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GENERALIZED DERIVATIONS ON IDEALS OF PRIME RINGS

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Abstract. Let R be a prime ring. By a generalized derivation we mean an additive mapping $g : R \to R$ such that $g(xy) = g(x)y + xd(y)$ for all $x, y \in R$ where d is a derivation of R. In the present paper our main goal is to generalize some results concerning derivations of prime rings to generalized derivations of prime rings.

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1. INTRODUCTION

Throughout this paper R always denotes an associative prime ring with center $Z(R)$, extended centroid C, Martindale quotients ring Q and Utumi quotients ring U. For any $x, y \in R$, the commutator of x and y denoted by $[x, y]$ is defined to be $xy - yx$. Recall that a ring R is prime if $xRy = 0$ implies $x = 0$ or $y = 0$. An additive mapping $\alpha : R \to R$ is called a derivation if $\alpha(xy) = \alpha(x)y + x\alpha(y)$ holds for all $x, y \in R$. The commutativity of prime rings with derivations was initiated by Posner [16]. Over the last two decades, a lot of work has been done on this subject (see [4, 7, 11, 16] where further references can be found). Following Brešar [4], $d : R \to R$ is called a *generalized derivation* if there exists a derivation α of R such that

$$d(xy) = d(x)y + x\alpha(y) \quad \text{for all } x, y \in R.$$

Hence the concept of generalized derivations covers both the concepts of a derivation and of a left multiplier that is, an additive mapping $f : R \to R$ satisfying $f(xy) = f(x)y$ for all $x, y \in R$. Basic examples are derivations and generalized inner derivations given by maps of type $f : R \ni x \mapsto ax + xb \in R$ for some $a, b \in R$.

In [9], Hvala initiated generalized derivations from the algebraic viewpoint. In [13], T.K. Lee extended the definition of generalized derivations as follows:

By a generalized derivation we mean an additive mapping $g : I \to U$ such that $g(xy) = g(x)y + xd(y)$ for all $x, y \in I$, where I is a dense right ideal of R and d is a derivation from I into U.

Moreover Lee also proved that every generalized derivation can be uniquely extended to a generalized derivation of U and thus all generalized derivations of R will be implicitly assumed to be defined on the whole U and obtained the following results:

Theorem 1 ([13], Theorem 3)**.** *Every generalized derivation g on a dense right ideal of R can be uniquely extended to U and assumes the form $g(x) = ax + d(x)$ for some $a \in U$ and a derivation d on U.*

In this paper we extend some well-known results concerning derivations of prime rings to generalized derivations of prime ring.

We note that if R has the property that $Rx = 0$ implies $x = 0$ and $h : R \to R$ is any function , $d : R \to R$ is any additive mapping such that $d(xy) = d(x)y + x\alpha(y)$ for all $x, y \in R$, then d is uniquely determined by h and moreover h must be a derivation (see [4], Remark 1).

In all that follows, unless stated otherwise, R will be a prime ring. The related object we need to mention is the two-sided Quotient ring Q of a ring R, the two-sided Utumi quotient U of a ring R (sometimes, as in [3], U is called the maximal ring of quotients). The definitions, the axiomatic formulations and the properties of these quotient rings U and Q can be found in [2] and [3].

We make a frequent use of the theory of generalized polynomial identities and of the theory of differential identities (see [3, 5, 10, 12, 15]). In particular we need to recall that when R is a prime ring and I a nonzero two-sided ideal of R, then I, R, Q and U satisfy the same polynomial identities [5] and also the same differential identities [12].

We will also make frequent use of the following result due to Kharchenko [10] (see also [12]):

Let R be a prime ring, d a nonzero derivation of R and I a nonzero two-sided ideal of R. Let $f(x_1,...,x_n,d(x_1),...,d(x_n))$ be a differential identity on I, that is the relation

$$f(r_1,...,r_n,d(r_1),...,d(r_n)) = 0$$

holds for all $r_1,...,r_n \in I$.

One of the following holds:

1) Either d is an inner derivation in Q, the Martindale quotient ring of R, in the sense that there exists $q \in Q$ such that $d(x) = [q,x]$, for all $x \in R$, and I satisfies the generalized polynomial identity

$$f(r_1,...,r_n,[q,r_1],...,[q,r_n]);$$

2) or I satisfies the generalized polynomial identity

$$f(x_1,...,x_n,y_1,...,y_n).$$

In [14], T.K. Lee and W.K. Shiue proved a version of Kharchenko's theorem for generalized derivations and presented some results concerning certain identities with generalized derivations. More detail about generalized derivations can be in [9, 13] and [14].

We recall some related known result in literature: We say that an additive map F acts as a homomorphism on a nonempty subset $T \subseteq R$, if $F(xy) = F(x)F(y)$ for all $x, y \in T$; F acts as an anti-homomorphism on T, if $F(xy) = F(y)F(x)$ for all $x, y \in T$; finally F acts as a Jordan homomorphism on T if $F(x^2) = F(x)^2$ for all $x, y \in T$. Obviously any additive mapping, which is a homomorphism or an anti-homomorphism, is a Jordan homomorphism. On the other hand, in [8] Herstein proved that in case R is a prime ring of characteristic different from 2, any Jordan homomorphism on R is either a homomorphism or an anti-homomorphism of R. In [17], Rehman proved:

Theorem 2 ([17], Theorem 1.2)**.** *Let R be a prime ring of characteristic different from* 2 *and F a nonzero generalized derivation of R, with associated derivation d. If F acts as homomorphism or anti-homomorphism on a two-sided ideal of R, then R is commutative unless $d = 0$.*

Recently in [6], De Filippis extended the Rehman's result as follows:

Theorem 3 ([6], Theorem 2)**.** *Let R be a prime ring, L a noncetral Lie ideal of R and F a nonzero generalized derivation of R. If F acts as a Jordan homomorphism on L, then either $F(x) = x$ for all $x \in R$, or $char(R) = 2$, R satisfies the standard identity $s_4(x_1, x_2, x_3, x_4)$, L is commutative and $u^2 \in Z(R)$, for all $u \in L$.*

By motivating above results, in the present paper our aim is to obtain a generalization of Rehman's one in [17], moreover this study is a partial generalization of the result in [6] (in case $I = L$ is a two-sided ideal of R).

Throughout the paper, we denote by I_{id} the identity map of a ring R (i.e., the map $I_{id} : R \to R$ defined by $I_{id}(x) = x$ for all $x \in R$).

2. Results

In the following, we assume that R is a prime ring and that $Z(R)$ is the center of R without stated otherwise.

For the proof of our main results we need the following lemma.

Lemma 1. *Let R be a noncommutative prime ring with a generalized derivation d associated with a derivation α of R. Suppose that $0 \neq c$ is an element of R such that $cd(x) \in Z(R)$ for all $x \in R$. Then there exists $q \in U$ such that $d(x) = qx$ and $cq = 0$.*

Proof. By Theorem 1 we can write d as the form $d(x) = qx + \alpha(x)$, where $q \in U$. By the hypothesis we have $c(qx + \alpha(x)) \in Z(R)$ for all $x \in R$. Since R and U satisfy the same differential identity [12] we get

$$c(qx + \alpha(x)) \in C \quad \text{for all } x \in U. \tag{2.1}$$

Suppose first that $\alpha \neq 0$. By the result of modulo Kharchenko's Theorem [10] we can divide the proof into two cases.
Assume first that α is an inner derivation of U induced by an element $b \in U$, that is $[b,x]$, for all $x \in U$. In this case $d(x) = qx + [b,x]$. By the hypothesis we have $c(qx + [b,x]) \in C$ for all $x \in U$. Hence above relation implies that

$$[r, c(qx + [b,x])] = 0 \quad \text{for all } r, x \in U \tag{2.2}$$

and in particular $cq \in C$. Replacing x by b we get $cq[r,b] = 0$ for all $r \in U$. By the primeness of R we obtain that either $cq = 0$ or $b \in C$. Since $\alpha \neq 0$ we are forced to consider the first case. Let $cq = 0$. By (2.2) we get $[r, c[b,x]] = 0$ for all $r, x \in U$. Substituting xb for x in the last relation we have

$$c[b,x][r,b] = 0 \quad \text{for all } r, x \in U.$$

By the primeness of U and by the supposing on α the above relation implies that $c = 0$, a contradiction.
Assume now that α is not an inner derivation of U. By Kharchenko's Theorem in [10, 12], we get $c(qx + y) \in C$ for all $x, y \in U$. In particular we obtain that $cqx \in C$ for all $x \in U$. Since R is noncommutative prime ring and $cq \in C$ we arrive at $cq = 0$. By the last relation we get $cy \in C$ implying that $c = 0$, a contradiction.
Thanks to two contradictions we are forced to assume that $\alpha = 0$. So we get $d(x) = qx$ and using (2.1) we also obtain that $cq = 0$, as asserted. □

Now we are ready to prove our main results. The following theorem may be considered as a generalization of [1], Theorem 3.4.

Theorem 4. *Let R be a prime ring with center $Z(R)$ and I be a nonzero ideal of R. If R admits a nonzero generalized derivation d of R, with associated derivation α such that $d(xy) - d(x)d(y) \in Z(R)$ or $d(xy) + d(x)d(y) \in Z(R)$ for all $x, y \in I$, then either R is commutative or $d = I_{id}$ or $d = -I_{id}$.*

Proof. As we have remarked above we may take a generalized derivation d as the form $d(x) = ax + \alpha(x)$ for all $x \in U$ where $a \in U$ and it is known that R and I satisfy the same differential identity [12]. So we may assume that R admits a generalized derivation such that $d(xy) - d(x)d(y) \in Z(R)$ or $d(xy) + d(x)d(y) \in Z(R)$ for all $x, y \in R$. For each $y \in R$ we consider two subsets $K_y = \{x \in R : d(xy) - d(x)d(y) \in Z(R)\}$ and $M_y = \{x \in R : d(xy) + d(x)d(y) \in Z(R)\}$. Then K_y and M_y are two additive subgroups of $(R, +)$ such that $(R, +) = K_y \cup M_y$; and since a group cannot be the union of two proper subgroups, we have that either $R = K_y$ or $R = M_y$ for all $y \in R$. Repeating the same above argument we obtain that either $R = \{y \in R : R = K_y\}$ or $R = \{y \in R : R = M_y\}$. Note that the second case can be reduced to the first case. Indeed, since $f = -d$ is also a generalized derivation of R associated with a derivation $\beta = -\alpha$ the latter case just means that $f(xy) - f(x)f(y) \in Z(R)$ for all $x, y \in R$. Thus we only need to handle the case that

$$d(xy) - d(x)d(y) \in Z(R) \quad \text{for all } x, y \in R.$$

If R is commutative we are done. So we may suppose that R is not commutative. For some $a \in U$ write $d(x) = ax + \alpha(x)$ in the last relation. Since R and U satisfy the same differential identity [12] we have

$$d(xy) - d(x)d(y) \in C \quad \text{for all } x, y \in U. \tag{2.3}$$

Take 1 instead of x in (2.3). Hence we get $(1-a)d(y) \in C$ for all $y \in U$.
First suppose that $a \neq 1$. In view of Lemma 1 there exists $q \in U$ such that $d(y) = qy$ for all $y \in U$ and $(1-a)q = 0$. By (2.3) we have $qxy - qxqy \in C$ and so $qx(1-q)y \in C$ for all $x, y \in U$. Since R is a noncommutative prime ring the last relation gives us that $q = 0$ or $q = 1$. The first case implies that $d = 0$, a contradiction. Moreover it is easily seen that $a = q$. Thus the second case gives a contradiction.
Now suppose that $a = 1$. By (2.3) we have

$$\alpha(x)\alpha(y) \in C \quad \text{for all } x, y \in U. \tag{2.4}$$

Applying Lemma 1 to (2.4), we obtain $\alpha(x)\alpha(y) = 0$ for all $x, y \in U$. Replacing x by xz in the last relation we get $\alpha(x)z\alpha(y) = 0$ for all $x, y, z \in U$. By the primeness of U we arrive at $\alpha = 0$. By the last relation and the assumption $a = 1$ we arrive at $d = I_{id}$, as asserted. □

Theorem 5. *Let R be a prime ring with center $Z(R)$ and I be a nonzero ideal of R. If R admits a nonzero generalized derivation d of R, with associated derivation α such that $d(xy) - d(y)d(x) \in Z(R)$ or $d(xy) + d(y)d(x) \in Z(R)$ for all $x, y \in I$, then R is commutative.*

Proof. In a similar manner as the proof of Theorem 4 we obtain that either $d(xy) - d(y)d(x) \in Z(R)$ for all $x, y \in R$ or $d(xy) + d(y)d(x) \in Z(R)$ for all $x, y \in R$. As stated before, since the second case can be reduced to the first case by using the observation in the proof of Theorem 4, we consider only the case

$$d(xy) - d(y)d(x) \in Z(R) \quad \text{for all } x, y \in R.$$

If R is commutative we are done. So we may suppose that R is not commutative. By Theorem 1, for some $a \in U$ write $d(x) = ax + \alpha(x)$ for all $x \in R$ and since R and U satisfy the same differential identity [12] we have

$$d(xy) - d(y)d(x) \in C \quad \text{for all } x, y \in U. \tag{2.5}$$

Substituting 1 for y in (2.5) we get $(1-a)d(x) \in C$ for all $x \in U$.
If $a \neq 1$, there exits $q \in U$ such that $d(x) = qx$ and $(1-a)q = 0$ by Lemma 1. Using this fact in (2.5) we have

$$qxy - qyqx \in C \quad \text{for all } x, y \in U.$$

Replacing x by xy we get $(qxy - qyqx)y \in C$ for all $x, y \in U$. Since $qxy - qyqx \in C$ and $(qxy - qyqx)y \in C$ for all $x, y \in U$, we see that for every $y \in U$, $qxy - qyqx = 0$ for all $x \in U$ or $y \in C$. Recall that R is noncommutative. So $qxy - qyqx = 0$ for all $x, y \in U$. Setting $x = 1$ in the last relation, we get $qU(1-q) = 0$. So the

last relation implies that $q=0$ or $q=1$. If $q=0$, then $d=0$, a contradiction to our hypothesis. If $q=1$, then $xy-yx=0$ for all $x,y\in U$ and hence R is commutative, a contradiction to our assumption.
Now let $a=1$. Then by the hypothesis we have $xy+\alpha(x)y+x\alpha(y)-yx-y\alpha(x)-\alpha(y)x-\alpha(y)\alpha(x)\in C$ for all $x,y\in U$ yielding that

$$[x,y]+[\alpha(x),y]+[x,\alpha(y)]-\alpha(y)\alpha(x)\in C \quad \text{for all } x,y\in U. \tag{2.6}$$

If $\alpha=0$, then (2.6) implies that $[x,y]\in C$ for all $x,y\in U$ which gives us that R is commutative, a contradiction. So we can assume that $\alpha\neq 0$. By Kharchenko's Theorem [10], if α is an inner derivation induced by an element $b\in U\setminus C$ such that $\alpha(x)=[b,x]$ for all $x\in U$ then replacing y by b in (2.6) we get $[x,b]+[\alpha(x),b]\in C$ for all $x\in U$. Taking xb instead of x we have $([x,b]+[\alpha(x),b])b\in C$ for all $x\in U$. Since $b\notin C$ we obtain $0=[x,b]+[\alpha(x),b]=\alpha(x)+\alpha^2(x)$. Replacing x by $\alpha(x)$ in (2.6) and using the last relation we have $\alpha(x)\alpha(y)\in C$. Replacing y by yb in the last relation and using $b\notin C$ we get $\alpha(x)\alpha(y)=0$ for all $x,y\in U$ yielding that $\alpha=0$, a contradiction. If α is not inner, then by Kharchenko's Theorem in [10, 12], we get

$$[x,y]+[z,y]+[x,w]-wz\in C \quad \text{for all } x,y,z,w\in U.$$

In particular we obtain $[x,y]\in C$ for all $x,y\in U$ yielding that R is commutative, a contradiction. □

Example 1. Let R_1 be any commutative and R_2 any noncommutative ring. Define the ring R as $R=R_1\oplus R_2=\{(a,b): a\in R_1 \, and \, b\in R_2\}$. It is clear that R is a noncommutative ring. Let δ be any derivation of R_1. Define an additive map $\alpha: R\to R$ as $\alpha((a,b))=(\delta(a),0)$, where $(a,b)\in R$. One can be easily shown that α is a derivation on R. Then the map $d: R\to R$ defined as $d((a,b))=(a+\delta(a),b)$ is a generalized derivation on R associated with the derivation α. It is easy to verify that d satisfies $d(xy)-d(x)d(y)\in Z(R)$ for all $x,y\in R$, but neither R is commutative, nor $d=0$ nor $d=I_{id}$.

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REFERENCES

[1] E. Albaş and N. Argaç, "Generalized derivations of prime rings," *Algebra Colloq.*, vol. 11, no. 3, pp. 399–410, 2004.
[2] K. I. Beidar, "Rings of quotients of semiprime rings," *Vestn. Mosk. Univ., Ser. I*, vol. 33, no. 5, pp. 36–43, 1978.
[3] K. I. Beidar, W. S. Martindale, and A. V. Mikhalev, *Rings with generalized identities*, ser. Pure and Applied Mathematics. New York: Marcel Dekker, 1996.
[4] M. Brešar, "On the distance of the composition of two derivations to the generalized derivations," *Glasg. Math. J.*, vol. 33, no. 1, pp. 89–93, 1991.

[5] C.-L. Chuang, "GPIs having coefficients in Utumi quotient rings," *Proc. Am. Math. Soc.*, vol. 103, no. 3, pp. 723–728, 1988.
[6] V. De Filippis, "Generalized derivations as Jordan homomorphisms on Lie ideals and right ideals," *Acta Math. Sin., Engl. Ser.*, vol. 25, no. 12, pp. 1965–1974, 2009.
[7] A. Giambruno and I. N. Herstein, "Derivations with nilpotent values," *Rend. Circ. Mat. Palermo, II. Ser.*, vol. 30, pp. 199–206, 1981.
[8] I. N. Herstein, *Topics in ring theory*, ser. Chicago Lectures in Mathematics. Chicago-London: The University of Chicago Press, 1969.
[9] B. Hvala, "Generalized derivations in rings," *Commun. Algebra*, vol. 26, no. 4, pp. 1147–1166, 1998.
[10] V. K. Kharchenko, "Differential identities of prime rings," *Algebra Logic*, vol. 17, pp. 155–168, 1978.
[11] P. H. Lee and T. K. Lee, "On derivations of prime rings," *Chin. J. Math.*, vol. 9, no. 2, pp. 107–110, 1981.
[12] T.-K. Lee, "Semiprime rings with differential identities," *Bull. Inst. Math., Acad. Sin.*, vol. 20, no. 1, pp. 27–38, 1992.
[13] T.-K. Lee, "Generalized derivations of left faithful rings," *Commun. Algebra*, vol. 27, no. 8, pp. 4057–4073, 1999.
[14] T.-K. Lee and W.-K. Shiue, "Identities with generalized derivations," *Commun. Algebra*, vol. 29, no. 10, pp. 4437–4450, 2001.
[15] W. S. Martindale, "Prime rings satisfying a generalized polynomial identity," *J. Algebra*, vol. 12, pp. 576–584, 1969.
[16] E. C. Posner, "Derivations in prime rings," *Proc. Am. Math. Soc.*, vol. 8, pp. 1093–1100, 1957.
[17] N. ur Rehman, "On generalized derivations as homomorphisms and anti-homomorphisms," *Glas. Mat., III. Ser.*, vol. 39, no. 1, pp. 27–30, 2004.

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