi(x) = \Psi(x)\}.$$

It is clear that $0 < x_1 < x_2$ and $\Phi(x_i) = \Psi(x_i)$, $i \in \{1, 2\}$. We point out that the two graphs are continuous. We show that the graph of one of the functions Φ and Ψ lies above the graph of the other on the interval $(0, x_1)$. In fact, without loss of generality we may assume on the contrary that $G_\Phi^{0\|x_1}$ lies both above and below $G_\Psi^{0\|x_1}$. Then necessarily the two graphs must cross each other in the interior of interval $(0, x_1)$, i. e. there is some $x_0 \in (0, x_1)$ such that $\Phi(x_0) = \Psi(x_0)$. This, however, is in contradiction with the minimality of x_1. Hence we can assume that $G_\Phi^{0\|x_1}$ lies above $G_\Psi^{0\|x_1}$. By the continuity and the fact that $\Phi(x_1) = \Psi(x_1)$ we note that G_Φ crosses G_Ψ at point $(x_1, \Phi(x_1))$. Nevertheless, since both Φ and Ψ are unbounded increasing functions and $\Phi(x_2) = \Psi(x_2)$, the graph G_Φ must cross the graph G_Ψ at point $(x_2,\ \Phi(x_2))$. This means that Φ must be convex on the interval $(x_1,\ x_2)$, which is absurd since these functions are concave. □

Corollary 2. *Let Φ and $\Psi \in \mathcal{Y}_{\text{conc}}$ be arbitrary. Then among the following three assertions exactly one fulfills*

(1) $\{x \in [0, \infty) : \Phi(x) = \Psi(x)\} = [0,\ \infty)$.
(2) $\{x \in (0,\ \infty) : \Phi(x) \neq \Psi(x)\} = (0,\ \infty)$.
(3) *There is a unique number $x^* \in (0, \infty)$ with $\Phi(x^*) = \Psi(x^*)$ such that*

$$\{x \in (0, \infty) \setminus \{x^*\} : \Phi(x) \neq \Psi(x)\} = (0, \infty) \setminus \{x^*\}.$$

Lemma 8. *Let Φ and $\Psi \in \mathcal{Y}_{\text{conc}}$ be arbitrary. Then in order that $\Phi(x) = \Psi(x)$ for all $x \in [0, \infty)$ it is necessary and sufficient that*

$$\int_{[0,\infty)} (\Phi - \Psi)^2 \, d\mu = 0.$$

Proof. We first note that the sufficiency is obvious. To show the necessity let us assume that

$$\int_{[0,\infty)} (\Phi - \Psi)^2 \, d\mu = 0.$$

Then, on the one hand,

$$\mu(\{x \in [0, \infty) : \Phi(x) = \Psi(x)\}) = \mu([0, \infty)) = \frac{1}{3}$$

so that necessarily $\{x \in [0, \infty) : \Phi(x) = \Psi(x)\} \neq \emptyset$. On the other hand

$$\mu(\{x \in (0, \infty) : \Phi(x) \neq \Psi(x)\}) = 0.$$

Note that both the sets $\{x \in [0, \infty) : \Phi(x) = \Psi(x)\}$ and $\{x \in (0, \infty) : \Phi(x) \neq \Psi(x)\}$ cannot be non-empty at the same time (because of Corollary 2). Consequently,

$$\{x \in (0,\ \infty) : \Phi(x) \neq \Psi(x)\} = \emptyset$$

and, therefore, $\{x \in [0, \infty) : \Phi(x) = \Psi(x)\} = [0,\ \infty)$. □

We are now in the position to state the result hereby.

Proposition 5. *The mapping $d : \mathcal{Y}_{\text{conc}} \times \mathcal{Y}_{\text{conc}} \to [0, \infty)$, defined by*

$$d(\Phi,\ \Psi) = \sqrt{\int_{[0,\infty)} (\Phi - \Psi)^2 \, d\mu} = \sqrt{\int_0^\infty \frac{(\Phi(x) - \Psi(x))^2}{(x+1)^4} dx},$$

satisfies the metric axioms, i. e. for any three functions Φ_1, Φ_2 and $\Phi_3 \in \mathcal{Y}_{\text{conc}}$

(1) $d(\Phi_1,\ \Phi_2) \geq 0$ *and* $d(\Phi_1,\ \Phi_2) = 0$ *if and only if* $\Phi_1 = \Phi_2$.
(2) $d(\Phi_1,\ \Phi_2) = d(\Phi_2,\ \Phi_1)$.
(3) $d(\Phi_1,\ \Phi_2) \leq d(\Phi_1,\ \Phi_3) + d(\Phi_3,\ \Phi_2)$.

The pair $(\mathcal{Y}_{\text{conc}},\ d)$ is a metric space and then by referring to Theorem C the pair $(\mathcal{A},\ d)$ is also a metric space.

Theorem 7. *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ and write $\Phi_n = \Phi^{n/(n+1)}$, $n \geq 1$. Then*

$$\lim_{n\to\infty} \int_{[0,\infty)} \Phi_n^2 d\mu = \int_{[0,\infty)} \Phi^2 d\mu.$$

Proof. For every index $n \geq 1$, define $\Phi_n^* := (\Phi_n(1))^{-1}\Phi_n$. Clearly $(\Phi_n) \subset \mathcal{Y}_{\text{conc}}$ (see Corollary 1) and hence $(\Phi_n^*) \subset \mathcal{A}$ because of Lemma 1. Via Theorem 6 it follows that sequence (Φ_n) converges to Φ pointwise, which in turn entails that sequence (Φ_n^*) converges to $(\Phi(1))^{-1}\Phi$ pointwise. Write the function $Z(x) := x + 1$, $x \in [0, \infty)$. We obtain (by Lemma 3) that

$$\sup_{n\geq 1} \Phi_n^*(x) \leq S(x) \leq Z(x), \qquad x \in [0,\infty).$$

Now, on the one hand, a simple computation shows that $Z \in L^2$. On the other hand, we can deduce from Lemma 6 that $(\Phi_n) \subset L^2$ and thus $(\Phi_n^*) \subset L^2$. Therefore, the Dominated Convergence Theorem guarantees that

$$\lim_{n\to\infty} \int_{[0,\infty)} \Phi_n^{*2} d\mu = (\Phi(1))^{-2} \int_{[0,\infty)} \Phi^2 d\mu.$$

Now we remark that for every index $n \geq 1$,

$$\int_{[0,\infty)} \Phi_n^2 d\mu = (\Phi(1))^2 \int_{[0,\infty)} \Phi_n^{*2} d\mu.$$

Passing to the limit we can conclude that

$$\lim_{n\to\infty} \int_{[0,\infty)} \Phi_n^2 d\mu = \int_{[0,\infty)} \Phi^2 d\mu.$$

This was to be proven. □

Theorem 8. *The subset $\mathcal{A}$ is a dense set in $\mathcal{Y}_{\text{conc}}$.*

Proof. Let $\Phi \in \mathcal{Y}_{\text{conc}}$ be arbitrary. For every index $n \geq 1$, set $\Phi_n^* := (\Phi_n(1))^{-1}\Phi_n$, where $\Phi_n = \Phi^{n/(n+1)}$. We need to prove that

$$\lim_{n\to\infty} d(\Phi, \Phi_n) = \lim_{n\to\infty} \int_{[0,\infty)} (\Phi - \Phi_n)^2 d\mu = 0.$$

In fact, fix arbitrarily an index $n \geq 1$. Then

$$\int_{[0,\infty)} (\Phi - \Phi_n)^2 d\mu = \int_{[0,\infty)} \Phi_n^2 d\mu + \int_{[0,\infty)} \Phi^2 d\mu - 2\int_{[0,\infty)} \Phi\Phi_n d\mu. \tag{4.4}$$

Then Lemma 3 entails that

$$(\Phi(1))^{-(2n+1)/(n+1)} \Phi\Phi_n \leq Z^{(2n+1)/(n+1)} \leq Z^2,$$

since $Z(x) \geq 1$ for all $x \in [0, \infty)$ and the sequence $\left(\frac{2n+1}{n+1}\right)$ tends increasingly to 2. On the other hand,

$$\lim_{n\to\infty} (\Phi(1))^{-(2n+1)/(n+1)} \Phi(x)\Phi_n(x) = (\Phi(1))^{-2}\Phi^2(x)$$

for all $x \in [0, \infty)$. Then by means of The Dominated Convergence Theorem it follows that

$$\lim_{n\to\infty} \int_{[0,\infty)} \Phi\Phi_n d\mu = \lim_{n\to\infty} (\Phi(1))^{(2n+1)/(n+1)} \int_{[0,\infty)} (\Phi(1))^{-\frac{2n+1}{n+1}} \Phi^{\frac{2n+1}{n+1}} d\mu \tag{4.5}$$
$$= \int_{[0,\infty)} \Phi^2 d\mu.$$

Finally we note that

$$\lim_{n\to\infty} \int_{[0,\infty)} \Phi_n^2 d\mu = \int_{[0,\infty)} \Phi^2 d\mu, \tag{4.6}$$

by Theorem 7. Therefore, combining the results established in (4.4)–(4.6), we get $\lim_{n\to\infty} d(\Phi, \Phi_n) = 0$. We can thus conclude on the validity of the theorem. □

If the integral representation (1.1) is such that its derivative is a right-continuous function, tending increasingly to infinity and assumes the value zero at the origin, then we speak of *convex Young-functions* (see, e. g. [2]). Clearly the inverse of every convex Young-function is a concave Young-function (and *vice versa*).

A convex Young-function Ψ is said to satisfy the growth condition if

$$\sup_{x>0} \frac{\Psi(\beta x)}{\Psi(x)} < \infty$$

for some number $\beta > 1$ which is equivalent to

$$\sup_{x>0} \frac{x\psi(x)}{\Psi(x)} := p < \infty$$

with ψ being the derivative of Ψ. (The quantity p is referred to as the power of Ψ.)

Open problem 1. *Let $\Phi \in \mathcal{Y}_{\text{conc}}$ be arbitrary. In order that $\Phi(x) = x^p$, $x \in [0, \infty)$ for some $p \in (0, 1)$ it is necessary and sufficient that both $\Phi \in \mathcal{A}_1$ and its inverse Φ^{-1} satisfy the growth condition together with the property $\Phi^{-1}(1) = 1$.*

Open problem 2. *The converse of Remark 5 holds true.*

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