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POSITIVE SOLUTIONS OF BVPS FOR INFINITE DIFFERENCE EQUATIONS WITH ONE-DIMENSIONAL p-LAPLACIAN

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Abstract. Sufficient conditions guaranteeing the existence of three positive solutions of the multi-point boundary value problem for the infinite difference equation

$$\begin{cases} \Delta[p(n)\phi(\Delta x(n))] + f(n,x(n),\Delta x(n)) = 0, \;\; n \in N_0, \\ x(0) - \sum_{n=1}^{\infty} \alpha_n x(n) = 0, \\ \lim_{n\to+\infty} \frac{x(n)}{1+\sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} - \sum_{n=1}^{\infty} \beta_n x(n) = 0, \end{cases}$$

are established using a fixed point theorem. It is the purpose of this paper to show that this approach of obtaining positive solutions of BVPs by using multi-fixed-point theorems can be extended to infinite difference equations containing the nonlinear operator $\Delta[p\phi(\Delta x)]$. The possible solutions of this BVP are not concave if $p(n) \not\equiv constant$.

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Keywords: one-dimension p-Laplacian infinite difference equation, multi-point boundary value problem, positive solution, fixed point theorem

1. INTRODUCTION

In recent years, there have been many papers interested in proving the existence of positive solutions of the boundary value problems (BVPs for short) of the finite difference equations since these BVPs have extensive applications, see the papers [1, 5–9, 11–15, 18, 19], [10] and the references therein.

Recently, the authors [2–4, 16, 17] studied the existence of solutions of the boundary value problems for infinite difference equations. In [17], the existence of multiple positive solutions of boundary value problems for the second-order discrete equations

$$\begin{cases} \Delta^2 x(n-1) - p\Delta x(n-1) - qx(n-1) + f(n,x(n)) = 0, \;\; n \in N, \\ \alpha x(0) - \beta\Delta x(0) = 0, \\ \lim_{n\to+\infty} x(n) = 0, \end{cases} \tag{1.1}$$

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was investigated by using the cone compression and expansion and fixed point theorems in Fréchet spaces, where $\alpha > 0, \beta > 0,\ p > 0, q > 0$ and f is a continuous function.

In paper [2], the existence of solutions of a class of the infinite time scale boundary value problems was considered. It is easy to see that the results of [2] can be applied to the following BVP for the infinite difference equation

$$\begin{cases} \Delta^2 x(n) + f(n, x(n)) = 0,\ n \in N_0, \\ x(0) = 0, \\ x(n) \text{ is bounded.} \end{cases}$$

The methods are based upon a growth argument and the upper and lower solutions methods.

In [12] the existence of at least three positive solutions for the following BVP of the finite difference equations

$$\begin{cases} \Delta[\phi(\Delta x(n))] + f(n, x(n), \Delta x(n)) = 0,\ n \in [0, N], \\ x(0) - \sum_{i=1}^{m} \alpha_i x(n_i) = 0, \\ x(N+2) - \sum_{i=1}^{m} \beta_i x(n_i) = 0 \end{cases} \tag{1.2}$$

was proved under some assumptions.

Motivated by above mentioned two papers, the purpose of this paper is to investigate the multi-point boundary value problem of the second order infinite p-Laplacian difference equation

$$\begin{cases} \Delta[p(n)\phi(\Delta x(n))] + f(n, x(n), \Delta x(n)) = 0,\ n \in N_0, \\ x(0) - \sum_{n=1}^{\infty} \alpha_n x(n) = 0, \\ \lim_{n \to +\infty} \frac{x(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} - \sum_{n=1}^{\infty} \beta_n x(n) = 0, \end{cases} \tag{1.3}$$

where N_0 denotes the set of all nonnegative integers, $p(n) > 0, \alpha_n \geq 0, \beta_n \geq 0$ for all $n \in N_0$ and satisfy

$$\sum_{n=1}^{\infty} \alpha_n < 1,$$

$$\sum_{n \in N} \alpha_n \sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))} < \infty$$

and

$$\frac{\sum_{n=1}^{\infty} \beta_n}{1 - \sum_{n=1}^{\infty} \alpha_n} \sum_{n=1}^{\infty} \alpha_n \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{1}{p(j)}\right) + \sum_{n=1}^{\infty} \beta_n \sum_{j=0}^{n-1} \phi^{-1}\left(\frac{1}{p(j)}\right) < 1,$$

and

$$\sum_{s=0}^{\infty} \frac{1}{\phi^{-1}(p(s))} = \infty, \tag{1.4}$$

f is a Caratheodory function, i.e., f satisfies that for each $r > 0$ there exist a real number sequence $\{\phi_r(n)\}$ with $\sum_{n=0}^{\infty}\phi_r(n) < \infty$ such that

$$f\left(n, \left(1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}\right)x, \frac{1}{\phi^{-1}(p(n))}y\right) \le \phi_r(n),\ \ n \in N_0,\ |x| \le r,\ |y| \le r,$$

ϕ is defined by $\phi(x) = |x|^{p-2}x$ with $p > 1$, its inverse function is denoted by $\phi^{-1}(x) = |x|^{q-2}x$ with $1/p + 1/q = 1$, $\Delta x(n) = x(n+1) - x(n)$. We establish sufficient conditions for the existence of at least three positive solutions of BVP(3).

It is easy to see that the positive solutions of BVP(1) are bounded, the positive solutions of BVP(3) may be unbounded since 1.4 holds. The results in this article generalize the theorems in [12] to the infinite case and the fixed point theorem used is different from those used in [3, 12, 17]. The most interesting part in this article is the construction of the nonlinear operator and the cone, this method is not found in known papers.

The remainder of this paper is organized as follows: in Section 2, we first give some lemmas, then the main result (Theorem 1) and its proof are presented. An example is given in Section 3 to illustrate the main result.

2. Main Results

In this section, we first present some background definitions in Banach spaces and state three important fixed point theorems and lemmas. Then the main result is given and proved.

Definition 1. Let X be a real Banach space. The nonempty convex closed subset P of X is called a cone in X if $ax \in P$ for all $x \in P$ and $a \ge 0$, $x \in X$ and $-x \in X$ imply $x = 0$.

Definition 2. A map $\psi : P \to [0, +\infty)$ is a nonnegative continuous concave or convex functional map provided ψ is nonnegative, continuous and satisfies $\psi(tx + (1-t)y) \ge t\psi(x) + (1-t)\psi(y)$, or $\psi(tx + (1-t)y) \le t\psi(x) + (1-t)\psi(y)$, for all $x, y \in P$ and $t \in [0, 1]$.

Definition 3. An operator $T; X \to X$ is completely continuous if it is continuous and maps bounded sets into pre-compact sets.

Definition 4. Let $a, b, c, d, h > 0$ be positive constants, α, ψ be two nonnegative continuous concave functionals on the cone P, γ, β, θ be three nonnegative continuous convex functionals on the cone P. Define the convex sets as follows:

$$P_c = \{x \in P : ||x|| < c\},$$
$$P(\gamma, \alpha; a, c) = \{x \in P : \alpha(x) \ge a,\ \gamma(x) \le c\},$$
$$P(\gamma, \theta, \alpha; a, b, c) = \{x \in P : \alpha(x) \ge a,\ \theta(x) \le b,\ \gamma(x) \le c\},$$
$$Q(\gamma, \beta; , d, c) = \{x \in P : \beta(x) \le d,\ \gamma(x) \le c\},$$

$$Q(\gamma,\beta,\psi;h,d,c)=\{x\in P:\psi(x)\geq h,\ \beta(x)\leq d,\ \gamma(x)\leq c\}.$$

Lemma 1 ([7]). *Let X be a real Banach space, P be a cone in X, α,ψ be two nonnegative continuous concave functionals on the cone P, γ,β,θ be three nonnegative continuous convex functionals on the cone P. There exist constant $M>0$ such that*

$$\alpha(x)\leq\beta(x),\ ||x||\leq M\gamma(x)\ \textit{for all}\ x\in P.$$

Furthermore, Suppose that $h,d,a,b,c>0$ are constants with $d<a$. Let $T:\overline{P_c}\to\overline{P_c}$ be a completely continuous operator. If

(C1) $\{y\in P(\gamma,\theta,\alpha;a,b,c)|\alpha(x)>a\}\neq\emptyset$ *and*

$$\alpha(Tx)>a\ \textit{for every}\ x\in P(\gamma,\theta,\alpha;a,b,c);$$

(C2) $\{y\in Q(\gamma,\theta,\psi;h,d,c)|\beta(x)<d\}\neq\emptyset$ *and*

$$\beta(Tx)<d\ \textit{for every}\ x\in Q(\gamma,\theta,\psi;h,d,c);$$

(C3) $\alpha(Ty)>a$ *for* $y\in P(\gamma,\alpha;a,c)$ *with* $\theta(Ty)>b$;

(C4) $\beta(Tx)<d$ *for each* $x\in Q(\gamma,\beta;,d,c)$ *with* $\psi(Tx)<h$,

then T has at least three fixed points y_1, y_2 and y_3 such that

$$\beta(y_1)<d,\ \alpha(y_2)>a,\ \beta(y_3)>d,\ \alpha(y_3)<a.$$

Choose

$$X=\left\{\{x(n)\}:\begin{array}{c}x(n)\in R,n\in N_0\\ \text{there exist the limits}\\ \lim_{n\to\infty}\frac{x(n)}{\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}+1},\\ \lim_{n\to\infty}\phi^{-1}(p(n))\Delta x(n)\end{array}\right\}.$$

Define the norm

$$||x||=\max\left\{\sup_{n\in N_0}\frac{|x(n)|}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}},\ \sup_{n\in N_0}\phi^{-1}(p(n))|\Delta x(n)|\right\}.$$

It is easy to see that X is a real Banach space.

Denote $[a,b]=\{a,a+1,\cdots,b\}$ for $a\leq b$ and $[a,b]=\emptyset$ for $a>b$ with $a,b\in N_0$, N denotes the set of all positive integeres. Let $k_1,k_2\in N$ with $k_1<k_2$. Let

$$\mu=\frac{1}{\max\{1,\ \phi^{-1}(p(k_1-1))\}\left(1+\sum_{n=0}^{k_2-1}\frac{1}{\phi^{-1}(p(n))}\right)}. \tag{2.1}$$

Choose

$$P=\left\{x\in X:\begin{array}{l}x(n)\geq 0\text{ for all }n\in N_0,\\ \min_{n\in[k_1,k_2]}\frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}\geq\mu\sup_{n\in N_0}\frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}.\end{array}\right\},$$

It is easy to see that P is a nontrivial cone in X.

Let $h(n) \not\equiv 0 (n \in N_0)$ be a nonnegative sequence with $\sum_{n \in N_0} h(n)$ converging, consider the following BVP

$$\begin{cases} \Delta[p(n)\phi(\Delta y(n))] + h(n) = 0, \ n \in N_0, \\ y(0) - \sum_{n=1}^{\infty} \alpha_n y(n) = 0, \\ \lim_{n \to \infty} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} - \sum_{n=1}^{\infty} \beta_n y(n) = 0, \end{cases} \tag{2.2}$$

Set

$$\delta = \frac{\phi\left(\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right) + \sum_{n=1}^{\infty}\beta_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)\right)}{1-\phi\left(\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right) + \sum_{n=1}^{\infty}\beta_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)\right)}. \tag{2.3}$$

Lemma 2. *If y is a solution of BVP(6), then $y(n) \geq 0$ and $\Delta y(n) \geq 0$ for all $n \in N_0$ and there exists an unique number $A_h \in \left[0, \delta \sum_{s=0}^{\infty} h(s)\right]$ such that*

$$y(n) = \frac{1}{1 - \sum_{n=1}^{\infty}\alpha_n}$$
$$\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A_h + \sum_{s=j}^{\infty} h(s)}{p(j)}\right) + \sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A_h + \sum_{s=j}^{\infty} h(s)}{p(j)}\right). \tag{2.4}$$

Proof. Since $\Delta[p(n)\phi(\Delta y(n))] = -h(n) \leq 0$ for all $n \in N_0$ and $\sum_{n \in N_0} h(n)$ converges, we get that $p(n)\phi(\Delta y(n))$ is decreasing and there exists the limit $\lim_{n\to\infty} p(n)\phi(\Delta y(n))$. Then $\phi^{-1}(p(n))\Delta y(n)$ is decreasing and there exists the limit $\lim_{n\to\infty} \phi^{-1}(p(n))\Delta y(n)$.

Since there exists the limit $\lim_{n\to\infty} \phi^{-1}(p(n))\Delta y(n) = c$, we can prove

$$\lim_{n \to \infty} \frac{y(n)}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} = c.$$

In fact, if $c = 0$, then for any $\epsilon > 0$ there exists $H > 0$ such that

$$\phi^{-1}(p(n))|\Delta y(n)| < \frac{\epsilon}{2}, n \geq H.$$

It follows that

$$|y(n)| \leq |y(H)| + \sum_{s=H}^{n-1} |\Delta y(s)| \leq |y(H)| + \frac{\epsilon}{2} \sum_{s=H}^{n-1} \frac{1}{\phi^{-1}(p(s))}, n \geq H.$$

Then

$$\frac{|y(n)|}{1 + \sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}}$$

$$\leq \frac{|y(H)|}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} + \frac{1}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}\frac{\epsilon}{2}\sum_{s=H}^{n-1}\frac{1}{\phi^{-1}(p(s))}$$
$$< \frac{|y(H)|}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} + \frac{\epsilon}{2}, \quad n \geq H.$$

Since $\sum_{s=0}^{\infty}\frac{1}{\phi^{-1}(p(s))} = \infty$, we can choose $H' > H$ large enough so that

$$\frac{|y(n)|}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \leq \frac{|y(H)|}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} + \frac{\epsilon}{2} < \epsilon, n \geq H',$$

which implies that

$$\lim_{n\to\infty}\frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} = 0.$$

If $c \neq 0$, then $\lim_{t\to\infty}\left(\phi^{-1}(p(n))\Delta y(n) - c\right) = 0$. It follows that

$$\lim_{t\to\infty}\phi^{-1}(p(n))\Delta\left[\ y(n) - c\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}\right] = 0.$$

Then we get similarly that

$$\lim_{t\to\infty}\frac{y(n) - c\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} = 0.$$

Together with $\sum_{s=0}^{\infty}\frac{1}{\phi^{-1}(p(s))} = \infty$, it follows that

$$\lim_{n\to\infty}\frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} = c.$$

It follows from 2.2 that there exist $A, B \in R$ such that

$$p(n)\phi(\Delta y(n)) = A + \sum_{s=n}^{\infty}h(s), \tag{2.5}$$

and

$$y(n) = B + \sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\left(A+\sum_{s=n}^{\infty}h(s)\right)\right), n \in N_0.$$

By the boundary conditions in 2.2, we get

$$B = B\sum_{n=1}^{\infty}\alpha_n + \sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\left(A+\sum_{s=j}^{\infty}h(s)\right)\right).$$

It follows that

$$B = \frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\left(A+\sum_{s=j}^{\infty}h(s)\right)\right).$$

So

$$\begin{aligned} y(n) = {} & \frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\left(A+\sum_{s=j}^{\infty}h(s)\right)\right) \\ & +\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\left(A+\sum_{s=j}^{\infty}h(s)\right)\right). \end{aligned}$$

From 2.5, one sees that $\lim_{n\to\infty} p(n)\phi(\Delta y(n)) = A$.
Then $\lim_{n\to\infty}\phi^{-1}(p(n))\Delta y(n) = \phi^{-1}(A)$. So

$$\lim_{n\to\infty}\frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} = \phi^{-1}(A).$$

Boundary conditions in 2.2 imply that

$$\begin{aligned} \phi^{-1}(A) = {} & \frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A+\sum_{s=j}^{\infty}h(s)}{p(j)}\right) \\ & +\sum_{n=1}^{\infty}\beta_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A+\sum_{s=j}^{\infty}h(s)}{p(j)}\right). \end{aligned} \tag{2.6}$$

Set

$$\begin{aligned} G(x) = 1 & - \frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\left(1+\frac{\sum_{s=j}^{\infty}h(s)}{x}\right)\right) \\ & -\sum_{n=1}^{\infty}\beta_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\left(1+\frac{\sum_{s=j}^{\infty}h(s)}{x}\right)\right). \end{aligned}$$

It is easy to see that $G(x)$ is continuous on $(-\infty,0)$ and $(0,\infty)$, respectively, $G(x)$ is increasing on $(-\infty,0)$ and $(0,\infty)$, respectively.

Since

$$\begin{aligned} & \lim_{x\to-\infty} G(x) \\ & = 1-\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)-\sum_{n=1}^{\infty}\beta_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right) > 0, \end{aligned}$$

and

$$\lim_{x\to 0^-} G(x) = +\infty,$$

we get $G(x)$ has no zero point on $(-\infty,0)$. On the other hand, we have

$$\begin{aligned}
&G\left(\delta\sum_{s=0}^{\infty}h(s)\right)\\
&= 1-\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\left(1+\frac{\sum_{s=j}^{\infty}h(s)}{\delta\sum_{s=0}^{\infty}h(s)}\right)\right)\\
&\quad -\sum_{n=1}^{\infty}\beta_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\left(1+\frac{\sum_{s=j}^{\infty}h(s)}{\delta\sum_{s=0}^{\infty}h(s)}\right)\right)\\
&\geq 1-\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\left(1+\frac{1}{\delta}\right)\right)\\
&\quad -\sum_{n=1}^{\infty}\beta_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\left(1+\frac{1}{\delta}\right)\right)\\
&= 1-\phi^{-1}\left(1+\frac{1}{\delta}\right)\left(\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)\right.\\
&\quad \left.+\sum_{n=1}^{\infty}\beta_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)\right)\\
&= 0,
\end{aligned}$$

and

$$\lim_{x\to 0^+} G(x) = -\infty,$$

Hence we find that there exists an unique $\xi\in\left(0,\delta\sum_{s=0}^{\infty}h(s)\right]$ such that $G(\xi)=0$. It follows that there exists unique $A_h\in\left[0,\delta\sum_{s=0}^{\infty}h(s)\right]$ satisfying 2. Hence 2.4 holds for $A_h\in\left[0,\delta\sum_{s=0}^{\infty}h(s)\right]$. It is easy to see from 2.4 that $y(n)\geq 0$ and $\Delta y(n)\geq 0$ for all $n\in N_0$. The proof is complete. □

Lemma 3. *Choose integers $k_1,k_2\in N_0$ with $1<k_1+1<k_2$. Let μ be defined by 2.1. Suppose y is a solution of BVP(6). Then*

$$\min_{n\in[k_1,k_2])}\frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}\geq\mu\sup_{n\in N_0}\frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}. \tag{2.7}$$

Proof. Since $\Delta[p(n)\phi(\Delta y(n))] = -h(n) \le 0$ for all $n \in N_0$, we see that $p(n)\phi(\Delta y(n))$ is decreasing. Then $\phi^{-1}(p(n))\Delta y(n)$ is decreasing.

It follows from Lemma 1 that $y(n) \ge 0$ and $\Delta y(n) \ge 0$ for all $n \in N_0$ and there exists the limit

$$\sup_{n\in N_0} \frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}.$$

To complete the proof of 2.7, we consider two cases:

Case 1. there is $n_0 \in N_0$ such that

$$\sup_{n\in N_0} \frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} = \frac{y(n_0)}{1+\sum_{s=0}^{n_0-1}\frac{1}{\phi^{-1}(p(s))}}.$$

For $n_1, n, n_2 \in N_0$ with $n_1 < n < n_2$, we have

$$\begin{aligned}
&\left(\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n_1-1}\frac{1}{\phi^{-1}(p(s))}\right) \\
&\frac{y(n_2)-y(n)}{\sum_{s=0}^{n_2-1}\frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} + y(n_1) - y(n) \\
&= \frac{\left(\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n_1-1}\frac{1}{\phi^{-1}(p(s))}\right)(y(n_2)-y(n))}{\sum_{s=0}^{n_2-1}\frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \\
&+ \frac{\left(\sum_{s=0}^{n_2-1}\frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}\right)(y(n_1)-y(n))}{\sum_{s=0}^{n_2-1}\frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \\
&= \frac{\left(\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n_1-1}\frac{1}{\phi^{-1}(p(s))}\right)\sum_{s=n}^{n_2-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}(p(s))\Delta y(s)}{\sum_{s=0}^{n_2-1}\frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \\
&- \frac{\left(\sum_{s=0}^{n_2-1}\frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}\right)\sum_{s=n_1}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}(p(s))\Delta y(s)}{\sum_{s=0}^{n_2-1}\frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}.
\end{aligned}$$

Since $\phi^{-1}(p(n))\Delta y(n)$ is decreasing, we get $\phi^{-1}(p(s))\Delta y(s) \le \phi^{-1}(p(k))\Delta y(k)$ for all $s \ge k$. So there there is λ such that

$$\phi^{-1}(p(s))\Delta y(s) \le \phi^{-1}(p(k))\Delta y(k),\ s \ge k.$$

Then we get

$$\frac{\left(\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n_1-1}\frac{1}{\phi^{-1}(p(s))}\right)\sum_{s=n}^{n_2-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}(p(s))\Delta y(s)}{\sum_{s=0}^{n_2-1}\frac{1}{\phi^{-1}(p(s))} - \sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}$$

$$
\begin{aligned}
&= \frac{\sum_{s=n_1}^{n-1}\frac{1}{\phi^{-1}(p(s))}\sum_{s=n}^{n_2-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}(p(s))\Delta y(s)}{\sum_{s=n}^{n_2-1}\frac{1}{\phi^{-1}(p(s))}}\\
&\le \frac{\lambda\sum_{s=n_1}^{n-1}\frac{1}{\phi^{-1}(p(s))}\sum_{s=n}^{n_2-1}\frac{1}{\phi^{-1}(p(s))}}{\sum_{s=n}^{n_2-1}\frac{1}{\phi^{-1}(p(s))}}\\
&= \frac{\lambda\sum_{s=n}^{n_2-1}\frac{1}{\phi^{-1}(p(s))}\sum_{s=n_1}^{n-1}\frac{1}{\phi^{-1}(p(s))}}{\sum_{s=n}^{n_2-1}\frac{1}{\phi^{-1}(p(s))}}\\
&\le \frac{\sum_{s=n}^{n_2-1}\frac{1}{\phi^{-1}(p(s))}\sum_{s=n_1}^{n-1}\frac{1}{\phi^{-1}(p(s))}\phi^{-1}(p(s))\Delta y(s)}{\sum_{s=n}^{n_2-1}\frac{1}{\phi^{-1}(p(s))}}.
\end{aligned}
$$

So

$$
\left(\sum_{s=n_1}^{n-1}\frac{1}{\phi^{-1}(p(s))}\right)\frac{y(n_2)-y(n)}{\sum_{s=n}^{n_2-1}\frac{1}{\phi^{-1}(p(s))}}+y(n_1)-y(n)\le 0.
$$

It follows that

$$
y(n)\ge \frac{\sum_{s=n}^{n_2-1}\frac{1}{\phi^{-1}(p(s))}}{\sum_{s=n_1}^{n_2-1}\frac{1}{\phi^{-1}(p(s))}}y(n_1)+\frac{\sum_{s=n_1}^{n-1}\frac{1}{\phi^{-1}(p(s))}}{\sum_{s=n_1}^{n_2-1}\frac{1}{\phi^{-1}(p(s))}}y(n_2). \tag{2.8}
$$

If $n_0 = k_1$, we get

$$
\begin{aligned}
\min_{n\in[k_1,k_2]}\frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} &\ge \frac{y(k_1)}{1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}}\\
&= \frac{y(n_0)}{1+\sum_{s=0}^{n_0-1}\frac{1}{\phi^{-1}(p(s))}}\frac{1+\sum_{s=0}^{k_1-1}\frac{1}{\phi^{-1}(p(s))}}{1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}}\\
\ge \frac{1}{\phi^{-1}(p(k_1-1))\left(1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}\right)} &\sup_{n\in N_0}\frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}\\
&\ge \mu\sup_{n\in N_0}\frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}.
\end{aligned}
$$

If $n_0 > k_1$, choose $n_1 = k_1 - 1$, $n = k_1$ and $n_2 = n_0$, by using 2.8 we have

$$
\begin{aligned}
y(k_1) &= \frac{\sum_{s=k_1}^{n_0-1}\frac{1}{\phi^{-1}(p(s))}}{\sum_{s=k_1-1}^{n_0-1}\frac{1}{\phi^{-1}(p(s))}}y(k_1-1)+\frac{\sum_{s=k_1-1}^{k_1-1}\frac{1}{\phi^{-1}(p(s))}}{\sum_{s=k_1-1}^{n_0-1}\frac{1}{\phi^{-1}(p(s))}}y(n_0)\\
&\ge \frac{\sum_{s=k_1-1}^{k_1-1}\frac{1}{\phi^{-1}(p(s))}}{\sum_{s=k_1-1}^{n_0-1}\frac{1}{\phi^{-1}(p(s))}}y(n_0).
\end{aligned}
$$

Then

$$\begin{aligned}
&\min_{n\in[k_1,k_2]} \frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \\
&\geq \frac{y(k_1)}{1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}} \\
&\geq \frac{1+\sum_{s=0}^{n_0-1}\frac{1}{\phi^{-1}(p(s))}}{1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}} \frac{\sum_{s=k_1-1}^{k_1-1}\frac{1}{\phi^{-1}(p(s))}}{\sum_{s=k_1-1}^{n_0-1}\frac{1}{\phi^{-1}(p(s))}} \frac{y(n_0)}{1+\sum_{s=0}^{n_0-1}\frac{1}{\phi^{-1}(p(s))}} \\
&\geq \frac{1}{\phi^{-1}(p(k_1-1))\left(1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}\right)} \sup_{n\in N_0} \frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \\
&\geq \mu \sup_{n\in N_0} \frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}.
\end{aligned}$$

If $n_0 < k_1$, we have

$$\begin{aligned}
&\min_{n\in[k_1,k_2]} \frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \\
&\geq \frac{y(k_1)}{1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}} \\
&\geq \frac{1}{1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}} y(n_0) \\
&\geq \frac{1+\sum_{s=0}^{n_0-1}\frac{1}{\phi^{-1}(p(s))}}{1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}} \frac{y(n_0)}{1+\sum_{s=0}^{n_0-1}\frac{1}{\phi^{-1}(p(s))}} \\
&\geq \frac{1}{1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}} \sup_{n\in N_0} \frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \\
&\geq \mu \sup_{n\in N_0} \frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}.
\end{aligned}$$

Case 2. $\sup_{n\in N_0} \frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} = \lim_{n\to\infty} \frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}$.
Choose $n' > k_2$, similarly to Case 1 we can prove that

$$\min_{n\in[k_1,k_2]} \frac{y(n))}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \geq \mu \frac{y(n')}{1+n'}.$$

Let $n' \to \infty$, one sees

$$\min_{n\in[k_1,k_2]} \frac{y(n))}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \geq \mu \sup_{n\in N_0} \frac{y(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}.$$

From Cases 1 and 2, we get 2.7. The proof is complete. □

Define the functionals on $P : P \to R$ by

$$\gamma(x) = \sup_{n\in N_0} \phi^{-1}(p(n))|\Delta x(n)|,\ x \in P,$$

$$\beta(x) = \sup_{n\in N_0} \frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}},\ x \in P,$$

$$\theta(x) = \sup_{n\in N_0} \frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}},\ x \in P,$$

$$\alpha(x) = \min_{n\in[k_1,k_2]} \frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}},\ x \in P,$$

$$\psi(x) = \min_{n\in[k_1,k_2]} \frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}},\ x \in P.$$

Lemma 4. *If y is a solution of BVP(6), we have $||y|| \leq M\gamma(y)$ for all $y \in P$, where*

$$M = \max\left\{1,\ \sup_{n\in N_0} \frac{\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))} + \frac{\sum_{n\in N}\alpha_n \sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}}{1-\sum_{i=1}^{\infty}\alpha_i}}{1+\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}}\right\}. \tag{2.9}$$

Proof. Since y is the solution of BVP(6), we get

$$\begin{aligned}
|y(n)| &= |y(n) - y(0) + y(0)| \\
&\leq \left|\sum_{i=0}^{n-1}\Delta y(i)\right| + \left|\frac{y(0) - \sum_{n\in N}\alpha_n y(0)}{1-\sum_{n\in N}\alpha_n}\right| \\
&= \left|\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}\phi^{-1}(p(i))\Delta y(i)\right| + \frac{\sum_{n\in N}\alpha_n|y(n)-y(0)|}{1-\sum_{n\in N}\alpha_n} \\
&\leq \sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}\sup_{n\in N_0}\phi^{-1}(p(n))|\Delta y(n)| + \left|\frac{\sum_{n\in N}\alpha_n[y(n)-y(0)]}{1-\sum_{i=1}^{\infty}\alpha_i}\right| \\
&\leq \left(\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))} + \frac{\sum_{n\in N}\alpha_n\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}}{1-\sum_{i=1}^{\infty}\alpha_i}\right)\sup_{n\in N_0}\phi^{-1}(p(n))|\Delta y(n)|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{y(n)}{1+\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}}\\
&\leq \frac{\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}+\frac{\sum_{n\in N}\alpha_n\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}}{1-\sum_{i=1}^{\infty}\alpha_i}}{1+\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}}\sup_{n\in N_0}\phi^{-1}(p(n))|\Delta y(n)|\\
&\leq \sup_{n\in N_0}\frac{\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}+\frac{\sum_{n\in N}\alpha_n\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}}{1-\sum_{i=1}^{\infty}\alpha_i}}{1+\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}}\sup_{n\in N_0}\phi^{-1}(p(n))|\Delta y(n)|.
\end{aligned}$$

we get that

$$\begin{aligned}
||y|| &= \max\left\{\sup_{n\in N_0}\frac{|y(n)|}{1+\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}},\ \sup_{n\in N_0}\phi^{-1}(p(n))|\Delta y(n)|\right\}\\
&\leq \max\left\{1,\ \sup_{n\in N_0}\frac{\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}+\frac{\sum_{n\in N}\alpha_n\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}}{1-\sum_{i=1}^{\infty}\alpha_i}}{1+\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}}\right\}\\
&\qquad \sup_{n\in N_0}\phi^{-1}(p(n))|\Delta y(n)|\\
&= M\gamma(y).
\end{aligned}$$

Then $||y||\leq M\gamma(y)$ for all solutions y of BVP(6). The proof is complete. □

For $x\in P$, define $(Tx)(n)$ by

$$\begin{aligned}
(Tx)(n) &= \frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A_x+\sum_{s=j}^{\infty}f(s,x(s),\Delta x(s))}{p(j)}\right)\\
&\quad +\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A_x+\sum_{s=j}^{\infty}f(s,x(s),\Delta x(s))}{p(j)}\right),\ n\in N_0,
\end{aligned}$$

where A_x satisfies

$$\begin{aligned}
\phi^{-1}(A_x) &= \frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A_x+\sum_{s=j}^{\infty}f(s,x(s),\Delta x(s))}{p(j)}\right)\\
&\quad +\sum_{n=1}^{\infty}\beta_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A_x+\sum_{s=j}^{\infty}f(s,x(s),\Delta x(s))}{p(j)}\right).
\end{aligned}$$

One sees easily that

$$\begin{cases} \Delta[\phi(\Delta(Tx)(n))] + f(n, x(n), \Delta x(n)) = 0, & n \in N_0, \\ (Tx)(0) - \sum_{i=1}^{\infty} \alpha_i (Tx)(i) = 0, \\ \lim_{n\to\infty} \frac{(Tx)(n)}{1+\sum_{i=0}^{n-1} \frac{1}{\phi^{-1}(p(i))}} - \sum_{i=1}^{\infty} \beta_i (Tx)(i) = 0. \end{cases} \tag{2.10}$$

Let δ be defined by 2.3. By Lemma 2, we have

$$A_x \in \left[0, \delta \sum_{n=0}^{\infty} f(n, x(n), \Delta x(n))\right]. \tag{2.11}$$

Lemma 5. *Let* $V = \{x \in X : ||x|| < l\}(l > 0)$. *If* $\left\{\frac{x(n)}{1+\sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} : x \in V\right\}$ *and* $\{\phi^{-1}(p(n))\Delta x(n) : x \in V\}$ *are both equiconvergent at infinity, where*

$$V_1 =: \left\{\frac{x(n)}{1+\sum_{s=0}^{n-1} \frac{1}{\phi^{-1}(p(s))}} : x \in V\right\} \bigcup \{\phi^{-1}(p(n))\Delta x(n) : x \in V\}$$

is called equiconvergent at infinity if and only if for all $\epsilon > 0$, *there exists* $N = N(\epsilon) > 0$ *such that for all* $x \in V$, *it holds that*

$$\left| \frac{x(n_1)}{1+\sum_{s=0}^{n_1-1} \frac{1}{\phi^{-1}(p(s))}} - \frac{x(n_2)}{1+\sum_{s=0}^{n_2-1} \frac{1}{\phi^{-1}(p(s))}} \right| < \epsilon,$$

and

$$|\phi^{-1}(p(n_1))\Delta x(n_1) - \phi^{-1}(p(n_2))\Delta x(n_2)| < \epsilon \;\; n_1, n_2 > N.$$

Then V *is pre-compact on* X.

Proof. The proof is similar to that of the proof of a Lemma in [14] and is omitted. □

Lemma 6. *The following facts hold.*

(i) $Tx \in P$ *for each* $x \in P$;

(ii) x *is a solution of BVP(3) if and only if* x *is a solution of the operator equation* $x = Tx$;

(iii) $T : P \to P$ *is completely continuous;*

Proof. (i) Note the definition of P. For $x \in P$, Lemma 2 and Lemma 3 imply that $Tx \in P$.

(ii) It is easy to see that x is a solution of BVP(3) if and only if x is a solution of the operator equation $x = Tx$.

(iii) It suffices to prove that T is continuous on P and T maps bounded subsets into pre-compact sets. We divide the proof into four steps:

Step 1. For each bounded subset $D \subset P$, prove that $\{A_x : x \in \overline{D}\}$ is bounded in R.

Denote

$$L_1 = \sup\left\{\max_{n\in N_0}\frac{|x(n)|}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}},\ \sup_{n\in N_0}\phi^{-1}(p(n))|\Delta x(n)|:\ x\in\overline{D}\right\}$$

and

$$f_{L_1}(j) = \max_{|x|,|y|\le L_1}\left|f\left(j,(1+\sum_{s=0}^{j-1}\frac{1}{\phi^{-1}(p(s))})x,\frac{y}{\phi^{-1}(p(j))}\right)\right|.$$

Since f is a Caratheodory function, it follows from 2.11 that

$$0\le A_x\le\delta\sum_{j=0}^{\infty}f_{L_1}(j)<\infty.$$

Hence $\{A_x : x\in\overline{D}\}$ is bounded in R.

Step 2. For each bounded subset $D\subset P$, and each $x_0\in D$, prove that T is continuous at x_0.

For $x_0\in D$ and $x_n\in D$ with $x_n\to x_0(n\to+\infty)$ in D.
Denote $u_n(k)=(Tx_n)(k), u_0(k)=(Tx_0)(k)$ for all $k\in N_0$. We prove that T is continuous at x_0, i.e., $u_n\to u_0(n\to+\infty)$. Let A_{x_0} be defined by

$$\begin{aligned}&\phi^{-1}(A_{x_0})\\ &=\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A_{x_0}+\sum_{s=j}^{\infty}f(s,x_0(s),\Delta x_0(s))}{p(j)}\right)\\ &\quad+\sum_{n=1}^{\infty}\beta_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A_{x_0}+\sum_{s=j}^{\infty}f(s,x_0(s),\Delta x_0(s))}{p(j)}\right).\end{aligned}$$

First, we prove that A_x are continuous in x, i.e.,

$$A_{x_n}\to A_{x_0},\ n\to+\infty.$$

It follows from Step 1 that A_{x_n} is bounded. Without loss of generality, suppose that $A_{x_n}\to\overline{A}\ne A_{x_0}$.

It is easy to see that

$$\begin{aligned}&\lim_{n\to+\infty}u_n(k)\\ &=\lim_{n\to+\infty}\left[\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A_{x_n}+\sum_{s=j}^{\infty}f(s,x_n(s),\Delta x_n(s)))}{p(j)}\right)\right.\\ &\quad\left.+\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A_{x_n}+\sum_{s=j}^{\infty}f(s,x_n(s),\Delta x_n(s)))}{p(j)}\right)\right]\end{aligned}$$

$$= \frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{\overline{A}+\lim_{n\to+\infty}\sum_{s=j}^{\infty}f(s,x_n(s),\Delta x_n(s)))}{p(j)}\right)$$
$$+\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{\overline{A}+\lim_{n\to+\infty}\sum_{s=j}^{\infty}f(s,x_n(s),\Delta x_n(s)))}{p(j)}\right)$$
$$= \frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{\overline{A}+\sum_{s=j}^{\infty}f(s,x_0(s),\Delta x_0(s)))}{p(j)}\right)$$
$$+\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{\overline{A}+\sum_{s=j}^{\infty}f(s,x_0(s),\Delta x_0(s)))}{p(j)}\right)$$
$$= \overline{u}(k).$$

One sees that $\overline{A}=\lim_{n\to\infty}\phi^{-1}(p(n))\Delta\overline{u}(n)$ and $\overline{u}$ satisfies

$$\overline{u}(0)-\sum_{n=1}^{\infty}\alpha_n\overline{u}(n)=0,\ \lim_{n\to\infty}\frac{\overline{u}(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}-\sum_{i=1}^{\infty}\beta_i\overline{u}(i)=0.$$

So

$$\phi^{-1}(\overline{A})=\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{\overline{A}+\sum_{s=j}^{\infty}f(s,x_0(s),\Delta x_0(s))}{p(j)}\right)$$
$$+\sum_{n=1}^{\infty}\beta_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{\overline{A}+\sum_{s=j}^{\infty}f(s,x_0(s),\Delta x_0(s))}{p(j)}\right).$$

It follows from Lemma 2 that $\overline{A}=A_{x_0}$. Hence

$$A_{x_n}\to\overline{A}=A_{x_0},\ \ n\to+\infty.$$

This together with the continuous property of f implies that T is continuous at x_0.

Step 3. For each bounded subset $\Omega\subset P$, prove that $T\Omega$ is bounded.

In fact, for each bounded subset $\Omega\subseteq D$, and $x\in\Omega$. Suppose

$$||x||=\max\left\{\sup_{n\in N_0}\frac{|x(n)|}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}},\ \max_{n\in[0,N+1]}\phi^{-1}(p(n))|\Delta x(n)|\right\}\le M_1$$

and Step 1 implies that there exist constants $M_2>0$ such that $|A_x|<M_2$ for all $x\in\Omega$. Then

$$\frac{|(Tx)(n)|}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}$$

$$= \frac{1}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}$$
$$\left(\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A_x+\sum_{s=j}^{\infty}f(s,x(s),\Delta x(s))}{p(j)}\right)\right.$$
$$\left.+\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A_x+\sum_{s=j}^{\infty}f(s,x(s),\Delta x(s))}{p(j)}\right)\right)$$
$$\leq \frac{1}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}$$
$$\left(\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{M_2+\sum_{s=j}^{\infty}f(s,x(s),\Delta x(s))}{p(j)}\right)\right.$$
$$\left.+\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{M_2+\sum_{s=j}^{\infty}f(s,x(s),\Delta x(s))}{p(j)}\right)\right)$$
$$\leq \phi^{-1}(M_2+\sum_{s=0}^{\infty}f_{M_1}(s))$$
$$\frac{\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)+\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}$$
$$\leq \phi^{-1}(M_2+\sum_{s=0}^{\infty}f_{M_1}(s))$$
$$\sup_{n\in N_0}\frac{\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)+\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}$$
$$=: M_3,$$

where $f_{M_1}(j)=\max_{|x|\leq M_1,|y|\leq M_1\leq M_1}\left|f\left(j,(1+\sum_{s=0}^{j-1}\frac{1}{\phi^{-1}(p(s))})x,\frac{y}{\phi^{-1}(p(j))}\right)\right|$. Similarly, one has that

$$\phi^{-1}(p(n))|\Delta(Tx)(n)|=\left|\phi^{-1}\left(A_x+\sum_{j=n}^{\infty}f(j,x(j),\Delta x(j))\right)\right|$$

$$\le \phi^{-1}\left(M_2+\sum_{j=0}^{\infty} f_{M_1}(j)\right) =: M_4.$$

It follows that $T\Omega$ is bounded.

Step 4. For each bounded subset $\Omega \subset P$, prove that $T\Omega$ is pre-compact.

Similarly to Step 3, we see $|A_x| \le M_2$, we get

$$A_x+\sum_{n=0}^{\infty} f(n,x(n),\Delta x(n)) \le \phi(M_4).$$

Note $\phi^{-1}(x)=|x|^{q-2}x$. Then there $\xi \in \left[A_x, A_x+\sum_{j=n}^{\infty} f(j,x(j),\Delta x(j))\right]$ such that

$$\begin{aligned}
&|\phi^{-1}(p(n))\Delta(Tx)(n)-\phi^{-1}(A_x)| \\
&= \left|\phi^{-1}\left(A_x+\sum_{j=n}^{\infty} f(j,x(j),\Delta x(j))\right)-\phi^{-1}(A_x)\right| \\
&= (q-1)\xi^{q-2}\sum_{j=n}^{\infty} f(j,x(j),\Delta x(j)) \\
&\le (q-1)\phi(M_4)^{q-2}\sum_{j=n}^{\infty} f(j,x(j),\Delta x(j)) \\
&\le (q-1)\phi(M_4)^{q-2}\sum_{j=n}^{\infty} f_{M_1}(j) \\
&\to 0 \text{ uniformly as } n\to\infty.
\end{aligned}$$

For any $\epsilon>0$, there exists $N_{1,\epsilon}>0$ such that

$$|\phi^{-1}(p(n_1))\Delta(Tx)(n_1)-\phi^{-1}(p(n_2))\Delta(Tx)(n_2)|<\epsilon,\ \ n>N_{1,\epsilon}. \tag{2.12}$$

Since $\lim_{n\to\infty}\phi^{-1}(p(n))\Delta(Tx)(n)=\phi^{-1}(A_x)$ uniformly, we know from the same methods used in the proof of Lemma 2 that

$$\lim_{n\to\infty}\frac{|(Tx)(n)|}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}=\phi^{-1}(A_x) \text{ uniformly.}$$

So there exists $N_{2,\epsilon}>0$ such that

$$\left|\frac{(Tx)(n_1)}{1+\sum_{s=0}^{n_1-1}\frac{1}{\phi^{-1}(p(s))}}-\frac{(Tx)(n_2)}{1+\sum_{s=0}^{n_2-1}\frac{1}{\phi^{-1}(p(s))}}\right|<\epsilon,\ \ n>N_{2,\epsilon}. \tag{2.13}$$

Choose $N_\epsilon = \max\{N_{1,\epsilon}, N_{2,\epsilon}\}$. Then

$$|\phi^{-1}(p(n_1)\Delta(Tx)(n_1) - \phi^{-1}(p(n_2)\Delta(Tx)(n_2)| < \epsilon,$$

and

$$\left|\frac{(Tx)(n_1)}{1+\sum_{s=0}^{n_1-1}\frac{1}{\phi^{-1}(p(s))}} - \frac{(Tx)(n_2)}{1+\sum_{s=0}^{n_2-1}\frac{1}{\phi^{-1}(p(s))}}\right| < \epsilon, \ \ n > N_\epsilon.$$

One knows that $T\Omega$ is pre-compact. Lemma 5 with Steps 1, 2, 3 and 4 imply that T is completely continuous. □

Theorem 1. *Choose $k_1, k_2 \in N$ with $k_1 < k_2$. Let μ be defined by 2.1, δ by 2.3 and M by 2.9. Suppose that there exist positive constants e_1, e_2, c such that*

$$c \geq \frac{e_2}{\mu} > e_2 > e_1 > 0.$$

Let

$$Q = \phi\left(\frac{c}{M}\right)\frac{1}{1+\delta};$$

$$W = \phi\left(\frac{e_2\left(1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}\right)}{\sum_{j=0}^{k_1-1}\phi^{-1}\left(\frac{\sum_{s=k_1}^{k_2}\frac{1}{2^{s+1}}}{p(j)}\right)}\right);$$

$$E = \frac{1}{1+\delta}\phi\left(\frac{e_1}{\sup_{n\in N_0}\frac{\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)+\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}}\right).$$

If $Q > W$ and

(A1) $f\left(n, \left(1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}\right)u, \frac{v}{\phi^{-1}(p(n))}\right) \leq \frac{Q}{2^{n+1}}$ *for all $n \in N_0, u \in [0,c], v \in [0,c]$;*

(A2) $f\left(n, \left(1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}\right)u, \frac{v}{\phi^{-1}(p(n))}\right) \geq \frac{W}{2^{n+1}}$ *for all $n \in [k_1,k_2], u \in [e_2, \frac{e_2}{\mu}], v \in [0,c]$;*

(A3) $f\left(n, \left(1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}\right)u, \frac{v}{\phi^{-1}(p(n))}\right) \leq \frac{E}{2^{n+1}}$ *for all $n \in N_0, u \in [0,e_1], v \in [0,c]$;*

then BVP(3) has at least three positive solutions x_1, x_2, x_3 such that

$$\sup_{n\in N_0}\frac{x_1(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} < e_1, \quad \min_{n\in[k_1,k_2]}\frac{x_2(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} > e_2, \tag{2.14}$$

and

$$\sup_{n\in N_0}\frac{x_3(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}>e_1,\quad \min_{n\in[k_1,k_2]}\frac{x_3(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}<e_2. \tag{2.15}$$

Proof. Let X, P and T be defined above. We complete the proof by using Lemma 1. By the definitions, it is easy to see that α,ψ are two nonnegative continuous concave functionals on the cone P, γ,β,θ are three nonnegative continuous convex functionals on the cone P.

One sees $\alpha(x)\le\beta(x)$ for all $x\in P$. From Lemma 4, we have $||x||\le M\gamma(x)$ for all $x\in P$.

Lemma 6 implies that $x=x(n)$ is a solution of BVP(3) if and only if x is a solution of the operator equation $x=Tx$ and $T:P\to P$ is completely continuous.

Corresponding to Lemma 1, choose

$$h=\mu e_1,\ d=e_1,\ a=e_2,\ b=\frac{e_2}{\mu},\ c=c.$$

Now, we prove that all conditions of Lemma 1 hold. One sees that $0<d<a$. The remainder is divided into five steps.

Step 1. Prove that $T:\ \overline{P_c}\to\overline{P_c}$;

For $x\in\overline{P_c}$, we have $||x||\le c$. Then

$$0\le\frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}\le c, n\in N_0,$$
$$0\le\phi^{-1}(p(n))\Delta x(n)\le c \text{ for } n\in N_0.$$

So (A1) implies that

$$\begin{aligned}&f(n,x(n),\Delta x(n))\\ =&f\left(n,\left(1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}\right)\frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}},\frac{1}{\phi^{-1}(p(n))}\phi^{-1}(p(n))\Delta x(n)\right)\\ \le&\frac{Q}{2^{n+1}},\ n\in N_0.\end{aligned}$$

It follows from Lemma 6 that $Tx\in P$. One sees from Lemma 1 that

$$0\le A_x\le\delta\sum_{j=0}^{\infty}f(j,x(j),\Delta x(j)). \tag{2.16}$$

We have that

$$\phi^{-1}(p(n))|\Delta(Tx)(n)|=\left|\phi^{-1}\left(A_x+\sum_{j=n}^{\infty}f(j,x(j),\Delta x(j))\right)\right|$$

$$\leq \phi^{-1}\left((1+\delta)\sum_{j=0}^{\infty} f(j,x(j),\Delta x(j))\right)$$
$$\leq \phi^{-1}\left((1+\delta)\sum_{j=0}^{\infty}\frac{Q}{2^{j+1}}\right)$$
$$\leq \phi^{-1}(Q(1+\delta)) \leq c.$$

From Lemma 5, we have

$$\frac{|(Tx)(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \leq M \sup_{n\in N_0} \phi^{-1}(p(n))|\Delta(Tx)(n)|$$
$$\leq M\phi^{-1}((1+\delta)Q) \leq c.$$

It follows that

$$||Tx|| = \max\left\{\max_{n\in N_0}\frac{|(Tx)(n)|}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}, \max_{n\in N_0}\phi^{-1}(p(n))|\Delta(Tx)(n)|\right\} \leq c.$$

Then $T:\ \overline{P_c} \to \overline{P_c}$.

Step 2. Prove that

$$\{y \in P(\gamma,\theta,\alpha;a,b,c)|\alpha(x) > a\}$$
$$= \left\{y \in P\left(\gamma,\theta,\alpha;e_2,\frac{e_2}{\mu},c\right)|\alpha(x) > e_2\right\} \neq \varnothing$$

and $\alpha(Tx) > e_2$ for every $x \in P\left(\gamma,\theta,\alpha;e_2,\frac{e_2}{\mu},c\right)$;

By the definition of μ, we can choose A, B such that

$$A \in \left[\left(1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}\right)e_2, \left(1+\sum_{s=0}^{k_1-1}\frac{1}{\phi^{-1}(p(s))}\right)\frac{e_2}{\mu}\right],$$

$$B \leq \min\left\{\frac{e_2}{\mu}, \frac{A}{\mu\left(1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}\right)}\right\}$$

and

$$|A-B| < \frac{c}{\max\{\phi^{-1}(p(k_1-1)),\phi^{-1}(p(k_2))\}}.$$

Let

$$x(n) = \begin{cases} A, n \in [k_1,k_2], \\ B, n \notin [k_1,k_2]. \end{cases}$$

Then $x \in P$ and

$$\alpha(x) = \min_{n\in[k_1,k_2]} \frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \geq \frac{A}{\left(1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}\right)} > e_2,$$

$$\theta(x) = \sup_{n\in N_0} \frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}$$
$$\leq \max\left\{B, \frac{A}{\left(1+\sum_{s=0}^{k_1-1}\frac{1}{\phi^{-1}(p(s))}\right)}, \frac{B}{\left(1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}\right)}\right\} \leq \frac{e_2}{\mu} = b,$$

and

$$\gamma(x) = \sup_{n\in N_0} \phi^{-1}(p(n))|\Delta x(n)| = \max\{\phi^{-1}(p(k_1-1)),\phi^{-1}(p(k_2))\}|A-B| < c.$$

It follows that $\{y \in P(\gamma,\theta,\alpha;a,b,c)|\alpha(x) > a\} \neq \emptyset$.

For $x \in P(\gamma,\theta,\alpha;a,b,c)$, one has that

$$\alpha(x) = \min_{n\in[k_1,k_2]} \frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \geq e_2,$$
$$\theta(x) = \sup_{n\in N_0} \frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \leq \frac{e_2}{\mu},$$

and

$$\gamma(x) = \sup_{n\in N_0} \phi^{-1}(p(n))|\Delta x(n)| \leq c.$$

Then

$$e_2 \leq \frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \leq \frac{e_2}{\mu},\ n \in [k_1,k_2],\ \ 0 \leq \phi^{-1}(p(n))\Delta x(n) \leq c.$$

Thus (A2) implies that

$$f(n,x(n),\Delta x(n)) \geq \frac{W}{2^{n+1}},\ \ n \in [k_1,k_2].$$

We get

$$\alpha(Tx) = \min_{n\in[k_1,k_2]} \frac{(Tx)(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}$$
$$> \frac{(Tx)(k_1)}{1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}}$$
$$= \frac{1}{1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}}\left[\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\right.$$

$$\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A_x+\sum_{s=j}^{\infty}f(s,x(s),\Delta x(s))}{p(j)}\right)$$
$$\left.+\sum_{j=0}^{k_1-1}\phi^{-1}\left(\frac{A_x+\sum_{s=j}^{\infty}f(s,x(s),\Delta x(s))}{p(j)}\right)\right]$$
$$\geq\frac{1}{1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}}\sum_{j=0}^{k_1-1}\phi^{-1}\left(\frac{\sum_{s=j}^{\infty}f(s,x(s),\Delta x(s))}{p(j)}\right)$$
$$\geq\frac{1}{1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}}\sum_{j=0}^{k_1-1}\phi^{-1}\left(\frac{\sum_{s=k_1}^{k_2}f(s,x(s),\Delta x(s))}{p(j)}\right)$$
$$\geq\frac{1}{1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}}\sum_{j=0}^{k_1-1}\phi^{-1}\left(\frac{\sum_{s=k_1}^{k_2}\frac{W}{2^{s+1}}}{p(j)}\right)$$
$$=e_2.$$

This completes Step 2.

Step 3. Prove that

$$\{y\in Q(\gamma,\theta,\psi;h,d,c)|\beta(x)<d\}=\{y\in Q\,(\gamma,\theta,\psi;\mu e_1,e_1,c)\,|\beta(x)<e_1\}\neq\emptyset$$

and

$$\beta(Tx)<e_1 \text{ for every } x\in Q(\gamma,\theta,\psi;h,d,c)=Q\,(\gamma,\theta,\psi;\mu e_1,e_1,c);$$

Similarly to Step 2, we can see that $\{y\in Q(\gamma,\theta,\psi;h,d,c)|\beta(x)<d\}\neq\emptyset$.

For $x\in Q(\gamma,\theta,\psi;h,d,c)$, one has that

$$\psi(x)=\min_{n\in[k_1,k_2]}\frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}\geq\mu e_1$$
$$\theta(x)=\sup_{n\in N_0}\frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}\leq d=e_1,$$

and

$$\gamma(x)=\sup_{n\in N_0}\phi^{-1}(p(n))|\Delta x(n)|\leq c.$$

Hence we get that

$$0\leq\frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}\leq e_1,\ n\in N_0;\ \ 0\leq\phi^{-1}(p(n))\Delta x(n)\leq c,\ n\in N_0.$$

Then (A3) implies that

$$f(n,x(n),\Delta x(n))\leq\frac{E}{2^{n+1}},\ \ n\in N_0.$$

So 2.16 implies that

$$\beta(Tx) = \sup_{n\in N_0} \frac{(Tx)(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}$$
$$= \sup_{n\in N_0} \frac{1}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}\left[\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\right.$$
$$\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A_x+\sum_{s=j}^{\infty}f(s,x(s),\Delta x(s))}{p(j)}\right)$$
$$\left.+\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{A_x+\sum_{s=j}^{\infty}f(s,x(s),\Delta x(s))}{p(j)}\right)\right]$$
$$< \sup_{n\in N_0}\frac{\phi^{-1}\left((1+\delta)\sum_{s=0}^{\infty}f(s,x(s),\Delta x(s))\right)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}\left[\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\right.$$
$$\left.\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)+\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)\right]$$
$$\le \sup_{n\in N_0}\frac{\phi^{-1}\left((1+\delta)\sum_{s=0}^{\infty}\frac{E}{2^{j+1}}\right)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}\left[\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)\right.$$
$$\left.+\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)\right]$$
$$\le \phi^{-1}((1+\delta)E)$$
$$\sup_{n\in N_0}\frac{\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)+\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}$$
$$\le e_1 = d.$$

This completes Step 3.

Step 4. Prove that $\alpha(Ty) > a$ for $y \in P(\gamma,\alpha;a,c)$ with $\theta(Ty) > b$;

For $x \in P(\gamma,\alpha;a,c) = P(\gamma,\alpha;e_2,c)$ with $\theta(Tx) = \beta(Tx) > b = \frac{e_2}{\mu}$, we have that

$$\alpha(x) = \min_{n\in[k_1,k_2]}\frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \ge e_2,$$
$$\gamma(x) = \sup_{n\in N_0}\phi^{-1}(p(n))|\Delta x(n)| \le c,$$

$$\sup_{n\in N_0} \frac{(Tx)(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} > \frac{e_2}{\mu}.$$

Then

$$\alpha(Tx) = \min_{n\in[k_1,k_2]} \frac{(Tx)(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \geq \mu\beta(Tx) > e_2 = a.$$

This completes Step 4.

Step 5. Prove that $\beta(Tx) < d$ for each $x \in Q(\gamma,\beta;d,c)$ with $\psi(Tx) < h$.

For $x \in Q(\gamma,\beta;d,c)$ with $\psi(Tx) < h$, we have

$$\gamma(x) = \sup_{n\in N_0} \phi^{-1}(p(n))|\Delta x(n)| \leq c,$$

$$\beta(x) = \sup_{n\in N_0} \frac{x(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \leq d = e_1,$$

$$\psi(Tx) = \min_{n\in[k_1,k_2]} \frac{(Tx)(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} < h = \mu e_1.$$

Then

$$\beta(Tx) = \sup_{n\in N_0} \frac{(Tx)(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} \leq \frac{1}{\mu} \min_{n\in[k_1,k_2]} \frac{(Tx)(n)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}} < e_1 = d.$$

This completes the Step 5.

Then Lemma 1 implies that T has at least three fixed points y_1, y_2 and y_3 such that

$$\beta(y_1) < e_1,\ \alpha(y_2) > e_2,\ \beta(y_3) > e_1,\ \alpha(y_3) < e_2.$$

Hence BVP(3) has three positive solutions y_1, y_2 and y_3 satisfying 2.14 and 2.15. The proof is complete. □

3. An example

In this section, we present an example to illustrate Theorem 1.

Example 1. Consider the following BVP

$$\begin{cases} \Delta^2 x(n) = -f(n,x(n),\Delta x(n)), & n \in N_0, \\ x(0) = \sum_{n=1}^{\infty} \frac{1}{2^{n+1}} x(n), \\ \lim_{n\to\infty} \frac{x(n)}{1+n} = \sum_{n=1}^{\infty} \frac{1}{3^{n+1}} x(n), \end{cases} \tag{3.1}$$

where $f : N_0 \times [0,+\infty)^2 \to [0,+\infty)$ is a Caratheodory function.

Corresponding to BVP(3), $p(n) \equiv 1$, $\alpha_n = \frac{1}{2^{n+1}}, \beta_n = \frac{1}{3^{n+1}}$ and $\phi(x) = x$.

Choose the constant $k_1 = 10, k_2 = 10000$, $e_1 = 100, e_2 = 5400, c = 3688400 \times 2^{10009}$. It is easy to see that

$$\mu = \frac{1}{\max\{\phi^{-1}(p(k_1-1)),1\}\left(1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}\right)} = \frac{1}{10001},$$

$$M = \max\left\{1,\ \sup_{n\in N_0}\frac{\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}+\frac{\sum_{n\in N}\alpha_n\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}}{1-\sum_{i=1}^{\infty}\alpha_i}}{1+\sum_{i=0}^{n-1}\frac{1}{\phi^{-1}(p(i))}}\right\} = \frac{5}{4},$$

$$\delta$$

$$= \frac{\phi\left(\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)+\sum_{n=1}^{\infty}\beta_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)\right)}{1-\phi\left(\frac{\sum_{n=1}^{\infty}\beta_n}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)+\sum_{n=1}^{\infty}\beta_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)\right)}$$

$$= 1,$$

$$Q = \phi\left(\frac{c}{M}\right)\frac{1}{1+\delta} = 1498400\times 2^{10009};$$

$$W = \phi\left(\frac{e_2\left(1+\sum_{s=0}^{k_2-1}\frac{1}{\phi^{-1}(p(s))}\right)}{\phi^{-1}\left(\sum_{s=k_1}^{k_2}\frac{1}{2^{s+1}}\right)\sum_{j=0}^{k_1-1}\phi^{-1}\left(\frac{1}{p(j)}\right)}\right) = \frac{10001\times 540\times 2^{10009}}{2^{10000}-2^9};$$

$$E = \frac{1}{1+\delta}\phi\left(\frac{e_1}{\sup_{n\in N_0}\frac{\frac{1}{1-\sum_{n=1}^{\infty}\alpha_n}\sum_{n=1}^{\infty}\alpha_n\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)+\sum_{j=0}^{n-1}\phi^{-1}\left(\frac{1}{p(j)}\right)}{1+\sum_{s=0}^{n-1}\frac{1}{\phi^{-1}(p(s))}}}\right) = 40.$$

So $Q > W$ and $c > \frac{e_2}{\mu} > e_2 > e_1 > 0$. If

$$f_0(x) = \begin{cases} 20, & x\in[0,100], \\ 20+(x-100)\frac{\frac{1498400\times 2^{10009}+\frac{10001\times 540\times 2^{10009}}{2^{10000}-2^9}}{2}-20}{5400-100}, & x\in[100,5400], \\ \frac{\frac{15300}{49}-\frac{5}{343}}{5404-1554}(x-1554)+\frac{5}{343}, & x\in[5400, 5400\times 10001], \\ \frac{1498400\times 2^{10009}+\frac{10001\times 540\times 2^{10009}}{2^{10000}-2^9}}{2}, & x\in[5400\times 10001,+\infty), \end{cases}$$

and

$$f(n,x,y) = \frac{f_0\left(\frac{x}{1+n}\right)}{2^{n+1}} + \frac{|y|}{343\times 3688400}.$$

Then

$$f\left(n,(1+n)x,y\right) = \frac{f_0(x)}{2^{n+1}} + \frac{|y|}{343\times 3688400}.$$

It is easy to check that if

(A1) $f(n,(1+n)u,v) \le \frac{Q}{2^{n+1}}$ for all $n \in N_0, u \in [0,c], v \in [0,c]$;
(A2) $f(n,(1+n)u,v) \ge \frac{w}{2^{n+1}}$ for all $n \in [k_1,k_2], u \in [e_2, e_2\mu], v \in [0,c]$;
(A3) $f(n,(1+n)u,v) \le \frac{E}{2^{n+1}}$ for all $n \in N_0, u \in [0,e_1], v \in [0,c]$;
then Theorem 1 implies that BVP(21) has at least three positive solutions such that

$$\sup_{n \in N_0} \frac{x_1(n)}{1+n} < 100, \quad \min_{n \in [10,10000]} \frac{x_2(n)}{1+n} > 5400,$$

and

$$\sup_{n \in N_0} \frac{x_3(n)}{1+n} > 100, \quad \min_{n \in [10,10000]} \frac{x_3(n)}{1+n} < 5400.$$

Remark 1. BVP(21) in Example 1 can not be solved by the theorems in [2–4, 16, 17].

References

[1] R. P. Agarwal, *Difference equations and inequalities: theory, methods, and applications. 2nd, revised and expanded ed.*, ser. Pure and Applied Mathematics. New York: Marcel Dekker, 2000, vol. 228.

[2] R. P. Agarwal, M. Bohner, and D. O'Regan, "Time scale boundary value problems on infinite intervals," *J. Comput. Appl. Math.*, vol. 141, no. 1-2, pp. 27–34, 2002.

[3] R. P. Agarwal and D. O'Regan, "Boundary value problems for general discrete systems on infinite intervals," *Comput. Math. Appl.*, vol. 33, no. 7, pp. 85–99, 1997.

[4] R. P. Agarwal and D. O'Regan, "Discrete systems on infinite intervals," *Comput. Math. Appl.*, vol. 35, no. 9, pp. 97–105, 1998.

[5] R. P. Agarwal and D. O'Regan, "Cone compression and expansion fixed point theorems in fréchet spaces with applications," *J. Differ. Equations*, vol. 171, no. 2, pp. 412–429, 2001.

[6] R. P. Agarwal and D. O'Regan, "Nonlinear Urysohn discrete equations on the infinite interval:a fixed-point approach," *Comput. Math. Appl.*, vol. 42, no. 3-5, pp. 273–281, 2001.

[7] R. I. Avery, "A generalization of the Leggett-Williams fixed point theorem," *Math. Sci. Res. Hot-Line*, vol. 3, no. 7, pp. 9–14, 1999.

[8] R. I. Avery and A. C. Peterson, "Three positive fixed points of nonlinear operators on ordered Banach spaces," *Comput. Math. Appl.*, vol. 42, no. 3-5, pp. 313–322, 2001.

[9] W.-S. Cheung, J. Ren, P. J. Wong, and D. Zhao, "Multiple positive solutions for discrete nonlocal boundary value problems," *J. Math. Anal. Appl.*, vol. 330, no. 2, pp. 900–915, 2007.

[10] J. R. Graef, L. Kong, and B. Yang, "Positive solutions for third order multi-point singular boundary value problems," *Czech. Math. J.*, vol. 60, no. 1, pp. 173–182, 2010.

[11] Y. Li and L. Lu, "Existence of positive solutions of p-Laplacian difference equations," *Appl. Math. Lett.*, vol. 19, no. 10, pp. 1019–1023, 2006.

[12] Y. Liu, "Positive solutions of multi-point BVPs for second-order p-Laplacian difference equations," *Commun. Math. Anal.*, vol. 4, no. 1, pp. 58–77, 2008.

[13] Y. Liu and W. Ge, "Twin positive solutions of boundary value problems for finite difference equations with p-Laplacian operator," *J. Math. Anal. Appl.*, vol. 278, no. 2, pp. 551–561, 2003.

[14] R. Ma and Y. N. Raffoul, "Positive solutions of three-point nonlinear discrete second order boundary value problem," *J. Difference Equ. Appl.*, vol. 10, no. 2, pp. 129–138, 2004.

[15] H. Pang, H. Feng, and W. Ge, "Multiple positive solutions of quasi-linear boundary value problems for finite difference equations," *Appl. Math. Comput.*, vol. 197, no. 1, pp. 451–456, 2008.

[16] A. S. V. Ravi Kanth and Y. N. Reddy, “A numerical method for solving two-point boundary value problems over infinite intervals,” *Appl. Math. Comput.*, vol. 144, no. 2-3, pp. 483–494, 2003.

[17] Y. Tian and W. Ge, “Multiple positive solutions of boundary value problems for second-order discrete equations on the half-line,” *J. Difference Equ. Appl.*, vol. 12, no. 2, pp. 191–208, 2006.

[18] P. J. Y. Wong and L. Xie, “Three symmetric solutions of Lidstone boundary value problems for difference and partial difference equations,” *Comput. Math. Appl.*, vol. 46, no. 6-9, pp. 1445–1460, 2003.

[19] J. Yu and Z. Guo, “On generalized discrete boundary value problems of Emden-Fowler equation,” *Science in China (Ser. A Mathematics)*, vol. 36, no. 7, pp. 721–732, 2006.

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