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Mehmet Ali Özarslan

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A-STATISTICAL CONVERGENCE OF MITTAG-LEFFLER OPERATORS

MEHMET ALI ÖZARSLAN

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Abstract. In this paper we introduce the Mittag-Leffler operators, which includes the modified Szász-Mirakjan operators. We obtain the transformation properties and compute the rate of convergence by using modulus of continuity. Furthermore we give the A-statistical approximation theorem for these operators.

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1. INTRODUCTION

The function defined by [11]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (z \in \mathbb{C}; \mathcal{R}(\alpha) > 0)$$

is known as the Mittag-Leffler function. The two-index Mittag-Leffler function is defined by [14]

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad (z, \beta \in \mathbb{C}; \mathcal{R}(\alpha) > 0).$$

Note that $E_{\alpha,1}(z) = E_\alpha(z)$ and

$$E_{1,1}(z) = e^z,\ E_{1,2}(z) = \frac{e^z - 1}{z},\ E_{1,m+1}(z) = \frac{e^z - \sum_{k=0}^{m-1} \frac{x^k}{k!}}{z^m}.$$

Moreover, for $|z| < 2\pi$, we have

$$\frac{1}{E_{1,2}(z)} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \qquad \frac{1}{E_{1,m+1}(z)} = \sum_{n=0}^{\infty} B_n^{(m)} \frac{z^n}{n!}$$

where the coefficients (B_n) are the familiar Bernoulli numbers and $\left(B_n^{(m)}\right)$ are the generalized Bernoulli numbers (see [2]).

Let (b_n) be a sequence of positive real numbers and let $\beta > 0$ be fixed. For all $n \in \mathbb{N}$, we introduce the Mittag-Leffler operators by

$$L_n^{(\beta)}(f;x) = \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}\right)} \sum_{k=0}^{\infty} f\left(\frac{k}{n}b_n\right) \frac{(nx)^k}{b_n^k \Gamma(k+\beta)}, \tag{1.1}$$

where $f \in E := \left\{ f \in C[0,+\infty) : \lim_{x\to+\infty} \frac{f(x)}{1+x^2} \text{is finite} \right\}$ and $C[0,+\infty)$ denotes the space of continuous functions defined on $[0,+\infty)$. Recall that the Banach lattice E is endowed with the norm

$$\|f\|_* := \sup_{x\in[0,+\infty)} \frac{|f(x)|}{1+x^2}.$$

It is obvious that the operators $L_n^{(\beta)}(f;x)$ defined in (1.1) are linear and positive.

Note that for $\beta = 1$, we have

$$L_n^{(1)}(f;x) = e^{-nx/b_n} \sum_{k=0}^{\infty} f\left(\frac{k}{n}b_n\right) \frac{(nx)^k}{b_n^k k!} = S_n(f;x)$$

where the operators S_n are the modified Szász-Mirakjan operators considered in [1].

By direct computations one can state the following lemma;

Lemma 1. *Let $\psi_x^2(t) = (t-x)^2$. Then, for each $x \geq 0$ and $n \in \mathbb{N}$, we have*

(a) $L_n^{(\beta)}(1;x) = 1$,

(b) $\left| L_n^{(\beta)}(t;x) - x \right| \leq \frac{|1-\beta| b_n}{n}$,

(c)

$$\left| L_n^{(\beta)}(t^2;x) - x^2 \right| \leq \frac{(2|1-\beta|+1) b_n}{n} x + \frac{\left(2(1-\beta)^2 + |1-\beta| + |1-\beta||\beta-2|\right) b_n^2}{n^2}$$

(d)

$$L_n^{(\beta)}\left(\psi_x^2;x\right) \leq \frac{(4|1-\beta|+1) b_n}{n} x + \frac{\left(2(1-\beta)^2 + |1-\beta| + |1-\beta||\beta-2|\right) b_n^2}{n^2}.$$

Proof. Since

$$\sum_{k=0}^{\infty} \frac{(nx)^k}{b_n^k \Gamma(k+\beta)} = E_{1,\beta}\left(\frac{nx}{b_n}\right),$$

then $L_n^{(\beta)}(1;x) = 1$. Using the fact that $\Gamma(k+\beta) = (k+\beta-1)\,\Gamma(k+\beta-1)$, we get

$$\begin{aligned}
L_n^{(\beta)}(t;x) &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}\right)} \sum_{k=1}^{\infty} \frac{kb_n}{n} \frac{(nx)^k}{b_n^k \Gamma(k+\beta)} \\
&= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}\right)} \sum_{k=1}^{\infty} \frac{[(k+\beta-1)+1-\beta]\,b_n}{n} \frac{(nx)^k}{b_n^k\,(k+\beta-1)\,\Gamma(k+\beta-1)} \\
&= x + \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}\right)} \sum_{k=1}^{\infty} \frac{1-\beta}{n} \frac{b_n (nx)^k}{b_n^k \Gamma(k+\beta)}. \qquad (1.2)
\end{aligned}$$

Hence

$$\left| L_n^{(\beta)}(t;x) - x \right| = \frac{|1-\beta|\,b_n}{n} \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}\right)} \sum_{k=1}^{\infty} \frac{(nx)^k}{b_n^k \Gamma(k+\beta)} \le \frac{|1-\beta|\,b_n}{n}.$$

Similarly, by $k(k-1) = (k+\beta-1)\,(k+\beta-2) + 2(1-\beta)k + (1-\beta)(\beta-2)$ and $\Gamma(k+\beta) = (k+\beta-1)\,(k+\beta-2)\Gamma(k+\beta-2)$, we get

$$\begin{aligned}
L_n^{(\beta)}(t^2;x) &= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}\right)} \sum_{k=1}^{\infty} \left(\frac{k}{n} b_n\right)^2 \frac{(nx)^k}{b_n^k \Gamma(k+\beta)} \\
&= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}\right)} \sum_{k=1}^{\infty} \frac{(k(k-1)+k)\,b_n^2}{n^2} \frac{(nx)^k}{b_n^k \Gamma(k+\beta)} \\
&= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}\right)} \sum_{k=2}^{\infty} \frac{k(k-1)b_n^2}{n^2} \frac{(nx)^k}{b_n^k \Gamma(k+\beta)} + \frac{b_n L_n^{(\beta)}(t;x)}{n} \\
&= \frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}\right)} \sum_{k=2}^{\infty} \frac{(k+\beta-1)\,(k+\beta-2)b_n^2}{n^2} \\
&\quad \times \frac{(nx)^k}{b_n^k\,(k+\beta-1)\,(k+\beta-2)\Gamma(k+\beta-2)}
\end{aligned}$$

$$+\frac{1}{E_{1,\beta}\left(\frac{nx}{b_n}\right)}\sum_{k=2}^{\infty}\frac{2(1-\beta)kb_n^2}{n^2}\frac{(nx)^k}{b_n^k\Gamma(k+\beta)}$$
$$+\frac{(1-\beta)(\beta-2)b_n^2}{n^2E_{1,\beta}\left(\frac{nx}{b_n}\right)}\sum_{k=2}^{\infty}\frac{(nx)^k}{b_n^k\Gamma(k+\beta)}+\frac{b_nL_n^{(\beta)}(t;x)}{n}.$$

Therefore

$$\left|L_n^{(\beta)}(t^2;x)-x^2\right|\leq\frac{2|1-\beta|b_n}{nE_{1,\beta}\left(\frac{nx}{b_n}\right)}\sum_{k=2}^{\infty}\frac{kb_n}{n}\frac{(nx)^k}{b_n^k\Gamma(k+\beta)}$$
$$+\frac{|1-\beta||\beta-2|b_n^2}{n^2E_{1,\beta}\left(\frac{nx}{b_n}\right)}\sum_{k=2}^{\infty}\frac{(nx)^k}{b_n^k\Gamma(k+\beta)}+\frac{b_n\left|L_n^{(\beta)}(t;x)\right|}{n}$$
$$\leq\frac{(2|1-\beta|+1)b_n}{n}\left|L_n^{(\beta)}(t;x)\right|+\frac{|1-\beta||\beta-2|b_n^2}{n^2}.$$

Using (1.2), we obtain

$$\left|L_n^{(\beta)}(t^2;x)-x^2\right|\leq\frac{(2|1-\beta|+1)b_n}{n}x$$
$$+\frac{\left(2(1-\beta)^2+|1-\beta|+|1-\beta||\beta-2|\right)b_n^2}{n^2}.$$

Finally,

$$L_n^{(\beta)}\left(\psi_x^2;x\right)$$
$$\leq\left|L_n^{(\beta)}(t^2;x)-x^2\right|+2x\left|L_n^{(\beta)}(t;x)-x\right|+x^2\left|L_n^{(\beta)}(1;x)-1\right|$$
$$\leq\frac{(4|1-\beta|+1)b_n}{n}x+\frac{\left(2(1-\beta)^2+|1-\beta|+|1-\beta||\beta-2|\right)b_n^2}{n^2}$$

which completes the proof. □

We organize the paper as follows: In Section 2, we give the transformation properties of the operators $L_n^{(\beta)}$ and compute the rate of convergence by using the modulus of continuity. In Section 3, we prove an A-statistical Korovkin type approximation theorem.

2. Transformation properties and Rate of Convergence

We start with the following lemma, which proves that $L_n^{(\beta)}$ maps E into itself.

Lemma 2. *Let* $\left(\frac{b_n}{n}\right)$ *be a bounded sequence of positive numbers and* $\beta > 0$ *be fixed. Then there exists a constant* $M(\beta)$ *such that, for* $w(x) = \left(1+x^2\right)^{-1}$, *we have*

$$w(x)L_n^{(\beta)}(\frac{1}{w};x) \leq M(\beta)$$

holds for all $x \in [0,\infty)$ *and* $n \in \mathbb{N}$. *Furthermore, for all* $f \in E$, *we have*

$$\left\| L_n^{(\beta)}(f) \right\|_* \leq M(\beta) \| f \|_* .$$

Proof. Using Lemma 1, we can write that

$$\begin{aligned} w(x)L_n^{(\beta)}(\frac{1}{w};x) &= \frac{1}{1+x^2}\left[L_n^{(\beta)}(1;x) + L_n^{(\beta)}(t^2;x)\right] \\ &\leq \frac{1}{1+x^2}\left[1+x^2+\frac{(2|1-\beta|+1)\,b_n}{n}x \right. \\ &\quad \left. +\frac{\left(2(1-\beta)^2+|1-\beta|+|1-\beta|\,|\beta-2|\right)b_n^2}{n^2}\right] \\ &\leq M(\beta). \end{aligned}$$

On the other hand

$$w(x)\left|L_n^{(\beta)}(f;x)\right| = w(x)\left|L_n^{(\beta)}(w\frac{f}{w};x)\right| \leq \|f\|_* \, w(x)L_n^{(\beta)}(\frac{1}{w};x) \leq M(\beta)\|f\|_* .$$

Taking supremum over $x \in [0,\infty)$ in the above inequality, gives the result. □

Now, recall that the usual modulus of continuity of f on the closed interval $[0,B]$ is defined by

$$\omega_B(f,\delta) = \sup_{\substack{|t-x|\leq\delta \\ x,t\in[0,B]}} |f(t)-f(x)| .$$

It is well known that, for a function $f \in E$, we have $\lim_{\delta\to\infty}\omega_B(f,\delta) = 0$.

The next theorem gives the rate of convergence of the operators $L_n^{(\beta)}(f;x)$ to $f(x)$, for all $f \in E$.

Theorem 1. *Let* $\beta > 0$ *be fixed,* $\left(\frac{b_n}{n}\right)$ *be a bounded sequence of positive numbers,* $f \in E$ *and* $\omega_{B+1}(f,\delta)$ $(B > 0)$ *be its modulus of continuity on the finite interval* $[0,B+1] \subset [0,\infty)$. *Then*

$$\left\| L_n^{(\beta)}(f;x) - f(x) \right\|_{C[0,B]} \leq M_f(\beta,B)\delta_n(\beta,B) + 2\omega_{B+1}(f,\delta_n^{1/2}(\beta,B))$$

where $\delta_n(\beta,B) = N_f(\beta,B)\frac{b_n}{n}\left[1+\frac{b_n}{n}\right]$,

$$N_f(\beta,B) = \max\left\{(4|1-\beta|+1)B, \left(2(1-\beta)^2+|1-\beta|+|1-\beta||\beta-2|\right)\right\}$$

and $M_f(\beta,B)$ *is an absolute constant depending on* f,β *and* B.

Proof. Let $\beta>0$ be fixed. For $x\in[0,B]$ and $t\le B+1$, we have the inequality

$$|f(t)-f(x)| \le \omega_{B+1}(f,|t-x|) \le \left(1+\frac{|t-x|}{\delta}\right)\omega_{B+1}(f,\delta) \tag{2.1}$$

where $\delta>0$. On the other hand, for $x\in[0,B]$ and $t>B+1$, using the fact that $t-x>1$, we have

$$|f(t)-f(x)| \le A_f(1+x^2+t^2) \le A_f(2+3x^2+2(t-x)^2) \le 6A_f\left(1+B^2\right)(t-x)^2 \tag{2.2}$$

By (2.1) and (2.2), we get for all $x\in[0,B]$ and $t\ge 0$ that

$$|f(t)-f(x)| \le 6A_f\left(1+B^2\right)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{B+1}(f,\delta).$$

Therefore

$$\left|L_n^{(\beta)}(f;x)-f(x)\right|$$
$$\le 6A_f\left(1+B^2\right)L_n^{(\beta)}((t-x)^2;x) + \left(1+\frac{L_n^{(\beta)}(|t-x|;x)}{\delta}\right)\omega_{B+1}(f,\delta).$$

Applying Cauchy-Schwarz inequality and Lemma 1, we get

$$\left|L_n^{(\beta)}(f;x)-f(x)\right|$$
$$\le 6A_f\left(1+B^2\right)L_n^{(\beta)}\left(\psi_x^2;x\right) + \left(1+\frac{\left[L_n^{(\beta)}\left(\psi_x^2;x\right)\right]^{1/2}}{\delta}\right)\omega_{B+1}(f,\delta)$$
$$\le 6A_f\left(1+B^2\right)$$
$$\times\left[(4|1-\beta|+1)B\frac{b_n}{n} + \left(2(1-\beta)^2+|1-\beta|+|1-\beta||\beta-2|\right)\frac{b_n^2}{n^2}\right]$$
$$+\left(1+\frac{\left[(4|1-\beta|+1)B\frac{b_n}{n} + \left(2(1-\beta)^2+|1-\beta|+|1-\beta||\beta-2|\right)\frac{b_n^2}{n^2}\right]^{1/2}}{\delta}\right)$$
$$\times\omega_{B+1}(f,\delta) \le M_f(\beta,B)\delta_n(\beta,B) + 2\omega_{B+1}(f,(\delta_n(\beta,B))^{1/2}),$$

where

$$N_f(\beta,B) = \max\left\{(4|1-\beta|+1)\,B, \left(2(1-\beta)^2+|1-\beta|+|1-\beta|\,|\beta-2|\right)\right\},$$

$M_f(\beta,B) = 6A_f\left(1+B^2\right)$ and $\delta_n(\beta,B) = N_f(\beta,B)\dfrac{b_n}{n}\left[1+\dfrac{b_n}{n}\right]$. Whence the result follows. □

3. A-STATISTICAL CONVERGENCE

Recently, A-statistical convergence of linear positive operators have been an active research area (see [3–5, 12]). We start to this section by recalling concepts of A-statistical convergence.

Let $A = (a_{jk})$ be a non-negative regular summability matrix.

Definition 1. The A-density of a subset K of $\mathbb{N}$ is given by

$$\delta_A(K) = \lim_j \sum_{k\in K} a_{j,k}, \tag{3.1}$$

provided that limit exists (see [7]).

Definition 2. A sequence $x = (x_n)$ is said to be A-statistically convergent to l and denoted by $\mathrm{st}_A\text{-}\lim x = l$ if for every $\varepsilon > 0$, $\delta_A\{n \in \mathbb{N} : |x_n - l| \geq \varepsilon\} = 0$ (see [6, 13]).

Taking $A = C_1$, the Cesaro matrix of order one in (3.1), A-statistical convergence reduces to statistical convergence [8, 10].Taking $A = I$, the identity matrix then A-statistical convergence reduces to ordinary convergence. Kolk [9] proved that in the case of $\lim_j \max_n \left|a_{j,n}\right| = 0$, A-statistical convergence is stronger than ordinary convergence.

Now let $A = (a_{jn})$ be a non-negative regular summability matrix. Assume that $(b_n)_{n\in\mathbb{N}}$ is a sequence in $[0,\infty)$ satisfying

$$st_A\text{-}\lim_n \frac{b_n}{n} = 0. \tag{3.2}$$

Then we have

$$st_A\text{-}\lim_n \left(\frac{b_n}{n}\right)^2 = 0. \tag{3.3}$$

Such a sequence $(b_n)_{n\in\mathbb{N}}$ satisfying (3.2), can be constructed as follows: Take $A = C_1$, and define

$$b_n := \begin{cases} n, & \text{if } n = m^2 \ (m \in \mathbb{N}) \\ n^{1/3}, & \text{otherwise.} \end{cases} \tag{3.4}$$

Then clearly $st_{C_1}\text{-}\lim \dfrac{b_n}{n} = st\text{-}\lim \dfrac{b_n}{n} = 0$.

Theorem 2. *Let $A=(a_{jk})$ be a non-negative regular summability matrix and $\beta>0$ be fixed. If*

$$st_A\text{-}\lim_n \frac{b_n}{n}=0$$

then

$$st_A\text{-}\lim_n \left\| L_n^{(\beta)}(f;x)-f(x)\right\|_{C[0,B]}=0$$

holds for every $f\in E$.

Proof. Given $r>0$ choose $\varepsilon>0$ such that $\varepsilon<r$. For fixed $\beta>0$, define the following sets:

$$U:=\{n:\delta_n(\beta,B)\geq r\},$$
$$U_1:=\left\{n:\frac{b_n}{n}\geq\frac{r-\varepsilon}{2N_f(\beta,B)}\right\},$$
$$U_2:=\left\{n:\left(\frac{b_n}{n}\right)^2\geq\frac{r-\varepsilon}{2N_f(\beta,B)}\right\},$$

where $N_f(\beta,B)$ and $\delta_n(\beta,B)$ be the same as in Theorem 1. Then it is clear that $U\subseteq U_1\cup U_2$, which gives

$$\sum_{k\in U}a_{jk}\leq\sum_{k\in U_1}a_{jk}+\sum_{k\in U_2}a_{jk}. \tag{3.5}$$

Letting $j\to\infty$ in (3.5) and using (3.2) and (3.3), we have $\lim_j\sum_{k\in U}a_{jk}=0$. This proves that $st_A\text{-}\lim_n\delta_n(\beta,B)=0$ which also implies

$$st_A\text{-}\lim_n\omega_{B+1}(f,\delta_n^{1/2}(\beta,B))=0.$$

Using Theorem 1 we get the result. □

Remark that choosing the sequence $(b_n)_{n\in\mathbb{N}}$ as in (3.4), the statistical approximation results in Theorem 2 works, however its classical case does not work since $\left(\frac{b_n}{n}\right)_{n\in\mathbb{N}}$ is not convergent in the ordinary sense.

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Author's address

Mehmet Ali Özarslan
Eastern Mediterranean University, Gazimagusa, TRNC, Mersin 10, Turkey
E-mail address: mehmetali.ozarslan@emu.edu.tr