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A NEW APPLICATION OF ALMOST INCREASING SEQUENCES

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Abstract. In the present paper we have proved a more general theorem dealing with $|A, p_n|_k$ summability by using almost increasing sequence. This theorem also includes several known results.

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1. INTRODUCTION

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Let $\sum a_n$ be a given infinite series with partial sums (s_n), and let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{1.1}$$

The series $\sum a_n$ is said to be summable $|A|_k, k \geq 1$, if (see [9])

$$\sum_{n=1}^{\infty} n^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{1.2}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1.3}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{1.4}$$

defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [6]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid \Delta\sigma_{n-1} \mid^k < \infty, \tag{1.5}$$

and it is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [8])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{1.6}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

In the special case when $p_n = 1$ for all n, $|A, p_n|_k$ summability is the same as $|A|_k$ summability. Also if we take $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n|_k$ summability is the same as $\left|\bar{N}, p_n\right|_k$ summability.

In [5], Bor has proved the following theorem for $\left|\bar{N}, p_n\right|_k$ summability factors of infinite series.

Theorem 1. *Let (X_n) be an almost increasing sequence and let there be sequences (β_n) and (λ_n) such that*

$$\mid \Delta\lambda_n \mid \leq \beta_n, \tag{1.7}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{1.8}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{1.9}$$

$$\mid \lambda_n \mid X_n = O(1). \tag{1.10}$$

and

$$\sum_{v=1}^{n} \frac{\mid t_v \mid^k}{v} = O(X_n) \quad as \quad n \to \infty, \tag{1.11}$$

where (t_n) is the n-th $(C,1)$ mean of the sequence (na_n). Suppose further, the sequence (p_n) is such that

$$P_n = O(np_n), \tag{1.12}$$

$$P_n \Delta p_n = O(p_n p_{n+1}). \tag{1.13}$$

Then the series $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$ is summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

Remark 1. It should be noted that, from the hypotheses of Theorem 1, (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ (see [3]).

2. The main result

The aim of this paper is to generalize Theorem 1 for absolute matrix summability. Before stating the main theorem we must first introduce some further notations. Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n,v = 0,1,... \tag{2.1}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1,2,... \tag{2.2}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v = \sum_{v=0}^{n} \bar{a}_{nv}a_v \tag{2.3}$$

and

$$\bar{\Delta}A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v. \tag{2.4}$$

Now, we shall prove the following theorem.

Theorem 2. *Let (X_n) be an almost increasing sequence. The conditions (1.7)–(1.13) of Theorem 1 and*

$$\sum_{v=1}^{n} \frac{p_v}{P_v} = O(X_n) \quad as \quad n \to \infty, \tag{2.5}$$

are satisfied. If $A = (a_{nv})$ is a positive normal matrix such that

$$\bar{a}_{no} = 1,\ n = 0,1,..., \tag{2.6}$$

$$a_{n-1,v} \geq a_{nv},\ for\ n \geq v+1, \tag{2.7}$$

$$a_{nn} = O(\frac{p_n}{P_n}), \tag{2.8}$$

$$|\ \widehat{a}_{n,v+1}\ | = O(v\ |\ \Delta_v(\widehat{a}_{nv})\ |), \tag{2.9}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$ is summable $|\ A,p_n\ |_k, k \geq 1$.

It should be noted that if we take $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 1.

We need the following lemmas for the proof of our theorem.

Lemma 1. *([7]) If (X_n) an almost increasing sequence, then under the conditions (1.8)–(1.9) we have that*

$$nX_n\beta_n = O(1), \tag{2.10}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{2.11}$$

Lemma 2 ([4])**.** *If the conditions (1.12) and (1.13) are satisfied, then $\Delta(P_n/p_n n^2) = O(1/n^2)$.*

Proof of Theorem 2. Let (T_n) denotes A-transform of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{np_n}$. Then we have by (2.3) and (2.4)

$$\bar{\Delta} T_n = \sum_{v=1}^{n} \hat{a}_{nv} \frac{a_v P_v \lambda_v}{v p_v}.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}
\bar{\Delta} T_n &= \sum_{v=1}^{n} \hat{a}_{nv} \frac{v a_v P_v \lambda_v}{v^2 p_v} \\
&= \sum_{v=1}^{n-1} \Delta_v \Big(\frac{\hat{a}_{nv} P_v \lambda_v}{v^2 p_v}\Big) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n^2 p_n} \sum_{r=1}^{n} r a_r \\
&= \sum_{v=1}^{n-1} \Delta_v \Big(\frac{\hat{a}_{nv} P_v \lambda_v}{v^2 p_v}\Big)(v+1) t_v + \frac{a_{nn} P_n \lambda_n}{n^2 p_n}(n+1) t_n \\
&= \frac{a_{nn} P_n \lambda_n}{n^2 p_n}(n+1) t_n + \sum_{v=1}^{n-1} \Delta_v(\hat{a}_{nv}) \frac{(v+1)}{v^2} \frac{P_v \lambda_v}{p_v} t_v \\
&\quad + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} P_v}{p_v} \Delta \lambda_v t_v \frac{(v+1)}{v^2} + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta\Big(\frac{P_v}{v^2 p_v}\Big) t_v (v+1) \\
&= T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4}.
\end{aligned}$$

Since

$$|T_n(1) + T_n(2) + T_n(3) + T_n(4)|^k \le 4^k (|T_n(1)|^k + |T_n(2)|^k + |T_n(3)| + |T_n(4)|^k)$$

to complete the proof of the theorem it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,r} \mid^k < \infty, \quad \text{for} \quad r = 1,2,3,4. \tag{2.12}$$

Firstly, by using Abel's transformation, we have that

$$\begin{aligned}\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} | T_{n,1} |^k &= O(1)\sum_{n=1}^{m}(\frac{P_n}{p_n})^{k-1}a_{nn}^k(\frac{P_n}{p_n})^k|\lambda_n|^k\frac{|t_n|^k}{n^k}\\ &= O(1)\sum_{n=1}^{m}|\lambda_n|^{k-1}|\lambda_n|\frac{|t_n|^k}{n}\\ &= O(1)\sum_{n=1}^{m}|\lambda_n|\frac{|t_n|^k}{n}\\ &= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{v=1}^{n}\frac{|t_v|^k}{v}+O(1)|\lambda_m|\sum_{n=1}^{m}\frac{|t_n|^k}{n}\\ &= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n+O(1)|\lambda_m|X_m\\ &= O(1)\sum_{n=1}^{m-1}\beta_n X_n+O(1)|\lambda_m|X_m=O(1) \quad \text{as} \quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of the theorem and Lemma 1.

Now, using the fact that $P_v = O(vp_v)$ by (1.12), we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} | T_{n,2} |^k= O(1)\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v||t_v|\right)^k$$

Now, applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, we have that

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} | T_{n,2} |^k &= O(1)\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right)\\ &\quad\times(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|)^{k-1}\\ &= O(1)\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{k-1}a_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right)\\ &= O(1)\sum_{v=1}^{m}|\lambda_v|^k|t_v|^k\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|\\ &= O(1)\sum_{v=1}^{m}|\lambda_v||t_v|^k a_{vv}=O(1)\sum_{v=1}^{m}|\lambda_v||t_v|^k\frac{p_v}{P_v}\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m-1} \Delta\,|\lambda_v| \sum_{r=1}^{v} \frac{p_r}{P_r}\,|t_r|^k + 0(1)\,|\lambda_m| \sum_{v=1}^{m} \frac{p_v}{P_v}\,|t_v|^k \\
&= O(1)\sum_{v=1}^{m-1} |\Delta\lambda_v|\,X_v + O(1)\,|\lambda_m|\,X_m \\
&= O(1)\sum_{v=1}^{m-1} \beta_v X_v + O(1)\,|\lambda_m|\,X_m \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of the theorem and Lemma 1.

Now, using Hölder's inequality we have that

$$
\begin{aligned}
&\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,3} \mid^k \\
&= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}|\beta_v|t_v|^k\right) \times \left(\sum_{v=1}^{n-1} |\hat{a}_{v,v+1}|\beta_v\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}|\beta_v|t_v|^k\right) \\
&= O(1)\sum_{v=1}^{m} \beta_v|t_v|^k \sum_{n=v+1}^{m+1} |\hat{a}_{n,v+1}| = O(1)\sum_{v=1}^{m} v\beta_v \frac{1}{v}|t_v|^k \\
&= O(1)\sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} \frac{1}{r}|t_r|^k + O(1)m\beta_m \sum_{v=1}^{m} \frac{1}{v}|t_v|^k \\
&= O(1)\sum_{v=1}^{m-1} |\Delta(v\beta_v)|X_v + O(1)m\beta_m X_m \\
&= O(1)\sum_{v=1}^{m-1} v|\Delta\beta_v|X_v + O(1)\sum_{v=1}^{m-1} \beta_{v+1}X_{v+1} + O(1)m\beta_m X_m \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of the theorem and Lemma 1.

Finally, since $\Delta(\frac{P_v}{v^2 p_v}) = O(\frac{1}{v^2})$, as in $T_{n,1}$ we have that

$$\sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{k-1} |T_n(4)|^k$$

$$= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}| \frac{|t_v|}{v} \right)^k$$

$$= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}|^k \frac{|t_v|^k}{v} \right) \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}| \frac{1}{v} \right)^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{k-1} \left(\sum_{v=1}^{n-1} |\hat{a}_{n,v+1}||\lambda_{v+1}|^k \frac{|t_v|^k}{v} \right) \left(\sum_{v=1}^{n-1} |\Delta_v \hat{a}_{nv}| \right)^{k-1}$$

$$= O(1) \sum_{n=2}^{m+1} (\frac{P_n}{p_n})^{k-1} a_{nn}^{k-1} \left(\sum_{v=1}^{n-1} |\Delta_v \hat{a}_{nv}||\lambda_{v+1}|^k \frac{|t_v|^k}{v} \right)$$

$$= O(1) \sum_{v=1}^{m} \frac{|\lambda_{v+1}|}{v} |t_v|^k \sum_{n=v+1}^{m+1} |\hat{a}_{n,v+1}|$$

$$= O(1) \sum_{v=1}^{m} \frac{|\lambda_{v+1}|}{v} |t_v|^k = O(1), \quad \text{as} \quad m \to \infty$$

Therefore we get

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n} \right)^{k-1} | T_{n,r} |^k = O(1) \quad as \quad m \to \infty, \quad \text{for} \quad r = 1, 2, 3, 4.$$

This completes the proof of Theorem 2. □

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