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Interval oscillation criteria for second order nonlinear differential equations with nonlinear damping

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INTERVAL OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH NONLINEAR DAMPING

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Abstract. Using generalized Riccati transformations, we derive new interval oscillation criteria for a class of general type second-order nonlinear differential equations with nonlinear damping. Examples are also given to illustrate the results.

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1. INTRODUCTION

In 1836 Sturm introduced the concept of oscillation when he studied the problem of the heat transmission, oscillation theory has been an important area of research in the qualitative theory of ordinary differential equations. Usually a qualitative approach is concerned with the behavior of solutions of a given differential equation and does not seek explicit solutions. Since then, oscillation behavior of solutions to different classes of linear and nonlinear ordinary differential equations have attracted the attention of many researchers.

In this paper we are concerned with the oscillatory behavior of the nonlinear second order differential equation with damping term

$$(r(t)k_1(x, x'))' + p(t)k_2(x, x')x' + q(t)f(x) = 0, \quad t \geq t_0, \quad (1.1)$$

where $t_0 \geq 0$ is a fixed real number, $p, q \in C([t_0, \infty), R)$, $r \in C^1([t_0, \infty), (0, \infty))$, $f \in C(R, R)$, $k_1 \in C^1(R^2, R)$ and $k_2 \in C(R^2, R)$. As usual, a function $x : [t_0, t_1) \rightarrow (-\infty, \infty)$, $t_1 > t_0$, is called a solution of Eq. (1.1) if $x(t)$ satisfies Eq. (1.1) for all $t \in [t_0, t_1)$. In what follows, we always assume that solutions of this equation are continuable, that is, they exist for all $t \geq t_0$. A nonconstant continuable solution $x(t)$ of Eq.(1.1) is called proper if $\sup\{|x(t)| : t \geq t_0\} > 0$. A proper solution $x(t)$ is called oscillatory if it does not have the largest zero, otherwise it is called nonoscillatory. The equation (1.1) is called oscillatory if all its proper solutions are oscillatory.

The oscillation problem for Eq. (1.1) and its various particular cases has been the subject of intensive studies in recent years (see, for example ([1–42]) and the references cited therein).

In 2005, Tiryaki and Zafer [30] established several new interval oscillation criteria for Eq. (1.1) by making use of a generalized Riccati substitutions

$$w(t) = \rho(t) \left[\frac{r(t)k_1(x(t), x'(t))}{x(t)} + g(t) \right] \quad (1.2)$$

and

$$w(t) = \rho(t) \left[\frac{r(t)k_1(x(t), x'(t))}{f(x(t))} + g(t) \right] \quad (1.3)$$

where $\rho \in C^1([t_0, \infty), (0, \infty))$ and $g \in C^1([t_0, \infty), R)$. The results in [30] extend, improve, and generalize some known oscillation criteria in the literature. As pointed out earlier Tiryaki and Zafer [29], equation (1.1) deserves attention simply because it includes (as special case) quite a large class of equations in the literature, and therefore more research is necessary to enlarge the families of the functions k_1 and k_2 . Motivated by the works of Tiryaki and Zafer ([29], [30]), we establish some sufficient conditions which guarantee the oscillatory of all the solutions of Eq. (1.1). We also want to emphasize that the results in this work neither include, nor are included by the results in [30] and are interesting in their own right. At the end, examples are given to illustrate the theoretical analysis of this work.

Let $D = \{(t, s) : t \geq s \geq t_0\}$. A function $H \in C(D, R)$ is said to belong to a function class F if

- (i) $H(t, t) = 0$ for $t \geq t_0$ and $H(t, s) > 0$ for $t > s \geq t_0$;
- (ii) H has continuous partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ satisfying

$$\frac{\partial H}{\partial t}(t, s) = h_1(t, s)\sqrt{H(t, s)} \quad \text{and} \quad \frac{\partial H}{\partial s}(t, s) = -h_2(t, s)\sqrt{H(t, s)},$$

where $h_1, h_2 \in L_{loc}(D, R)$.

2. OSCILLATION RESULTS FOR $f(x)$ WITHOUT MONOTONOCITY

In this section we consider the oscillation of Eq. (1.1) when the function $f(x)$ is not monotone. In this case we shall assume that the following conditions are satisfied:

- (A1) $\frac{f(x)}{x} \geq K$ for some constant $K > 0$ and all $x \in R \setminus \{0\}$;
- (A2) $k_1^2(u, v) \leq \alpha_1 v k_1(u, v)$ for some constant $\alpha_1 > 0$ and all $(u, v) \in R^2$;
- (A3) $uv k_2(u, v) \geq \alpha_2 u k_1(u, v)$ for some constant $\alpha_2 > 0$ and all $(u, v) \in R^2$.

Theorem 1. *Let (A1)-(A3) hold. Suppose that there exists an interval $(a, b) \subset [t_0, \infty)$ such that $p(t) \geq 0$ and $q(t) \geq 0$ for all $t \in (a, b)$, and that there exist $c \in (a, b)$,*

$H \in F$ and $\rho \in C^1([t_0, \infty), (0, \infty))$ such that

$$\begin{aligned} & \frac{1}{H(c, a)} \int_a^c H(s, a) K \rho(s) q(s) ds + \frac{1}{H(b, c)} \int_c^b H(b, s) K \rho(s) q(s) ds \\ & > \frac{\alpha_1}{4H(c, a)} \int_a^c \rho(s) r(s) \left[h_1(s, a) + Q_1(s) \sqrt{H(s, a)} \right]^2 ds \\ & + \frac{\alpha_1}{4H(b, c)} \int_c^b \rho(s) r(s) \left[h_2(b, s) - Q_1(s) \sqrt{H(b, s)} \right]^2 ds, \end{aligned} \quad (2.1)$$

where

$$Q_1(s) = \left(\frac{\rho'(s)}{\rho(s)} - \frac{\alpha_2 p(s)}{r(s)} \right). \quad (2.2)$$

Then every solution of Eq. (1.1) has a zero in (a, b) .

Proof. For the sake of contradiction, assume that there exists a solution $x(t)$ of Eq. (1.1) such that $x(t) \neq 0$ for all $t \in (a, b)$. Introduce a new function w by

$$w(t) = \rho(t) \frac{r(t) k_1(x(t), x'(t))}{x(t)}, \quad t \in (a, b). \quad (2.3)$$

Note that w is well-defined because $x(t) \neq 0$ for all $t \in (a, b)$. Differentiating (2.3), we obtain

$$\begin{aligned} w'(t) &= \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t) p(t) \frac{k_2(x(t), x'(t)) x'(t)}{x(t)} - \rho(t) q(t) \frac{f(x(t))}{x(t)} \\ &\quad - \rho(t) \frac{r(t) k_1(x(t), x'(t)) x'(t)}{x^2(t)}. \end{aligned} \quad (2.4)$$

Using (A1)-(A3) in (2.4), one has

$$w'(t) \leq -K \rho(t) q(t) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{\alpha_2 p(t)}{r(t)} \right) w(t) - \frac{1}{\alpha_1 \rho(t) r(t)} w^2(t). \quad (2.5)$$

Multiplying (2.5) by $H(t, s)$, integrating it with respect to s from c to t for $t \in [c, b)$, and using (i) and (ii), we easily obtain

$$\begin{aligned} & \int_c^t H(t, s) K \rho(s) q(s) ds \\ & \leq - \int_c^t H(t, s) w'(s) ds + \int_c^t H(t, s) Q_1(s) w(s) ds - \int_c^t \frac{H(t, s) w^2(s)}{\alpha_1 \rho(s) r(s)} ds \\ & = H(t, c) w(c) \end{aligned}$$

$$\begin{aligned}
& - \int_c^t \left\{ H(t,s) \frac{w^2(s)}{\alpha_1 \rho(s)r(s)} + [h_2(t,s)\sqrt{H(t,s)} - Q_1(s)H(t,s)]w(s) \right\} ds \\
& = H(t,c)w(c) \\
& - \int_c^t \left[\sqrt{\frac{H(t,s)}{\alpha_1 \rho(s)r(s)}} w(s) + \frac{1}{2} \sqrt{\alpha_1 \rho(s)r(s)} (h_2(t,s) - Q_1(s)\sqrt{H(t,s)}) \right]^2 ds \\
& + \frac{\alpha_1}{4} \int_c^t \rho(s)r(s) [h_2(t,s) - Q_1(s)\sqrt{H(t,s)}]^2 ds \\
& \leq H(t,c)w(c) + \frac{\alpha_1}{4} \int_c^t \rho(s)r(s) [h_2(t,s) - Q_1(s)\sqrt{H(t,s)}]^2 ds. \tag{2.6}
\end{aligned}$$

Similarly, multiplying (2.5) by $H(s,t)$ and integrating from t to c for $t \in (a, c]$, one obtains

$$\begin{aligned}
& \int_t^c H(s,t) K\rho(s)q(s)ds \\
& \leq -H(c,t)w(c) + \frac{\alpha_1}{4} \int_t^c \rho(s)r(s) [h_1(s,t) + Q_1(s)\sqrt{H(s,t)}]^2 ds. \tag{2.7}
\end{aligned}$$

Letting $t \rightarrow a^+$ in (2.7) and $t \rightarrow b^-$ in (2.6), dividing (2.6) and (2.7) by $H(b,c)$ and $H(c,a)$, respectively, and then adding them, we have that

$$\begin{aligned}
& \frac{1}{H(c,a)} \int_a^c H(s,a) K\rho(s)q(s)ds + \frac{1}{H(b,c)} \int_c^b H(b,s) K\rho(s)q(s)ds \\
& \leq \frac{\alpha_1}{4H(c,a)} \int_a^c \rho(s)r(s) [h_1(s,a) + Q_1(s)\sqrt{H(s,a)}]^2 ds \\
& + \frac{\alpha_1}{4H(b,c)} \int_c^b \rho(s)r(s) [h_2(b,s) - Q_1(s)\sqrt{H(b,s)}]^2 ds,
\end{aligned}$$

which contradicts the assumption (2.1). The proof is complete. \square

Thus by Theorem 1, we have the following oscillation result:

Theorem 2. *Let (A1)-(A3) hold. Suppose that there exists an interval $(a, b) \subset [t_0, \infty)$ such that $p(t) \geq 0$ and $q(t) \geq 0$ for all $t \in (a, b)$. If, for every $T \geq t_0$, there*

exist $H \in F$, $\rho \in C^1([t_0, \infty), (0, \infty))$, and $a, b, c \in R$ such that $T \leq a < c < b$ and (2.1) holds, then every solution of Eq. (1.1) is oscillatory.

Theorem 3. Let (A1)-(A3) hold, and $p(t) \geq 0$ and $q(t) \geq 0$ for all $t \in [t_0, \infty)$. If

$$\limsup_{t \rightarrow \infty} \int_l^t \left\{ H(s, l) K\rho(s)q(s) - \frac{\alpha_1\rho(s)r(s)}{4} \left[h_1(s, l) + Q_1(s)\sqrt{H(s, l)} \right]^2 \right\} ds > 0 \tag{2.8}$$

and

$$\limsup_{t \rightarrow \infty} \int_l^t \left\{ H(t, s) K\rho(s)q(s) - \frac{\alpha_1\rho(s)r(s)}{4} \left[h_2(t, s) - Q_1(s)\sqrt{H(t, s)} \right]^2 \right\} ds > 0, \tag{2.9}$$

for some $H \in F$, $\rho \in C^1([t_0, \infty), (0, \infty))$ and for each $l \geq t_0$, where $Q_1(s)$ is as in (2.2), then every solution of Eq. (1.1) is oscillatory.

Proof. For any $T \geq t_0$, let $a = T$. In (2.8), we choose $l = a$. Then there exists $c > a$ such that

$$\int_a^c \left\{ H(s, a) K\rho(s)q(s) - \frac{\alpha_1\rho(s)r(s)}{4} \left[h_1(s, a) + Q_1(s)\sqrt{H(s, a)} \right]^2 \right\} ds > 0. \tag{2.10}$$

In (2.9), we choose $l = c$. Then there exists $b > c$ such that

$$\int_c^b \left\{ H(b, s) K\rho(s)q(s) - \frac{\alpha_1\rho(s)r(s)}{4} \left[h_2(b, s) - Q_1(s)\sqrt{H(b, s)} \right]^2 \right\} ds > 0. \tag{2.11}$$

Combining (2.10) and (2.11), we obtain (2.1). The required conclusion thus follows from Theorem 2. The proof is complete. \square

Corollary 1. If the inequality in (A3) is actually an equality, we can eliminate the sign condition on $p(t)$ in the above theorems.

Remark 1. If (A3) hold with equality ($\alpha_2 = 1$), then Theorem 1, 2 and 3 reduce to Theorem 3.3, 3.4 and Corollary 3.2, respectively, obtained by Tiryaki and Zafer [30] with $g(t) = 0$ in the substitution (1.2).

Remark 2. If we use the substitution (1.2) instead of (2.3) and the condition (A3) is replaced with

$$uvk_2(u, v) \geq \alpha_2 k_1^2(u, v), \alpha_2 > 0, \tag{C1}$$

then Theorem 1, 2 and 3 becomes Theorem 3.1, Corollary 3.1 and Theorem 3.2 of Tiryaki and Zafer [30], respectively.

Remark 3. We note that the case $p(t) \leq 0$ can be handled in a similar manner by reversing the inequality sign in (A3).

Remark 4. If $k_1(u, v) = v$, $p(t) \equiv 0$, $\rho(t) = 1$ and $f(x) = x$, we also recover Corollary 2.2, Theorem 2.1 and Corollary 2.4 of Kong [12], from Theorem 1, 2 and 3, respectively.

It is easy to see that when $H(t, s) = H(t - s) \in F$, $h_1(t - s) \equiv h_2(t - s)$. From now on, we denote them by $h(t - s)$. Furthermore, the subclass of F containing such $H(t - s)$ will be denoted by F_0 . Applying Theorem 2 to F_0 we have the following result:

Theorem 4. *Suppose that (A1)-(A3) hold, and $p(t) \geq 0$ and $q(t) \geq 0$ for all $t \in (a, b)$. If for each $T \geq t_0$, there exist $\rho \in C^1([t_0, \infty), (0, \infty))$, $H \in F_0$, and real numbers a, c such that $T \leq a < c$ and*

$$\begin{aligned} & \int_a^c H(s-a) K[\rho(s)q(s) + \rho(2c-s)q(2c-s)] ds \\ & > \frac{\alpha_1}{4} \int_a^c [\rho(s)r(s) + \rho(2c-s)r(2c-s)] h^2(s-a) ds \\ & + \frac{\alpha_1}{2} \int_a^c [\rho(s)r(s)Q_1(s) - \rho(2c-s)r(2c-s)Q_2(s)] h(s-a) \sqrt{H(s-a)} ds \\ & + \frac{\alpha_1}{4} \int_a^c [\rho(s)r(s)Q_1^2(s) + \rho(2c-s)r(2c-s)Q_2^2(s)] H(s-a) ds, \end{aligned} \quad (2.12)$$

where $Q_2(s) = \left(\frac{\rho'(2c-s)}{\rho(2c-s)} - \frac{\alpha_2 p(2c-s)}{r(2c-s)} \right)$, then every solution of Eq. (1.1) is oscillatory.

Proof. Let $c = \frac{a+b}{2}$. Then

$$H(b-c) = H(c-a) = H\left(\frac{b-a}{2}\right),$$

and for any $w \in L[a, b]$, one has

$$\int_c^b w(s) ds = \int_a^c w(2c-s) ds.$$

Therefore,

$$\int_c^b H(b-s)K\rho(s)q(s)ds = \int_a^c H(s-a)K\rho(2c-s)q(2c-s)ds$$

and

$$\int_c^b \rho(s)r(s)h^2(b-s)ds = \int_a^c \rho(2c-s)r(2c-s)h^2(s-a)ds$$

Thus, (2.12) yields (2.1), and Eq. (1.1) is oscillatory by Theorem 2. □

For different choices of $H(t, s)$, we will obtain different sufficient conditions for oscillatory behavior of solutions of Eq. (1.1). Let $H(t, s) = (t - s)^\lambda, t \geq s \geq t_0$, where $\lambda > 1$ is a constant. Then, we obtain the following useful oscillation criterion.

Corollary 2. *Assume that (A1)-(A3) hold, and $p(t) \geq 0$ and $q(t) \geq 0$ for all $t \in [t_0, \infty)$. Then every solution of Eq. (1.1) is oscillatory provided that for each $l \geq t_0$ and for some $\lambda > 1$, there exists a function $\rho \in C^1([t_0, \infty), (0, \infty))$ such that the following two inequalities hold:*

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_l^t (s-l)^\lambda \left\{ K\rho(s)q(s) - \frac{\alpha_1 \rho(s)r(s)}{4} \left[\frac{\lambda}{(s-l)} + Q_1(s) \right]^2 \right\} ds > 0 \tag{2.13}$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_l^t (t-s)^\lambda \left\{ K\rho(s)q(s) - \frac{\alpha_1 \rho(s)r(s)}{4} \left[\frac{\lambda}{(t-s)} - Q_1(s) \right]^2 \right\} ds > 0. \tag{2.14}$$

Define

$$R(t) = \int_l^t \frac{1}{r(s)} ds, \quad t \geq l \geq t_0,$$

and set

$$H(t, s) = [R(t) - R(s)]^\lambda, \quad t \geq t_0$$

where $\lambda > 1$ is a constant.

If we choose $\rho(t) = 1$, by Theorem 3, we have the following oscillatory criterion.

Theorem 5. *Let (A1)-(A3) hold, $p(t) \geq 0$ and $q(t) \geq 0$ for all $t \in [t_0, \infty)$, and $\lim_{t \rightarrow \infty} R(t) = \infty$. Then every solution of Eq. (1.1) is oscillatory provided that for*

each $l \geq t_0$ and for some $\lambda > 1$, the following two inequalities hold:

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ \left(Kq(s) - \frac{\alpha_1 \alpha_2^2 p^2(s)}{4r(s)} \right) [R(s) - R(l)]^\lambda \right. \\ & \left. + \frac{\alpha_1 \alpha_2 \lambda p(s)}{2r(s)} [R(s) - R(l)]^{\lambda-1} \right\} ds > \frac{\alpha_1 \lambda^2}{4(\lambda-1)} \end{aligned} \quad (2.15)$$

and

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ \left(Kq(s) - \frac{\alpha_1 \alpha_2^2 p^2(s)}{4r(s)} \right) [R(t) - R(s)]^\lambda \right. \\ & \left. - \frac{\alpha_1 \alpha_2 \lambda p(s)}{2r(s)} [R(t) - R(s)]^{\lambda-1} \right\} ds > \frac{\alpha_1 \lambda^2}{4(\lambda-1)} \end{aligned} \quad (2.16)$$

Proof. Since $H(t, s) = [R(t) - R(s)]^\lambda$, it is easy to see that

$$h_1(t, s) = \lambda [R(t) - R(s)]^{(\lambda-2)/2} \frac{1}{r(t)},$$

and

$$h_2(t, s) = \lambda [R(t) - R(s)]^{(\lambda-2)/2} \frac{1}{r(s)}.$$

Noting that

$$\begin{aligned} \int_l^t r(s) h_1^2(s, l) ds &= \int_l^t r(s) \lambda^2 [R(s) - R(l)]^{\lambda-2} \frac{1}{r^2(s)} ds \\ &= \frac{\lambda^2}{\lambda-1} [R(t) - R(l)]^{\lambda-1} \end{aligned}$$

and

$$\begin{aligned} \int_l^t r(s) h_2^2(t, s) ds &= \int_l^t r(s) \lambda^2 [R(t) - R(s)]^{\lambda-2} \frac{1}{r^2(s)} ds \\ &= \frac{\lambda^2}{\lambda-1} [R(t) - R(l)]^{\lambda-1}, \end{aligned}$$

In view of $\lim_{t \rightarrow \infty} R(t) = \infty$, it follows that

$$\lim_{t \rightarrow \infty} \frac{\alpha_1}{4R^{\lambda-1}(t)} \int_l^t r(s) h_1^2(s, l) ds = \frac{\alpha_1 \lambda^2}{4(\lambda-1)} \quad (2.17)$$

and

$$\lim_{t \rightarrow \infty} \frac{\alpha_1}{4R^{\lambda-1}(t)} \int_l^t r(s)h_2^2(t,s)ds = \frac{\alpha_1\lambda^2}{4(\lambda-1)}. \tag{2.18}$$

From (2.15) and (2.17) we have that

$$\begin{aligned} & \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ H(s,l)Kq(s) - \frac{\alpha_1 r(s)}{4} \left[h_1(s,l) - \frac{\alpha_2 p(s)}{r(s)} \sqrt{H(s,l)} \right]^2 \right\} ds \\ &= \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ [R(s) - R(l)]^\lambda Kq(s) + \frac{\alpha_1 \alpha_2 \lambda p(s)}{2r(s)} [R(s) - R(l)]^{\lambda-1} \right. \\ & \quad \left. - \frac{\alpha_1 \alpha_2^2 p^2(s)}{4r(s)} [R(s) - R(l)]^\lambda \right\} ds - \lim_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \frac{\alpha_1}{4} r(s)h_1^2(s,l)ds \\ &= \limsup_{t \rightarrow \infty} \frac{1}{R^{\lambda-1}(t)} \int_l^t \left\{ \left(Kq(s) - \frac{\alpha_1 \alpha_2^2 p^2(s)}{4r(s)} \right) [R(s) - R(l)]^\lambda \right. \\ & \quad \left. + \frac{\alpha_1 \alpha_2 \lambda p(s)}{2r(s)} [R(s) - R(l)]^{\lambda-1} \right\} ds - \frac{\alpha_1 \lambda^2}{4(\lambda-1)} > 0, \end{aligned}$$

i.e., (2.8) holds. Similarly, (2.16) implies that (2.9) holds. By Theorem 3, Eq. (1.1) is oscillatory. The proof is complete. \square

Remark 5. If $k_1(u, v) = v$, $p(t) \equiv 0$, $\rho(t) = 1$ and $f(x) = x$, we recover Theorem 2.2 and Theorem 2.3 of Kong [12], from Theorem 4 and 5, respectively.

3. OSCILLATION RESULTS FOR $f(x)$ WITH MONOTONOCITY

We shall assume that the following conditions are satisfied:

- (A4) $f(x)$ is differentiable, $xf(x) \neq 0$ and $f'(x) \geq L$ for some constant $L > 0$ and all $x \in R \setminus \{0\}$;
- (A5) $vf(u)k_2(u, v) \geq \alpha_3 f(u)k_1(u, v)$ for some $\alpha_3 > 0$ and for all $(u, v) \in R^2$.

Theorem 6. *Let (A2), (A4) and (A5) hold. Suppose that there exists an interval $(a, b) \subset [t_0, \infty)$ such that $p(t) \geq 0$ for all $t \in (a, b)$, and that there exist $c \in (a, b)$, $H \in F$ and $\rho \in C^1([t_0, \infty), (0, \infty))$ such that*

$$\frac{1}{H(c,a)} \int_a^c H(s,a)\rho(s)q(s)ds + \frac{1}{H(b,c)} \int_c^b H(b,s)\rho(s)q(s)ds$$

$$\begin{aligned}
&> \frac{\alpha_1}{4LH(c,a)} \int_a^c \rho(s)r(s) \left[h_1(s,a) + Q_3(s)\sqrt{H(s,a)} \right]^2 ds \\
&+ \frac{\alpha_1}{4LH(b,c)} \int_c^b \rho(s)r(s) \left[h_2(b,s) - Q_3(s)\sqrt{H(b,s)} \right]^2 ds, \quad (3.1)
\end{aligned}$$

where

$$Q_3(s) = \left(\frac{\rho'(s)}{\rho(s)} - \frac{\alpha_3 p(s)}{r(s)} \right). \quad (3.2)$$

Then every solution of (1.1) has a zero in (a, b) .

Proof. For the sake of contradiction, assume that there exists a solution $x(t)$ of (1.1) such that $x(t) \neq 0$ for all $t \in (a, b)$. Introduce a new function w by

$$w(t) = \rho(t) \frac{r(t)k_1(x(t), x'(t))}{f(x(t))}, \quad t \in (a, b). \quad (3.3)$$

In view of Eq. (1.1), it follows from (3.3) that

$$\begin{aligned}
w'(t) &= \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t)p(t) \frac{k_2(x(t), x'(t))x'(t)}{f(x(t))} - \rho(t)q(t) \\
&\quad - \rho(t) \frac{r(t)k_1(x(t), x'(t))f'(x(t))x'(t)}{f^2(x(t))}. \quad (3.4)
\end{aligned}$$

Using (A2), (A4) and (A5) in (3.4), we easily obtain

$$w'(t) \leq -\rho(t)q(t) + \left(\frac{\rho'(t)}{\rho(t)} - \frac{\alpha_3 p(t)}{r(t)} \right) w(t) - \frac{L}{\alpha_1 \rho(t)r(t)} w^2(t). \quad (3.5)$$

The rest of the proof is similar to that of Theorem 1. \square

Theorem 7. Suppose that (A2), (A4) and (A5) hold, and $p(t) \geq 0$ for all $t \in (a, b)$. If, for every $T \geq t_0$, there exist $H \in F$, $\rho \in C^1([t_0, \infty), (0, \infty))$, and $a, b, c \in \mathbb{R}$ such that $T \leq a < c < b$ and (3.1) holds, then every solution of Eq. (1.1) is oscillatory.

Theorem 8. Let (A2), (A4) and (A5) hold, and $p(t) \geq 0$ for all $t \in [t_0, \infty)$. If

$$\limsup_{t \rightarrow \infty} \int_l^t \left\{ H(s, l) \rho(s)q(s) - \frac{\alpha_1 \rho(s)r(s)}{4L} \left[h_1(s, l) + Q_3(s)\sqrt{H(s, l)} \right]^2 \right\} ds > 0$$

and

$$\limsup_{t \rightarrow \infty} \int_l^t \left\{ H(t, s) \rho(s)q(s) - \frac{\alpha_1 \rho(s)r(s)}{4L} \left[h_2(t, s) - Q_3(s)\sqrt{H(t, s)} \right]^2 \right\} ds > 0,$$

for some $H \in F$, $\rho \in C^1([t_0, \infty), (0, \infty))$ and for each $l \geq t_0$, then every solution of Eq. (1.1) is oscillatory.

Corollary 3. *If the inequality in (A5) is actually an equality, we can eliminate the sign condition on $p(t)$ in the above theorems.*

Remark 6. If (A5) hold with equality ($\alpha_3 = 1$), then Theorem 6, 7 and 8 reduce to Theorem 2.3, 2.4 and Corollary 2.2, respectively, obtained by Tiryaki and Zafer [30] with $g(t) = 0$ in the substitution (1.3).

Remark 7. If we use the substitution (1.3) instead of (3.3) and the condition (A5) is replaced with

$$vf(u)k_2(u, v) \geq \alpha_3 k_1^2(u, v), \alpha_3 > 0, \tag{C2}$$

then Theorem 6, 7 and 8 becomes Theorem 2.1, Corollary 2.1 and Theorem 2.2 of Tiryaki and Zafer [30], respectively.

Remark 8. We note that the case $p(t) \leq 0$ can be handled in a similar manner by reversing the inequality sign in (A5).

Theorem 9. *Suppose that (A2), (A4) and (A5) hold, and $p(t) \geq 0$ for all $t \in (a, b)$. If for each $T \geq t_0$, there exist $\rho \in C^1([t_0, \infty), (0, \infty))$, $H \in F_0$, and real numbers a, c such that $T \leq a < c$ and*

$$\begin{aligned} & \int_a^c H(s-a) [\rho(s)q(s) + \rho(2c-s)q(2c-s)] ds \\ & > \frac{\alpha_1}{4L} \int_a^c [\rho(s)r(s) + \rho(2c-s)r(2c-s)] h^2(s-a) ds \\ & + \frac{\alpha_1}{2L} \int_a^c [\rho(s)r(s)Q_3(s) - \rho(2c-s)r(2c-s)Q_4(s)] h(s-a) \sqrt{H(s-a)} ds \\ & + \frac{\alpha_1}{4L} \int_a^c [\rho(s)r(s)Q_3^2(s) + \rho(2c-s)r(2c-s)Q_4^2(s)] H(s-a) ds, \end{aligned}$$

where $Q_4(s) = \left(\frac{\rho'(2c-s)}{\rho(2c-s)} - \frac{\alpha_3 p(2c-s)}{r(2c-s)} \right)$, then every solution of Eq. (1.1) is oscillatory.

Corollary 4. *Assume that (A2), (A4) and (A5) hold, and $p(t) \geq 0$ for all $t \in [t_0, \infty)$. Then every solution of Eq. (1.1) is oscillatory provided that for each $l \geq t_0$ and for some $\lambda > 1$, there exists a function $\rho \in C^1([t_0, \infty), (0, \infty))$ such that the following two inequalities hold:*

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_l^t (s-l)^\lambda \left\{ \rho(s)q(s) - \frac{\alpha_1 \rho(s)r(s)}{4L} \left[\frac{\lambda}{(s-l)} + Q_3(s) \right]^2 \right\} ds > 0$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{t^{\lambda-1}} \int_l^t (t-s)^\lambda \left\{ \rho(s)q(s) - \frac{\alpha_1 \rho(s)r(s)}{4L} \left[\frac{\lambda}{(t-s)} - Q_3(s) \right]^2 \right\} ds > 0.$$

Remark 9. If $k_1(u, v) = v$, $p(t) \equiv 0$, $\rho(t) = 1$ and $f(x) = x$, we also recover Corollary 2.2, Theorem 2.1 and Corollary 2.4, Theorem 2.2, Theorem 2.3 of Kong [12], from Theorem 6, 7, 8 and 9, Corollary 3, respectively.

Example 1. Consider the equation

$$\left((1 + \cos^2 t) \frac{x'}{1+x^2} \right)' + \frac{1}{1+x^2} x' + q(t)f(x) = 0 \quad (3.6)$$

where

$$k_1(u, v) = \frac{v}{1+u^2}, \quad k_2(u, v) = \frac{1}{1+u^2}, \quad f(x) = x(1+x^4), \quad (\alpha_1 = \alpha_2 = \alpha_3 = 1),$$

$$q(t) = \begin{cases} 7(t-3n), & 3n \leq t \leq 3n+1 \\ 7(-t+3n+2), & 3n+1 < t \leq 3n+2 \\ g(t), & 3n+2 < t \leq 3n+3, \end{cases}$$

$g(t) = -n(3n+3-t)(t-3n-2)$ and $n \in N_0 = \{0, 1, \dots\}$. For any $T \geq 0$, there exists $n \in N_0$ such that $3n \geq T$. Let $a = 3n$, $c = 3n+1$, $b = 2c - a$, $\rho(t) = 1$ and $H(t-s) = (t-s)^2$. Since $f'(x) = 1 + 5x^4 \geq 1 = L$ and $h(t-s) = 2$, we get

$$\begin{aligned} & \int_a^c H(s-a)[q(s) + q(2c-s)] ds \\ &= \int_{3n}^{3n+1} (s-3n)^2 [7(s-3n) + 7(3n+2-s)] ds \\ &= 14 \int_{3n}^{3n+1} (s-3n)^2 ds = 14/3 \end{aligned}$$

and

$$\begin{aligned} & \frac{\alpha_1}{4L} \int_a^c [r(s) + r(2c-s)] h^2(s-a) ds \\ &+ \frac{\alpha_1}{2L} \int_a^c [-p(s) + p(2c-s)] h(s-a) \sqrt{H(s-a)} ds \end{aligned}$$

$$\begin{aligned}
 & + \frac{\alpha_1}{4L} \int_a^c \left[\frac{p^2(s)}{r(s)} + \frac{p^2(2c-s)}{r(2c-s)} \right] H(s-a) ds \\
 & = \int_{3n}^{3n+1} [2 + \cos^2 s + \cos^2 (6n + 2 - s)] ds \\
 & + \frac{1}{4} \int_{3n}^{3n+1} \left[\frac{1}{1 + \cos^2 s} + \frac{1}{1 + \cos^2 (6n + 2 - s)} \right] (s - 3n)^2 ds \\
 & \leq \int_{3n}^{3n+1} 4 ds + \frac{1}{2} \int_{3n}^{3n+1} (s - 3n)^2 ds = 4 + \frac{1}{6} = 25/6 < 14/3.
 \end{aligned}$$

Hence, (3.6) is oscillatory by Theorem 9. The condition (C2) in Remark 7 is not satisfied. Therefore, the results of [30] can not be used to Eq. (3.6).

If we take $f(x) = x(3 + 2 \cos x)$ in Eq. (3.6), we find that every solution of (3.6) is oscillatory by Theorem 4. Since the condition (C1) in Remark 2 is not satisfied, the results of [30] can not be used to Eq. (3.6) with $f(x) = x(3 + 2 \cos x)$.

Example 2. Consider the equation

$$\left((2 + \sin t) \frac{x'}{1 + x^2} \right)' - \frac{4 + 2 \sin t}{1 + x^2} x' + (2 + \sin t) x(7 + x^4) = 0 \quad (3.7)$$

where

$$\begin{aligned}
 k_1(u, v) &= \frac{v}{1 + u^2}, \quad k_2(u, v) = \frac{1}{1 + u^2}, \quad (\alpha_1 = \alpha_2 = 1), \\
 f(x) &= x(7 + x^4), \quad (K = 7).
 \end{aligned}$$

Let $H(t, s) = (t - s)^2$, $\rho(t) = 1$. Since $p(t) = -4 - 2 \sin t < 0$ and $\alpha_1 \alpha_2 p(t) + r(t) < 0$, we see that Eq. (3.7) is oscillatory by Theorem 3 and Remark 3.

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