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WENDROFF TYPE INTEGRO–SUM INEQUALITIES AND APPLICATIONS

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Abstract. We present some new nonlinear Wendroff type integral inequalities with respect to two independent variables for discontinuous functions. The results obtained are applied to investigate properties of solutions of a certain class of hyperbolic equations with impulse influence on certain surfaces.

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1. **Introduction**

Differential models of various real processes are described by systems of ordinary differential equations with impulse influence [1-3]. Among the methods of investigating the properties of these systems (such as existence, uniqueness, boundedness, and stability of solutions), the method of integro-sum inequalities [4-15] plays an important role. Note that applications and generalizations of this method to estimate solutions of systems of partial differential equations with impulse perturbations were not considered.

The study of this kind of problems is motivated by models of real processes in the dynamics of hydromechanical systems which are described by certain classes of hyperbolic partial differential equations with impulse perturbations, where the impulse effect is concentrated on surfaces transversal to characteristics.

Integro-sum representations of the solutions of such systems contain Lebesgue–Stiltjes measure concentrated on the curves of jumps of the solutions.

2. **Main results**

Let us assume that:

(a) D^* is an open set in R^2 $(D^* \subset R^2)$;
(b) $D^* = D \setminus \Gamma$, where $D = \bigcup_j D_j$, $j = 1, 2, \ldots$;
(c) $\Gamma = \bigcup_j \Gamma_j$, $\Gamma_j = \{(x, y) : \varphi_j(x, y) = 0, j = 1, 2, \ldots\}$;
(d) $\varphi_j(x, y)$ are real-valued continuously differentiable functions such that $\operatorname{grad} \varphi_j(x, y) > 0$ for all $j = 1, 2, \ldots$;
(e) $D_1 \stackrel{\text{def}}{=} \{(x, y) : x \geq 0,\ y \geq 0,\ \varphi_1(x, y) < 0\}$;
$D_k \stackrel{\text{def}}{=} \{(x, y) : x \geq 0,\ y \geq 0,\ \varphi_{k-1}(x, y) \geq 0, \varphi_k(x, y) < 0,\ \ \forall k > 2,\ k \in N\}$;
(f) $G_j = \{(u, v) : (x, y) \in D_j,\ 0 \leq u \leq x,\ 0 \leq v \leq y,\ j \in N\}$;
(g) μ_{φ_k} is the Lebesgue-Stiltjes measure concentrated on the curves Γ_k.

Let us consider a real-valued nonnegative continuous function $u(x, y)$ on D^*, which has finite jumps on the curves $\{\Gamma_j\}$.

Denote by $\Phi(x, y, u)$ a nonnegative, continuous, and nondecreasing function of u, with $x, y \in D^*$ fixed. We shall consider the functions Φ of one of the following types $\Phi = \Phi_j$ $(j = 1, 4)$:

(i) $\Phi_1(x, y, u) = f_1(x, y)u(x, y)$,
(ii) $\Phi_2(x, y, u) = f_2(x, y)u(x, y) + f_3(x, y)$,
(iii) $\Phi_3(x, y, u) = f_4(x, y)u^\alpha(x, y)$, $\alpha = \text{const} > 0, \alpha \neq 1$,
(iv) $\Phi_4(x, y, u) = f_5(x, y)u(x, y) + f_6(x, y)u^\alpha(x, y)$, where $f_j(x, y)(j = \overline{1, 6})$ are continuous nonnegative functions in $R^2_+ = \{(x, y) : x \geq 0,\ y \geq 0\}$.

Let $W(x, y, u(x, y))$ denote a function of one of the following two types W_1 and W_2:

$$W_1 = \beta_j u(x, y), \quad j \in N$$

where $\beta_j \geq 0$ are constants, and

$$W_2 = \beta_j(x, y)u(x, y), \quad j \in N,$$

where $\beta_j(x, y)$ are continuous functions nonnegative for all $(x, y) \in R^2_+$.

Let $g(x, y)$ be a positive, nondecreasing continuous function in R^2_+. Assume that $u(x, y)$ satisfies the following integro-sum inequality in D^*:

$$u(x, y) \leq g(x, y) + \iint_{G_n} \Phi(\tau, s, u(\tau, s))d\tau ds + \sum_{j=1}^{n-1} \int_{\Gamma_j \cap G_n} W(x, y, u(x, y))d\mu_{\varphi_j}, \quad (2.1)$$

which is satisfied for all $(x,y)\in D^*$. Then the following results are true.

Proposition 1. *The estimate*

$$u(x,y)\le g(x,y)\exp[F_1(x,y)]\prod(\beta_j(x,y)),\ \textit{if } \Phi=\Phi_1,\ W=W_2; \tag{2.2}$$

$$u(x,y)\le g(x,y)\exp[F_1(x,y)]\prod(\beta_j),\ \textit{if } \Phi=\Phi_1,\ W=W_1$$

holds for all $(x,y)\in D^*$. *Here,*

$$\int_0^x\int_0^y f_i(\tau,s)d\tau ds \overset{\text{def}}{=} F_i(x,y),\ i=1,3,5,$$

$$\prod_{j=1}^{n-1}\int_{\Gamma_j\cap G_n}(1+\beta_j(x,y))d\mu_{\varphi_j}\overset{\text{def}}{=}\prod(\beta_j(x,y)),\quad \prod_{j=1}^{n-1}(1+\beta_j)\bigvee_{\Gamma_j\cap G_n}(\mu_{\varphi_j})\overset{\text{def}}{=}\prod(\beta_j),$$

where $\bigvee_{\Gamma_j\cap G_n}(\mu_{\varphi_j})$ *is the complete variation of* μ_{φ_j}.

Proposition 2. *The estimate*

$$u(x,y)\le g(x,y)\exp[F_2(x,y)]\prod(\beta_j(x,y))\times$$
$$\times\left(1+\int_0^x\int_0^y\frac{f_3(\tau,s)}{g(\tau,s)}\exp[-F_2(\tau,s)]d\tau ds\right)$$

holds if $\Phi=\Phi_2$, $W=W_2$, *and*

$$u(x,y)\le g(x,y)\exp[F_2(x,y)]\prod(\beta_j)\left(1+\int_0^x\int_0^y\frac{f_3(\tau,s)}{g(\tau,s)}\exp[-F_2(\tau,s)]d\tau ds\right),$$

if $\Phi=\Phi_2$, $W=W_1$.

Proposition 3. *The following assertions hold.*

(A) *The estimate*

$$u(x,y)\le g(x,y)\prod(\beta_j(x,y)\left[1+(1-\alpha)\int_0^x\int_0^y f_4(\tau,s)g^{\alpha-1}(\tau,s)d\tau ds\right]^{\frac{1}{1-\alpha}}$$

is true if $0<\alpha<1$, $\Phi=\Phi_3$, *and* $W=W_2$. *When* $\beta_j(x,y)=\beta_j=\text{const}>0$, *the expression* $\prod(\beta_j(x,y))$ *in estimate (A) is replaced by* $\prod(\beta_j)$.

(B) *The estimate*

$$u(x,y)\le g(x,y)\prod(\beta_j(x,y))\Big[1-(\alpha-1)\times$$
$$\times\prod\nolimits^{\alpha-1}\beta_j(x,y)\int_0^x\int_0^y f_y(\tau,s)g^{\alpha-1}(\tau,s)d\tau,ds\Big]^{-\frac{1}{\alpha-1}}$$

holds for $\alpha>1$ *and for arbitrary* $(x,y)\in D^*$ *such that*

$$\int_0^x\int_0^y f_4(\tau,s)g^{\alpha-1}(\tau,s)d\tau ds<[(\alpha-1)\prod\nolimits^{\alpha-1}(\beta_j(x,y))]^{-1}.$$

If $\beta_j(x,y)=\beta_j=\text{const}>0$, *then* $\prod(\beta_j(x,y))$ *in estimate (B) is replaced by* $\prod(\beta_j)$.

Proposition 4. *The following assertions hold.*

(C) *The estimate*

$$u(x,y) \leq g(x,y)\prod(\beta_j(x,y))\exp[F_5(x,y)] \times \left[1+(1-\alpha)\int_0^x\int_0^y f_6(\tau,s)g^{\alpha-1}(\tau,s)\times\right.$$

$$\left.\times\exp[(\alpha-1)F_5(\tau,s)]d\tau ds\right]^{\frac{1}{1-\alpha}}$$

holds for $0<\alpha<1$, $\Phi=\Phi_4$, $W=W_2$. *If* $\beta_j(x,y)=\beta_j$, *the expression* $\prod(\beta_j(x,y))$ *in (C) changes to* $\prod(\beta_j)$.

(D) *The estimate*

$$u(x,y) \leq g(x,y)\prod(\beta_j(x,y))\exp[F_5(x,y)]\left[1-(\alpha-1)\prod\nolimits^{\alpha-1}(\beta_j(x,y))\times\right.$$

$$\left.\times\int_0^x\int_0^y f_6(\tau,s)\times g^{\alpha-1}(\tau,s)\exp[(\alpha-1)F_5(\tau,s)]d\tau ds\right]^{-\frac{1}{\alpha-1}}$$

is true for $\alpha>1$ *and arbitrary* $(x,y)\in D^*$ *such that*

$$\int_0^x\int_0^y f_6 g^{\alpha-1}\exp[(\alpha-1)F_5]d\tau ds < \left[(\alpha-1)\prod\nolimits^{\alpha-1}(\beta_j(x,y))\right]^{-1}.$$

If $\beta_j(x,y)=\beta_j$, *the expression* $\prod(\beta_j(x,y))$ *in (D) changes to* $\prod(\beta_j)$.

The proofs of Propositions 1–4 are obtained by using the induction method.

Proof. Let us establish Proposition 1 (Propositions 2–4 are proved analogously).

Let $(x,y)\in D_1$. Inequality (2.1) reduces to the form

$$u(x,y)\leq g(x,y)+\iint_{G_1}\Phi_1(\tau,s,u(\tau,s))d\tau ds.$$

We can suppose that $g(x,y)=C>0$, a constant function, because otherwise, if $g(x,y)\neq C$ then, by using the fact that g is positive and nondecreasing, it is possible to obtain the comparison inequality

$$\mu(x,y)\leq 1+\iint_{G_1}\Phi_1(\tau,s,\mu(\tau,s))d\tau ds,$$

where $\mu=\frac{u}{g}$.

Obviously, the following estimate holds in D_1:

$$u(x,y)\leq C\exp[F_1(x,y)].$$

Consider the domain D_2. Let $G_2=G_2^1\cup G_2^2$, where

$$G_2^2=\{(x,y):(x,y)\in D_2\cap G_2\}.$$

Then, for all $(x,y)\in D_2$, the inequality

$$u(x,y) \le C + \iint_{G_2^1} \Phi_1(\tau,s,u(\tau,s))d\tau ds + \int_{\Gamma_1\cap G_2} \beta_1(x,y)u(x,y)d\mu_{\varphi_1} + \iint_{G_2^2} \Phi_1(\tau,s,u(\tau,s))d\tau ds$$

holds.

On the curves $\Gamma_1 \cap G_2$, we select some points $A_i(x_i^1, y_i^1),\ i=\overline{0,\ n-1}$, and consider the inequality

$$u(x,y) \le \sum_{i=0}^{n-1}\Big(C + \int_0^{x_1^1}\int_0^{y_i^1} \Phi_1(\tau,s,u(\tau,s))d\tau ds + u(x_i^1,y_i^1)\beta_1(x_i^1,y_i^1)\Delta\mu_{\varphi_i^i} + \int_{x_i^1}^{x}\int_{y_i^1}^{y} \Phi_1(\tau,s,u(\tau,s))d\tau ds\Big),$$

where $\Delta\mu_{\varphi_1^i}$ is the variation of the measure function φ_1 on the segment A_iA_{i+1}. Using (2.2), we obtain

$$\begin{aligned} u(x,y) &\le \sum_{i=0}^{n-1}\{C\exp[F_1(x_i^1,y_i^1)] + C\beta_1(x_i^1,y_i^1)\Delta\mu_{\varphi_1}^i\exp[F_1(x_i^1,y_i^1)]+ \\ &+ \int_{x_i^1}^{x}\int_{y_i^1}^{y}\Phi_1 d\tau ds\} \le \sum_{i=0}^{n-1}\{C(1+\beta_1(x_i^1,y_i^1)\Delta\mu_{\varphi_1}^i\exp[F_1(x_i^1,y_i^1)]+ \\ &+ \int_{x_i^1}^{x}\int_{y_i^1}^{y}\Phi_1 d\tau ds)\} \le \sum_{i=0}^{n-1}[C(1+\beta_1(x_i^1,y_i^1))\Delta_{\varphi_1}^i\exp[F_1(x,y)]]. \end{aligned} \tag{2.3}$$

When $\max_{0\le i\le n-1}\Delta\mu_{\varphi_1}^i \to 0$, we obtain

$$u(x,y) \le C\exp[F_1(x,y)]\int_{\Gamma_1\cap G}(1+\beta_1(x,y))d\mu_{\varphi_1}.$$

Let (2.2) be satisfied for all $(x,y)\in D_k$. Let us consider $(x,y)\in D_{k+1}$ and put $G = G_{k+1}^1 \cup G_{k+1}^2$,

$$G_{k+1}^1 = \{(x,y) : (x,y)\in D_{k+1}\cap G\},\ G_{k+1}^2 = G\setminus G_{k+1}^1.$$

Then

$$u(x,y) \le C + \iint_{G_{k+1}^1}\Phi_1 d\tau ds + \int_{\Gamma_k\cap G}\beta_k(x,y)u(x,y)d\mu_{\varphi_k} + \iint_{G_{k+1}^2}\Phi_1 d\tau ds.$$

On the curve $\Gamma_k\cap G$, we select some points $B_i(x_i^k, y_i^k), i=\overline{0,\ n-1}$. Denote by $\Delta\mu_{\varphi_k}^i$ the variation of the measure on the segment B_iB_{i+1}. Similarly to the consideration above, we obtain

$$u(x,y) \le \sum_{i=0}^{n-1}[C + \int_0^{x_i^k}\int_0^{y_i^k}\Phi_1 d\tau ds + u(x_i^k,y_i^k)\beta_k(x_i^k,y_i^k)\Delta_{\varphi_k}^i +$$

$$+u(x_i^k,y_i^k)\beta_k(x_i^k,y_i^k)\Delta^i_{\varphi_k}+\int_{x_i^k}^{x}\int_{y_i^k}^{y}\Phi_1 d\tau ds].$$

Then

$$u(x,y)\leq\sum_{i=0}^{n-1}[C(1+\beta_k(x_i^k,y_i^k))\times\Delta\mu^i_{\varphi_k}\exp[F_1(x,y)]\prod_{j=1}^{n-1}\int_{\Gamma_j\cap G}(1+\beta_j(x,y))d\mu_{\varphi_j}].$$

When $\max_{0\leq i\leq n-1}\Delta\mu^i_{\varphi_K}\to 0$, we get

$$u(x,y)\leq C\exp[F_1(x,y)]\prod_{j=1}^{k}\int_{\Gamma_j\cap G}(1+\beta_j(x,y))d\mu_{\varphi_j}$$

for all $(x,y)\in D_{k+1}$. This completes the proof of Proposition 1. □

3. Applications

Consider the model of a hydraulic system consisting of a tank, a delivery pipeline, an auger centrifugal pump, and a pressure head pipeline giving a liquid in gasogenerator and the chamber output pressure. Such systems were considered in [16, 17, 20]. Note that there may be auto-oscillations stipulated by the feedback in a hydraulic part of the system due to the fact that the system is located on a stand.

Experiments showed that, in certain cases, there may be a significant increase of the amplitude of auto-oscillations of the stand. Thus, the oscillation frequencies of the liquid in the tank are characterized by the eigenfrequencies of the stand that coincide with the eigen-oscillations of the output pressure. The amplitude of the oscillations may have white-noise-like-behaviour.

The main problem in the study of such systems is to describe the deviation of the system state from the equilibrium.

3.1. Dynamics of fluid in the pipeline.

In the one-dimensional case, the dynamics of a compressed liquid in the homogeneous pipeline is usually described by a system consisting of the following equations:

A) Equation of motion of the liquid:

$$\frac{\partial V}{\partial t}+u\frac{\partial V}{\partial x}+\frac{1}{\gamma}\frac{\partial p}{\partial x}+\frac{\lambda}{2d}V|V|=0.$$

Here, λ is the resistance coefficient depending on the Reynolds number, x represents the coordinate of an axis of the pipeline, γ is the density of the liquid, d is the diameter of the pipeline, p,V are the instantaneous pressure and speed of the liquid, respectively.

B) Equation of continuity:

$$\frac{\partial\gamma}{\partial t}+V\frac{\partial\gamma}{\partial x}+\gamma\frac{\partial V}{\partial x}=0.$$

C) The equation of state for the liquid, whose role is played by Hook's law, is

$$\gamma = \gamma_0 \left(1 + \frac{p - p_0}{\gamma_0 c^2}\right).$$

For the majority of hydraulic systems, the convection terms can be neglected. For the turbulence analysis, the nonlinear term admits linearization, and the equations with the distributed parameters reduce to the form

$$-\frac{\partial p}{\partial x} = \frac{\gamma}{F_\tau}\left(\frac{\partial q}{\partial t} + kq\right),$$

$$\frac{\partial p}{\partial t} = \frac{c^2\gamma}{F_\tau}\frac{\partial q}{\partial x}.$$

Here, q is the volumetric intensity of the liquid.

3.2. **The description of the centrifugal pump and pressure head sprocket.** Among the main characteristic features describing the mode of operation of the cavitational centrifugal pump is the dependence of the sizes of the cavitational cavities V_k on the input and output pressure $V_k = f(p, q)$. This dependence is related to the cavitation number $k(V_k, q)$ defined as the ratio of the pressure difference to the velocity head $p = p_n + k(V_k, q)\gamma\frac{\omega^2}{2}$, where p_n is the pressure of the saturated steam of the liquid, ω is the speed of the blade-to-blade flow in the auger channels, γ is the density of the liquid steam. When the liquid passes through the auger, the pressure varies according to the rule

$$p_n = (a_i n^2 + b_i n q^{(1)} + c_i (q^{(1)})^2) f_1(V),$$

where n is the number of revolutions of the shaft, a_i, b_i, c_i are the parameters describing the operation of the pump and the pressure head sprocket, $f_1(V)$ is an experimentally obtained function describing the influence of the size of the cavitational cavities on the operation of the auger of the pump and the pressure of the head sprocket.

Thus, a simplified model of the dynamics of the system is as follows:

tank → delivery pipeline → auger of the pump → pressure head pipeline → chamber of combustion positioned on the sliding stand.

This reduces to the following mathematical problems.

3.3. **Model without sharing cavitational cavities.** The corresponding system contains the following equations:

1) Equation of mechanical oscillations of the stand:

$$\ddot{z}_i + b_i(t)\dot{z}_i + \omega_i^2(t) z_i = \frac{R(t)}{m(t)\overline{p}_6} p_6, \quad i = 1, 2, \ldots, 6;$$

2) Equation of motion of the liquid in the pipeline:

$$-\frac{\partial p_1}{\partial x} = \frac{\gamma_1}{F_1}\frac{\partial q_1}{\partial t},$$

$$-\frac{\partial p_1}{\partial t} = \frac{c^2\gamma_1}{F_1}\frac{\partial q_1}{\partial x},$$

with the boundary conditions

$$p_1(0,t) + R_1 q_1(0,t) = \frac{\gamma_1 H(t)}{g}\sum_{i=1}^{6}\alpha_{il}\ddot{z}_i + \beta\gamma_1 F_1\sum_{i=1}^{6}\alpha_{i2}\dot{z}_i,$$

$$p_1(l_1,t) = B_1 V_k + B_2 q_1(l_1,t) + E\dot{V}_k + I_k\dot{q}(l_1,t).$$

Here, l_1 is the length of the pipeline and, in the quasistationary model, $E = I_k = 0$.

3) Equation of motion of the liquid in the pressure head pipeline:

$$-\frac{\partial p_2}{\partial x} = \frac{\gamma_2}{F_2}\frac{\partial q_2}{\partial t},$$

$$-\frac{\partial p_2}{\partial t} = \frac{c^2\gamma_2}{F_2}\frac{\partial q_2}{\partial x}$$

with the boundary conditions:

$$p_2(l_1+0,t) + R_2 q_2(l_1+0,t) = p_1(l_1-0,t) + s_1 q_1(l_1-0,t) + \varepsilon V_k, \tag{3.1}$$

$$\tau\dot{p}_2(l_1+l_2,t) + p_2(l_1+l_2,t) = A_2 q_2(l_1+l_2,t),$$

$$p_2(l_1+l_2,t) = p_2(t).$$

4) Equation of material balance describing summarized size of cavitational cavities in the pump:

$$\gamma_1\dot{V}_k = q_2(l_1+0,t) - q_1(l_1-0,t). \tag{3.2}$$

Relations (3.1), (3.2) can be considered to be the impulse effect concentrated on the curves transversal to the characteristics. Thus, we have to study solutions of the hyperbolic equations with impulse effect concentrated on surfaces transversal to characteristics [18, 19].

3.4. **Reduction to the integral equations and inequalities.** The reduced model is represented by an equation of hyperbolic type with, generally speaking, a nonlinear right-hand member and boudary conditions acting as the impulse effect.

Let us assume $s_i = c_i$ and $z_i = \frac{c_i F_i}{\gamma_i}$. The general solution of the partial differential equations considered has the form $p_i(x,t) = \nu(x - s_i t) + u(x + s_i t)$ and $q_i(x,t) = \frac{1}{z_i}(\nu(x-s_i t) - u(x+s_i t))$. In view of the boundary conditions at $t = 0$, by making the standard substitution $\xi = x - st$, and $\eta = x + st$, applying the d'Alembert formula in the domain $0 < \xi + \eta < 2l_1$, we obtain the following representation of the solutions:

$$u_1(\xi,\eta) = \frac{\varphi_1(\xi) + \varphi_1(\eta)}{2} + \frac{1}{2z_i}\int_\xi^\eta \psi_1(s)ds + \frac{z_i}{2}\iint_D f_1(s,t)u_1(s,t)|u_1(s,t)|dsdt.$$

Here, the functions φ, ψ are determined from the initial and boundary conditions. The integration is carried out in the domain

$$D_1 = \{s,t : s+t > 0, \quad s < \xi, \quad t < \eta, \ \xi + \eta < 2l_1\}.$$

Similarly, in the domain $2l_1 < \xi + \eta < 2l_2$, the formula of representation for solutions has the form

$$u_2(\xi,\eta) = \frac{\varphi_2(\xi)+\varphi_2(\eta)}{2} + \frac{1}{2z_i}\int_\xi^\eta \psi_2(s)ds + \frac{z_i}{2}\iint_{D_2} f_2(s,t)u_2(s,t)|u_2(s,t)|dsdt + \\ + \int_{D_2\cap\gamma} u(s,t)\varphi_2(s,t)d\mu.$$

Here,

$$D_2 = \{s,t : s+t>0, \quad s<\xi, \quad t<\eta, \quad 2l_1 < \xi+\eta < 2l_2,\}$$

the curve $\gamma = \{\xi,\ \eta : \xi+\eta = 2l_1\}$ represents the curve of rupture of the stream stipulated by the presence of the pump. The functions are constant along the characteristics in the domain $D_2 \setminus D_1$; their change is determined by a Stieltjes integral on the curve γ and depends on the constant characteristics of the pump. We do not consider the process of reflection of a wave from the blades of the pump.

Thus, the problem considered can be studied by using the multi-dimensional integral inequalities considered in Section 2.

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