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GLOBAL HÖLDER ESTIMATES FOR HYPOELLIPTIC OPERATORS WITH DRIFT ON HOMOGENEOUS GROUPS

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Abstract. Let $X_0, X_1, \dots, X_q$ be left invariant real vector fields on the homogeneous group G, satisfying Hörmander's condition on $\mathbb{R}^N$. Assume that $X_1, \dots, X_q$ are homogeneous of degree one and X_0 is homogeneous of degree two. In this paper we consider the following hypoelliptic operator with drift

$$L = \sum_{i,j=1}^{q} a_{ij} X_i X_j + a_0 X_0,$$

where (a_{ij}) is a $q \times q$ positive constant matrix and $a_0 \neq 0$, and obtain Global Hölder estimates for L on G by establishing several estimates of singular integrals.

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Keywords: homogeneous group, singular integral, global Hölder estimate

1. INTRODUCTION

Let G be a homogeneous group and $X_0, X_1, ..., X_q$ be left invariant real vector fields on $\mathbb{R}^N (q < N)$. Assume that $X_1, ..., X_q$ are homogeneous of degree one and X_0 is homogeneous of degree two, satisfying Hörmander's condition

$$rank\,\mathcal{L}(X_0, X_1, ..., X_q)(x) = N, x \in \mathbb{R}^N,$$

where $\mathcal{L}(X_0, X_1, ..., X_q)$ denotes the Lie algebra generated by $X_0, X_1, ..., X_q$. In this paper we are interested in the following hypoelliptic operator with drift

$$L = \sum_{i,j=1}^{q} a_{ij} X_i X_j + a_0 X_0, \tag{1.1}$$

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where $a_0 \neq 0, (a_{ij})_{i,j=1}^q$ is a constant matrix satisfying

$$\mu^{-1}|\xi|^2 \leq \sum_{i,j=1}^{q} a_{ij}\xi_i\xi_j \leq \mu|\xi|^2, \xi \in R^q, \tag{1.2}$$

for a constant $\mu > 0$.

Many authors paid attention to the hypoelliptic operator. The outstanding result in [8] points out that Hörmander's condition implies (actually, is equivalent to) the hypoellipticity of L in (1.1). The existence of fundamental solutions for homogeneous hypoelliptic operators on nilpotent Lie groups was investigated by Folland in [6]. Bramanti and Brandolini in [2] proved the uniqueness of homogeneous fundamental solutions for L. Let us note that L includes the classic Laplace operator and parabolic operator on Euclidean spaces. Another special case of L is

$$L_1 = \sum_{i,j=1}^{q} a_{ij}\partial^2_{x_i x_j} + \sum_{i,j=1}^{n} b_{ij}x_i\partial_{x_j} - \partial_t,$$

where $(x,t) \in \mathbb{R}^{n+1}$, $X_0 = \sum_{i,j=1}^{n} b_{ij}x_i\partial_{x_j} - \partial_t$ and $X_i = \partial_{x_i}, i = 1,2,\ldots,q$,$(a_{ij})_{i,j=1}^q$ is a positive matrix in R^q, $(b_{ij})$is a constant matrix with a suitable upper triangular structure. Note that L_1 belongs to a class of Kolmogorov-Fokker-Planck ultraparabolic operators. The operator L_1 appears in many research fields, for instance, in stochastic processes and kinetic models (see [3–5]), and in mathematical finance theory (see [1, 12]). After the previous study on L_1 in [9, 10], the authors of [7, 11, 13] established an invariant Harnack inequality for the non-negative solution of $L_1u = 0$ by applying the mean value formula. With the theory of singular integral, Polidoro and Ragusa in [14] concluded some Morrey-type imbedding results and gave a local Hölder continuity of the solution.

The aim of the paper is to prove global Hölder estimates on the homogeneous group G for L by applying the properties of the fundamental solution for L and several estimates of singular integrals on the homogeneous space . The method here is inspired by that used in [14]. Our results reflect the relations between the Morrey norms of Lu and Hölder exponents for u and X_iu, $i = 1,2,\ldots,q$. In order to state our main results, we first introduce the definition of Morrey space.

Definition 1. For $p \in (1,\infty)$,$\lambda \in [0,Q)$, the Morrey space on homogeneous group G is defined by

$$L^{p,\lambda}(G) = \left\{g \in L^p_{loc}(G) : \|g\|_{L^{p,\lambda}(G)} < \infty\right\},$$

where

$$\|g\|_{L^{p,\lambda}(G)} = \left(\sup_{r>0,x\in G}\int_{B_r(x)}\frac{1}{r^\lambda}|g(y)|^p dy\right)^{1/p},$$

$B_r(x)$ and Q will be given in (2.1) and (2.2), respectively. Here $L^{p,0}(G)$=$L^p(G)$.

The main results of this paper are as follows. For the case $\lambda \neq 0$, we have

Theorem 1. (1) *If* $1 < p < \frac{Q}{2}$, $Q-2p < \lambda < Q-p$, *then there exists a positive constant* $c = c(p,\lambda)$ *such that for every* $u \in C_0^\infty(G)$ *and any* $x, z \in G$, $x \neq z$,

$$\frac{|u(x)-u(z)|}{\left\|z^{-1}\circ x\right\|^{\theta}} \leq c\,\|Lu\|_{L^{p,\lambda}(G)}, \tag{1.3}$$

where $\theta = \frac{2p+\lambda-Q}{p}$;

(2) *If* $1 < p < \frac{Q}{2}$, $Q-p < \lambda < Q$, *then there exists a positive constant* $c = c(p,\lambda)$ *such that for every* $u \in C_0^\infty(G)$ *and any* $x, z \in G$, $x \neq z$,

$$\frac{|X_i u(x)-X_i u(z)|}{\left\|z^{-1}\circ x\right\|^{\theta}} \leq c\,\|Lu\|_{L^{p,\lambda}(G)}, \tag{1.4}$$

where $i = 1,\cdots,q$ *and* $\theta = \frac{p+\lambda-Q}{p}$.

For $\lambda = 0$, we have the following results, which restores the known result previously proved in [1].

Remark 1. (1) Assume $\frac{Q}{2} < p < Q$. Then there exists a positive constant $c = c(p)$ such that for every $u \in C_0^\infty(G)$ and any $x, z \in G$, $x \neq z$,

$$\frac{|u(x)-u(z)|}{\left\|z^{-1}\circ x\right\|^{\theta}} \leq c\,\|Lu\|_{L^{p}(G)}, \tag{1.5}$$

where $\theta = \frac{2p-Q}{p}$;

(2) Assume $p > Q$. Then there exists a positive constant $c = c(p)$ such that for every $u \in C_0^\infty(G)$ and any $x, z \in G$, $x \neq z$,

$$\frac{|X_i u(x)-X_i u(z)|}{\left\|z^{-1}\circ x\right\|^{\theta}} \leq c\,\|Lu\|_{L^{p}(G)}, \tag{1.6}$$

where $i = 1,\cdots,q$ and $\theta = \frac{p-Q}{p}$.

The plan of the paper is as follows: in Section 2 we introduce some knowledge of homogeneous group and related lemmas. Estimates of two integral operators are proved. Section 3 is devoted to the proof of the main result.

2. Preliminary

Given a pair of mappings:

$$[(x,y) \mapsto x\circ y] : \mathbb{R}^N \times \mathbb{R}^N \mapsto \mathbb{R}^N; [x \mapsto x^{-1}] : \mathbb{R}^N \mapsto \mathbb{R}^N,$$

which are smooth, it follows that $\mathbb{R}^N$ with these mappings forms a group, and the identity is the origin. If there exist $0 < \omega_1 \leq \omega_2 \leq \ldots \leq \omega_N$, such that the dilations

$$D(\lambda) : (x_1, \ldots, x_N) \mapsto (\lambda^{\omega_1} x_1, \ldots, \lambda^{\omega_N} x_N), \lambda > 0,$$

are group automorphisms, then the space $\mathbb{R}^N$ with this structure is called a homogeneous group and denoted by G.

Definition 2. We define a homogeneous norm $\|\cdot\|$ in G by the following way: if for any $x \in G, x \neq 0$, it holds

$$\|x\| = \rho \quad \Leftrightarrow \quad |D(1/\rho)x| = 1,$$

where $|\cdot|$ denotes the Euclidean norm ; also, let $\|0\| = 0$.

It is not difficult to derive that the homogeneous norm satisfies

(1) $\|D(\lambda)x\| = \lambda \|x\|$ for every $x \in G, \lambda > 0$;
(2) there exists $c(G) \geq 1$, such that for every $x, y \in G$,

$$\left\|x^{-1}\right\| \leq c \|x\| and \|x \circ y\| \leq c(\|x\| + \|y\|).$$

In view of the above properties, it is natural to define the quasidistance d :

$$d(x, y) = \left\|y^{-1} \circ x\right\|.$$

The ball with respect to d is denoted by

$$B(x, r) \equiv B_r(x) = \{y \in G : d(x, y) < r\}. \tag{2.1}$$

Note $B(0, r) = D(r)B(0, 1)$, therefore

$$|B(x, r)| = r^Q |B(0, 1)|, x \in G, r > 0,$$

where

$$Q = \omega_1 + \ldots + \omega_N. \tag{2.2}$$

We will call that Q is the homogeneous dimension of G. In general, $Q \geq 3$.

Definition 3. A differential operators Y on G is said homogeneous of degree $\beta (\beta > 0)$, if for every test function φ,

$$Y(\varphi(D(\lambda)x)) = \lambda^\beta (Y\varphi)(D(\lambda)x), \lambda > 0, x \in G;$$

A function f is called homogeneous of degree α, if

$$f((D(\lambda)x)) = \lambda^\alpha f(x), \lambda > 0, x \in G.$$

Remark 2. Clearly, if Y is a differential operators of homogeneous of degree β and f is a function of homogeneous of degree α, then Yf is homogeneous of degree $\alpha - \beta$.

Lemma 1. *([2]) The operator L possesses a unique fundamental solution $\Gamma(\cdot)$, such that for every test function $u \in C_0^\infty(G)$ and every $x \in G$, it holds*

(1) $\Gamma(\cdot) \in C^\infty(G \backslash \{0\})$;

(2) $\Gamma(\cdot)$ *is homogeneous of degree* $2-Q$;
(3) $u(x) = (Lu * \Gamma)(x) = \int_{\mathbb{R}^N} \Gamma(y^{-1} \circ x) Lu(y) dy$;
(4) $X_i u(x) = \int_{R^N} X_i \Gamma(y^{-1} \circ x) Lu(y) dy$.

Remark 3. If we set $\Gamma_i = X_i \Gamma, i = 1, \cdots, q$, then it is obvious from Remark 2 that $\Gamma_i(\cdot)$ is homogeneous of degree $1-Q$.

Proposition 1. *([2]) Let* $f \in C^1(R^N \backslash 0)$ *is a homogeneous function of degree* $\lambda < 1$*. Then there exist two constants* $c = c(G, f) > 0$ *and* $M = M(G) > 1$*, such that for any* x, y *satisfying* $\|x\| \geq M \|y\|$,

$$|f(x \circ y) - f(x)| + |f(y \circ x) - f(x)| \leq c \|y\| \|x\|^{\lambda - 1},$$

where $c = c(G, f) \sup\limits_{z \in \Sigma_N} |\nabla f(z)|$, Σ_N *is the unit sphere of* $\mathbb{R}^N$.

From Proposition 1, it follows

Lemma 2. *If* $K \in C^1(G \backslash \{0\})$ *is a homogeneous function of degree* $\alpha < 1$ *with respect to the group* $(D(\lambda))_{\lambda > 0}$*, then there exist two constants* $c > 0$ *and* $M > 1$*, such that if* $\|x\| \geq M \left\|x^{-1} \circ z\right\|$*, then*

$$|K(z) - K(x)| \leq \frac{c \left\|x^{-1} \circ z\right\|}{\|x\|^{1-\alpha}}.$$

By Lemma 1 and Lemma 2, we have immediately

Lemma 3. *For every* $x, y, z \in G$*, it holds*

(1) *there exists a constant* $c > 0$*, such that*

$$\Gamma(y^{-1} \circ x) \leq \frac{c}{\left\|y^{-1} \circ x\right\|^{Q-2}};$$

$$\Gamma_i(y^{-1} \circ x) \leq \frac{c}{\left\|y^{-1} \circ x\right\|^{Q-1}}.$$

(2) *there exist two constants* $c > 0$ *and* $M > 1$*, such that if* $\left\|y^{-1} \circ x\right\| \geq M \left\|x^{-1} \circ z\right\|$*, then*

$$\left|\Gamma(y^{-1} \circ x) - \Gamma(y^{-1} \circ z)\right| \leq \frac{c \left\|x^{-1} \circ z\right\|}{\left\|y^{-1} \circ x\right\|^{Q-1}};$$

$$\left|\Gamma_i(y^{-1} \circ x) - \Gamma_i(y^{-1} \circ z)\right| \leq \frac{c \left\|x^{-1} \circ z\right\|}{\left\|y^{-1} \circ x\right\|^{Q}}.$$

Now let us introduce two integral operators. For $p \in (1, \infty)$ and $\lambda \in [0, Q)$, fixed $z \in G$ and $\sigma > 0$, we define for every $g \in L^{p,\lambda}(G)$ that

$$T_\alpha g(x) = \int_{\left\|y^{-1} \circ x\right\| \geq \sigma \left\|z^{-1} \circ x\right\|} \frac{g(y)}{\left\|y^{-1} \circ x\right\|^{Q-\alpha}} dy, \alpha \in [0, Q);$$

$$T^{\beta}g(x)=\int_{\|y^{-1}\circ x\|<\sigma\|z^{-1}\circ x\|}\frac{g(y)}{\|y^{-1}\circ x\|^{Q-\beta}}dy,\beta\in(0,Q).$$

Lemma 4. *If $\lambda+p\alpha<Q$, then there exists $c=c(p,\lambda,\alpha,\sigma)>0$, such that*

$$|T_{\alpha}g(x)|\le c\,\|g\|_{L^{p,\lambda}(G}\left\|z^{-1}\circ x\right\|^{\frac{p\alpha+\lambda-Q}{p}}; \tag{2.3}$$

if $\lambda+p\beta>Q$, then there exists $c=c(p,\lambda,\beta,\sigma)>0$, such that

$$\left|T^{\beta}g(x)\right|\le c\,\|g\|_{L^{p,\lambda}(G)}\left\|z^{-1}\circ x\right\|^{\frac{p\beta+\lambda-Q}{p}}. \tag{2.4}$$

Proof. We follow the idea of Polidoro and Ragusa in [14]. If $\lambda+p\alpha<Q$, then it obtains by decomposing the domain of integration and applying the Hölder inequality that

$$\begin{aligned}|T_{\alpha}g(x)|&\le\sum_{k=1}^{\infty}\int_{2^{k-1}\sigma\|z^{-1}\circ x\|\le\|y^{-1}\circ x\|<2^{k}\sigma\|z^{-1}\circ x\|}\frac{g(y)}{\|y^{-1}\circ x\|^{Q-\alpha}}dy\\
&\le\sum_{k=1}^{\infty}\left(\frac{1}{2^{k-1}\sigma\left\|z^{-1}\circ x\right\|}\right)^{Q-\alpha}\int_{B_{2^{k}\sigma\|z^{-1}\circ x\|}(x)}|g(y)|dy\\
&\le\sum_{k=1}^{\infty}\left(\frac{1}{2^{k-1}\sigma\left\|z^{-1}\circ x\right\|}\right)^{Q-\alpha}\left(\int_{B_{2^{k}\sigma\|z^{-1}\circ x\|}(x)}|g(y)|^{p}dy\right)^{\frac{1}{p}}\\
&\quad\left|B_{2^{k}\sigma\|z^{-1}\circ x\|}(x)\right|^{\frac{p-1}{p}}\\
&\le c\sum_{k=1}^{\infty}\left(\frac{1}{2^{k-1}\sigma\left\|z^{-1}\circ x\right\|}\right)^{Q-\alpha}\left(2^{k}\sigma\left\|z^{-1}\circ x\right\|\right)^{\frac{\lambda}{p}}\|g\|_{L^{p,\lambda}(G)}\\
&\quad\left(2^{k}\sigma\left\|z^{-1}\circ x\right\|\right)^{\frac{(p-1)Q}{p}}\\
&\le c\,\|g\|_{L^{p,\lambda}(G)}\left\|z^{-1}\circ x\right\|^{\frac{p\alpha+\lambda-Q}{p}}\sum_{k=1}^{\infty}\left(2^{\frac{p\alpha+\lambda-Q}{p}}\right)^{k}.\end{aligned}$$

So (2.3) is proved, since the above series is convergent.

Similarly, if $\lambda+p\beta>Q$, then

$$\begin{aligned}\left|T^{\beta}g(x)\right|&\le\sum_{k=1}^{\infty}\int_{2^{-k}\sigma\|z^{-1}\circ x\|\le\|y^{-1}\circ x\|<2^{1-k}\sigma\|z^{-1}\circ x\|}\frac{g(y)}{\|y^{-1}\circ x\|^{Q-\beta}}dy\\
&\le\sum_{k=1}^{\infty}\left(\frac{1}{2^{-k}\sigma\left\|z^{-1}\circ x\right\|}\right)^{Q-\beta}\int_{B_{2^{1-k}\sigma\|z^{-1}\circ x\|}(x)}|g(y)|dy\end{aligned}$$

$$\leq \sum_{k=1}^{\infty} \left(\frac{1}{2^{-k}\sigma \left\| z^{-1} \circ x \right\|} \right)^{Q-\beta} \left(\int_{B_{2^{1-k}\sigma \left\| z^{-1} \circ x \right\|}(x)} |g(y)|^p dy \right)^{\frac{1}{p}}$$
$$\left| B_{2^{1-k}\sigma \left\| z^{-1} \circ x \right\|}(x) \right|^{\frac{p-1}{p}}$$
$$\leq c \sum_{k=1}^{\infty} \left(\frac{1}{2^{-k}\sigma \left\| z^{-1} \circ x \right\|} \right)^{Q-\beta} \left(2^{1-k}\sigma \left\| z^{-1} \circ x \right\| \right)^{\frac{\lambda}{p}} \|g\|_{L^{p,\lambda}(G)}$$
$$\left(2^{1-k}\sigma \left\| z^{-1} \circ x \right\| \right)^{\frac{(p-1)Q}{p}}$$
$$\leq c \|g\|_{L^{p,\lambda}(G)} \left\| z^{-1} \circ x \right\|^{\frac{p\beta+\lambda-Q}{p}} \sum_{k=1}^{\infty} \left(2^{\frac{Q-p\beta-\lambda}{p}} \right)^k .$$

This proves (2.4). □

Remark 4. In particular, when $\lambda = 0$, we see that if $p\alpha < Q$, then there exists a constant $c = c(p,\alpha,\sigma) > 0$, such that

$$|T_\alpha g(x)| \leq c \|g\|_{L^p(G)} \left\| z^{-1} \circ x \right\|^{\frac{p\alpha-Q}{p}} ; \tag{2.5}$$

if $p\beta > Q$, then there exists a constant $c = c(p,\beta,\sigma) > 0$, such that

$$\left| T^\beta g(x) \right| \leq c \|g\|_{L^p(G)} \left\| z^{-1} \circ x \right\|^{\frac{p\beta-Q}{p}} . \tag{2.6}$$

3. Proofs of the main results

Proof of Theorem 1. (1) With the help of (3) in Lemma 1 and Lemma 3, we know that there exist constants $c > 0$ and $M > 1$ such that

$$\begin{aligned}
|u(x) - u(z)| &= \left| \int_{\mathbb{R}^N} \Gamma(y^{-1} \circ x) - \Gamma(y^{-1} \circ z) Lu(y) dy \right| \\
&\leq \int_{\mathbb{R}^N} \left| \Gamma(y^{-1} \circ x) - \Gamma(y^{-1} \circ z) \right| |Lu(y)| dy \\
&\leq \int_{\|y^{-1}\circ x\| \geq M \|x^{-1}\circ z\|} \left| \Gamma(y^{-1} \circ x) - \Gamma(y^{-1} \circ z) \right| |Lu(y)| dy \\
&\quad + \int_{\|y^{-1}\circ x\| < M \|x^{-1}\circ z\|} \left| \Gamma(y^{-1} \circ x) - \Gamma(y^{-1} \circ z) \right| |Lu(y)| dy \\
&\leq \int_{\|y^{-1}\circ x\| \geq M \|x^{-1}\circ z\|} \left| \Gamma(y^{-1} \circ x) - \Gamma(y^{-1} \circ z) \right| |Lu(y)| dy \\
&\quad + \int_{\|y^{-1}\circ x\| < M \|x^{-1}\circ z\|} \left| \Gamma(y^{-1} \circ x) \right| |Lu(y)| dy
\end{aligned}$$

$$
\begin{aligned}
&+\int_{\left\|y^{-1}\circ x\right\|<M\left\|x^{-1}\circ z\right\|}\left|\Gamma(y^{-1}\circ z)\right||Lu(y)|dy\\
\leq&\int_{\left\|y^{-1}\circ x\right\|\geq M\left\|x^{-1}\circ z\right\|}\frac{c\left\|x^{-1}\circ z\right\|}{\left\|y^{-1}\circ x\right\|^{Q-1}}|Lu(y)|dy\\
&+\int_{\left\|y^{-1}\circ x\right\|<M\left\|x^{-1}\circ z\right\|}\frac{c}{\left\|y^{-1}\circ x\right\|^{Q-2}}|Lu(y)|dy\\
&+\int_{\left\|y^{-1}\circ x\right\|<M\left\|x^{-1}\circ z\right\|}\frac{c}{\left\|y^{-1}\circ z\right\|^{Q-2}}|Lu(y)|dy.
\end{aligned}
$$

Noting that if $\left\|y^{-1}\circ x\right\|\geq M\left\|x^{-1}\circ z\right\|$, then

$$\left\|y^{-1}\circ x\right\|\geq M\left\|x^{-1}\circ z\right\|\geq\frac{M}{c}\left\|z^{-1}\circ x\right\|;$$

if $\left\|y^{-1}\circ x\right\|<M\left\|x^{-1}\circ z\right\|$, then

$$\left\|y^{-1}\circ x\right\|<Mc\left\|z^{-1}\circ x\right\|$$

and

$$
\begin{aligned}
\left\|y^{-1}\circ z\right\|&\leq c\left(\left\|y^{-1}\circ x\right\|+\left\|x^{-1}\circ z\right\|\right)<c\left(M\left\|x^{-1}\circ z\right\|+\left\|x^{-1}\circ z\right\|\right)\\
&=c\,(1+M)\left\|x^{-1}\circ z\right\|,
\end{aligned}
$$

it follows

$$
\begin{aligned}
|u(x)-u(z)|\leq&\int_{\left\|y^{-1}\circ x\right\|\geq\frac{M}{c}\left\|z^{-1}\circ x\right\|}\frac{c\left\|x^{-1}\circ z\right\|}{\left\|y^{-1}\circ x\right\|^{Q-1}}|Lu(y)|dy\\
&+\int_{\left\|y^{-1}\circ x\right\|<Mc\left\|z^{-1}\circ x\right\|}\frac{c}{\left\|y^{-1}\circ x\right\|^{Q-2}}|Lu(y)|dy\\
&+\int_{\left\|y^{-1}\circ z\right\|<c(1+M)\left\|x^{-1}\circ z\right\|}\frac{c}{\left\|y^{-1}\circ z\right\|^{Q-2}}|Lu(y)|dy\\
\doteq&\ I_1+I_2+I_3.
\end{aligned}
$$

Applying Lemma 4 ($\alpha=1$ and $\sigma=\frac{M}{c}$) and noting $\lambda+p<Q$, there exists a constant $c=c(p,\lambda,\sigma)>0$ such that

$$I_1\leq c\left\|Lu\right\|_{L^{p,\lambda}(G)}\left\|z^{-1}\circ x\right\|\left\|z^{-1}\circ x\right\|^{\frac{p+\lambda-Q}{p}}=c\left\|Lu\right\|_{L^{p,\lambda}(G)}\left\|z^{-1}\circ x\right\|^{\frac{2p+\lambda-Q}{p}};$$

from Lemma 4 ($\beta=2$ and $\sigma=Mc$; $\beta=2$ and $\sigma=c(1+M)$, respectively) and $\lambda+2p>Q$, it follows

$$I_2\leq c\left\|Lu\right\|_{L^{p,\lambda}(G)}\left\|z^{-1}\circ x\right\|^{\frac{2p+\lambda-Q}{p}}$$

and

$$I_3 \leq c\,\|Lu\|_{L^{p,\lambda}(G)} \left\|z^{-1}\circ x\right\|^{\frac{2p+\lambda-Q}{p}}.$$

In conclusion, we deduce (1.3).

(2) We know from (4) in Lemma 1 and Lemma 3 that there exist two constants $c>0$ and $M>1$ such that

$$\begin{aligned}
|X_i u(x)-X_i u(z)| &= \left|\int_{\mathbb{R}^N} \Gamma_i(y^{-1}\circ x)-\Gamma_i(y^{-1}\circ z)Lu(y)dy\right| \\
&\leq \int_{\mathbb{R}^N} \left|\Gamma_i(y^{-1}\circ x)-\Gamma_i(y^{-1}\circ z)\right||Lu(y)|dy \\
&\leq \int_{\|y^{-1}\circ x\|\geq M\|x^{-1}\circ z\|} \left|\Gamma_i(y^{-1}\circ x)-\Gamma_i(y^{-1}\circ z)\right||Lu(y)|dy \\
&\quad+\int_{\|y^{-1}\circ x\|< M\|x^{-1}\circ z\|} \left|\Gamma_i(y^{-1}\circ x)-\Gamma_i(y^{-1}\circ z)\right||Lu(y)|dy \\
&\leq \int_{\|y^{-1}\circ x\|\geq M\|x^{-1}\circ z\|} \left|\Gamma_i(y^{-1}\circ x)-\Gamma_i(y^{-1}\circ z)\right||Lu(y)|dy \\
&\quad+\int_{\|y^{-1}\circ x\|< M\|x^{-1}\circ z\|} \left|\Gamma_i(y^{-1}\circ x)\right||Lu(y)|dy \\
&\quad+\int_{\|y^{-1}\circ x\|< M\|x^{-1}\circ z\|} \left|\Gamma_i(y^{-1}\circ z)\right||Lu(y)|dy \\
&\leq \int_{\|y^{-1}\circ x\|\geq M\|x^{-1}\circ z\|} \frac{c\left\|x^{-1}\circ z\right\|}{\left\|y^{-1}\circ x\right\|^{Q}}|Lu(y)|dy \\
&\quad+\int_{\|y^{-1}\circ x\|< M\|x^{-1}\circ z\|} \frac{c}{\left\|y^{-1}\circ x\right\|^{Q-1}}|Lu(y)|dy \\
&\quad+\int_{\|y^{-1}\circ x\|< M\|x^{-1}\circ z\|} \frac{c}{\left\|y^{-1}\circ z\right\|^{Q-1}}|Lu(y)|dy.
\end{aligned}$$

Let us remark that if $\left\|y^{-1}\circ x\right\| \geq M\left\|x^{-1}\circ z\right\|$, then

$$\left\|y^{-1}\circ x\right\| \geq \frac{M}{c}\left\|z^{-1}\circ x\right\|;$$

if $\left\|y^{-1}\circ x\right\| < M\left\|x^{-1}\circ z\right\|$, then

$$\left\|y^{-1}\circ x\right\| < Mc\left\|z^{-1}\circ x\right\|$$

and

$$\begin{aligned}
\left\|y^{-1}\circ z\right\| &\leq c\left(\left\|y^{-1}\circ x\right\|+\left\|x^{-1}\circ z\right\|\right) < c\left(M\left\|x^{-1}\circ z\right\|+\left\|x^{-1}\circ z\right\|\right) \\
&= c\,(1+M)\left\|x^{-1}\circ z\right\|.
\end{aligned}$$

It implies

$$
\begin{aligned}
|X_i u(x) - X_i u(z)| \le & \int_{\|y^{-1}\circ x\| \ge \frac{M}{c}\|z^{-1}\circ x\|} \frac{c\left\|x^{-1}\circ z\right\|}{\left\|y^{-1}\circ x\right\|^{Q}} |Lu(y)|dy \\
& + \int_{\|y^{-1}\circ x\| < Mc\|z^{-1}\circ x\|} \frac{c}{\left\|y^{-1}\circ x\right\|^{Q-1}} |Lu(y)|dy \\
& + \int_{\|y^{-1}\circ z\| < c(1+M)\|x^{-1}\circ z\|} \frac{c}{\left\|y^{-1}\circ z\right\|^{Q-1}} |Lu(y)|dy \\
\doteq & \, I_4 + I_5 + I_6.
\end{aligned}
$$

Applying Lemma 4 ($\alpha = 0$ and $\sigma = \frac{M}{c}$) and $\lambda < Q$, there exists a constant $c = c(p, \lambda, \sigma) > 0$ such that

$$
I_4 \le c\,\|Lu\|_{L^{p,\lambda}(G)} \left\|z^{-1}\circ x\right\| \left\|z^{-1}\circ x\right\|^{\frac{\lambda-Q}{p}} = c\,\|Lu\|_{L^{p,\lambda}(G)} \left\|z^{-1}\circ x\right\|^{\frac{p+\lambda-Q}{p}};
$$

from Lemma 4 ($\beta = 1$ and $\sigma = Mc$; $\beta = 1$ and $\sigma = c(1+M)$, respectively) and $\lambda + p > Q$, it gets

$$
I_5 \le c\,\|Lu\|_{L^{p,\lambda}(G)} \left\|z^{-1}\circ x\right\|^{\frac{p+\lambda-Q}{p}}
$$

and

$$
I_6 \le c\,\|Lu\|_{L^{p,\lambda}(G)} \left\|z^{-1}\circ x\right\|^{\frac{p+\lambda-Q}{p}}.
$$

In conclusion we reach to (1.4).

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