Miskolc Mathematical Notes
Vol. 13 (2012), No 1, pp. 53-65

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2012.345

Formulas for weighted binomial sums using the powers of terms of binary recurrences

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FORMULAS FOR WEIGHTED BINOMIAL SUMS USING THE POWERS OF TERMS OF BINARY RECURRENCES

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Received 30 March, 2011

Abstract. In this paper, we give general formulas for some weighted binomial sums, using the powers of terms of certain binary recurrences. As an applications of our results, we show that the weighted binomial sums of the generalized Fibonacci and Lucas numbers can be expressed via the second kinds of Chebyshev polynomials and Stirling numbers.

2000 *Mathematics Subject Classification:* 11B37; 11B39; 05A19

Keywords: binary recurrences, weighted binomial sums, Stirling numbers of the second kind, Chebyshev polynomials of the second kind

1. Introduction

For $n > 1$, define the binary recurrences $\{U_n\}$ and $\{V_n\}$ by

$$U_n = pU_{n-1} - U_{n-2} \quad \text{and} \quad V_n = pV_{n-1} - V_{n-2}$$

where $U_0 = 0$, $U_1 = 1$ and $V_0 = 2$, $V_1 = p$, respectively.

The Fibonacci subsequence $\{F_{2n}\}$ and the Pell subsequence $\{P_{2n}\}$ are the special cases of the sequence $\{U_n\}$ for $p = 3$ and $p = 6$, respectively. It is also known that the natural numbers are special cases of the sequence $\{U_n\}$ for $p = 2$.

The Binet formulas of $\{U_n\}$ and $\{V_n\}$ are as follows:

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n, \tag{1.1}$$

where $\alpha, \beta = \left(p \pm \sqrt{p^2 - 4}\right)/2$.

From [4], we have that for $k \geq 0$ and $n > 1$,

$$U_{kn} = V_k U_{k(n-1)} - U_{k(n-2)},$$
$$V_{kn} = V_k V_{k(n-1)} - V_{k(n-2)}.$$

Wiemann and Cooper [9] mentioned some conjectures of Melham for the sum:

$$\sum_{h=1}^{n} F_{2h}^{2m+1},$$

where F_n stands for the n^{th} Fibonacci number.

Ozeki [6] considered Melham's sum and gave an explicit expansion for it as a polynomial in F_{2n+1}.

In general, Prodinger [7] derived a general formula for the sum:

$$\sum_{h=0}^{n} F_{2h+\delta}^{2m+\varepsilon},$$

where $\varepsilon, \delta \in \{0,1\}$, as well as for the evaluations of the corresponding sums for Lucas numbers.

In [5], we considered alternating Melham's sums for Fibonacci and Lucas numbers of the form $\sum_{h=1}^{n} (-1)^h F_{2h+\delta}^{2m+\varepsilon}$ and $\sum_{h=1}^{n} (-1)^h L_{2h+\delta}^{2m+\varepsilon}$, where $\varepsilon, \delta \in \{0,1\}$.

Recently Khan and Kwong [3] considered the sums

$$\sum_{h=0}^{n} h^m \binom{n}{h} U_h \text{ and } \sum_{h=0}^{n} (-1)^{n+h} h^m \binom{n}{h} U_h, \tag{1.2}$$

and expressed them in terms of two associated sequences. The special cases of $m = 2,3$ leads to interesting binomial and Fibonacci identities.

In this paper, we shall give general formulas for the sums

$$\sum_{h=0}^{n} \binom{n}{h} h^m U_{ht}^{2m+\varepsilon}, \quad \sum_{h=0}^{n} \binom{n}{h} h^m V_{ht}^{2m+\varepsilon},$$
$$\sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h} h^m U_{ht}^{2m+\varepsilon}, \quad \sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h} h^m V_{ht}^{2m+\varepsilon},$$

where t is a positive integer and $\varepsilon \in \{0,1\}$. In order to do this, firstly we will consider general cases of the sums given by (1.2) and we shall derive similar formulas for their Lucas counterparts in the second section. After this, by using these results, we state our main results in the third section.

2. Generalized weighted binomial identities

In this section, we will give generalizations of the results of [3] by considering the sequence $\{U_{nk}\}$ instead of the sequence $\{U_n\}$. We also give similar formulas for the Lucas sequence $\{V_n\}$. While deriving these results, we follow the proof strategy of [3].

Define the sequences $\{X_{kn}\}$, $\{Y_{kn}\}$, $\{W_{kn}\}$ and $\{Z_{kn}\}$ for $n \geq 2$ as follows:

$$X_0 = 0,\ X_k = U_k, \quad X_{kn} = (V_k + 2)\left(X_{k(n-1)} - X_{k(n-2)}\right),$$
$$Y_0 = 0,\ Y_k = U_k, \quad Y_{kn} = (V_k - 2)\left(Y_{k(n-1)} + Y_{k(n-2)}\right),$$
$$W_0 = 2,\ W_k = V_k + 2, \quad W_{kn} = (V_k + 2)\left(W_{k(n-1)} - W_{k(n-2)}\right),$$
$$Z_0 = 2,\ Z_k = V_k - 2, \quad Z_{kn} = (V_k - 2)\left(Z_{k(n-1)} + Z_{k(n-2)}\right).$$

The Binet formulas of $\{X_{kn}\}$, $\{Y_{kn}\}$, $\{W_{kn}\}$ and $\{Z_{kn}\}$ are

$$X_{kn} = \frac{\left(1+\alpha^k\right)^n - \left(1+\beta^k\right)^n}{\alpha-\beta},\ Y_{kn} = \frac{\left(\alpha^k-1\right)^n - \left(\beta^k-1\right)^n}{\alpha-\beta},$$

$$W_{kn} = \left(1+\alpha^k\right)^n + \left(1+\beta^k\right)^n, \text{ and } Z_{kn} = \left(\alpha^k-1\right)^n + \left(\beta^k-1\right)^n,$$

where $\alpha^k,\ \beta^k = \left(V_k \pm \sqrt{V_k^2-4}\right)/2$.

Lemma 1. *For $n \geq 0$, we have*

$$\sum_{h=0}^{n} \binom{n}{h} U_{hk} = X_{nk}, \tag{2.1}$$

$$\sum_{h=0}^{n} \binom{n}{h} hU_{hk} = n\left(X_{kn} - X_{k(n-1)}\right), \tag{2.2}$$

$$\sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h}\, U_{hk} = Y_{kn}, \tag{2.3}$$

$$\sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h}\, hU_{hk} = n\left(Y_{kn} + Y_{k(n-1)}\right). \tag{2.4}$$

Proof. Since

$$\sum_{h=0}^{n} \binom{n}{h} \alpha^{hk} = \left(1+\alpha^k\right)^n \text{ and } \sum_{h=0}^{n} \binom{n}{h} \beta^{hk} = \left(1+\beta^k\right)^n,$$

the first claim follows from the Binet formula of $\{U_{hk}\}$. Similarly by considering

$$\sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h}\, \alpha^{hk} = \left(\alpha^k-1\right)^n,\ \sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h}\, \beta^{hk} = \left(\beta^k-1\right)^n,$$

we have the third claim.

Considering

$$\sum_{h=0}^{n} \binom{n}{h} h\alpha^{hk} = \frac{\alpha}{k} \cdot \frac{d}{d\alpha}\left[\sum_{h=0}^{n} \binom{n}{h} \alpha^{hk}\right] = \frac{\alpha}{k}\frac{d}{d\alpha}\left(\left(1+\alpha^k\right)^n\right)$$

$$= n\alpha^k\left(1+\alpha^k\right)^{n-1} = n\left(\left(1+\alpha^k\right)^n - \left(1+\alpha^k\right)^{n-1}\right), \tag{2.5}$$

and similarly

$$\sum_{h=0}^{n} \binom{n}{h} h\beta^{hk} = n\left(\left(1+\beta^k\right)^n - \left(1+\beta^k\right)^{n-1}\right), \tag{2.6}$$

one can easily obtain the rest of claimed identities. □

Define the operators D_U and Δ_U on X_{kn} and Y_{kn} for $n \geq 1$, respectively, as follows:

$$D_U(X_{kn}) = n\left(X_{kn} - X_{k(n-1)}\right),$$
$$\Delta_U(Y_{kn}) = n\left(Y_{kn} + Y_{k(n-1)}\right).$$

Lemma 2. *For $n \geq 0$*

$$\sum_{h=0}^{n} \binom{n}{h} V_{hk} = W_{nk},$$
$$\sum_{h=0}^{n} \binom{n}{h} h V_{hk} = n\left(W_{kn} - W_{k(n-1)}\right),$$
$$\sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h} V_{hk} = Z_{kn},$$
$$\sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h} h V_{hk} = n\left(Z_{kn} + Z_{k(n-1)}\right).$$

Proof. The proof is similar to the proof of Lemma 1. □

Define the operators D_V and Δ_V on W_{kn} and Z_{kn} for $n \geq 1$, respectively, as follows:

$$D_V(W_{kn}) = n\left(W_{kn} - W_{k(n-1)}\right),$$
$$\Delta_V(Z_{kn}) = n\left(Z_{kn} + Z_{k(n-1)}\right).$$

In [3], the authors stated that if $\sum_{h=0}^{n} h^m \binom{n}{h} U_h$ is of the form $\sum_{k\geq 0} a_k X_k$, then $\sum_{h=0}^{n} h^m \binom{n}{h} U_h = D\left(\sum_{k\geq 0} a_k X_k\right)$. Hence the coefficients a_k can be computed iteratively as follows.

For $m \geq 0$, define the polynomials $a_{m,r}(n)$ recursively as follows [3]:

$$a_{m,r}(n) = (n-r)\, a_{m-1,r}(n) - (n-r+1)\, a_{m-1,r-1}(n), \quad m \geq 1, \tag{2.7}$$

with the initial value $a_{0,0}(n) = 1$ and the convention that $a_{m,r}(n) = 0$ if $r < 0$ or $r > m$.

Thus, we see that a similar statement is also valid for the sum $\sum_{h=0}^{n} \binom{n}{h} h^m U_{hk}$, so we can give the following result.

Theorem 1. *For $n \geq 0$*

$$\sum_{h=0}^{n} \binom{n}{h} h^m U_{hk} = \sum_{r=0}^{m} a_{m,r}(n)\, X_{k(n-r)}, \tag{2.8}$$

$$\sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h} h^m U_{hk} = \sum_{r=0}^{m} (-1)^r a_{m,r}(n)\, Y_{k(n-r)}, \tag{2.9}$$

$$\sum_{h=0}^{n}\binom{n}{h}h^m V_{hk}=\sum_{r=0}^{m}a_{m,r}(n)\,W_{k(n-r)}, \tag{2.10}$$

$$\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^m V_{hk}=\sum_{r=0}^{m}(-1)^r a_{m,r}(n)\,Z_{k(n-r)}. \tag{2.11}$$

Proof. It is known that

$$\sum_{h=0}^{n}\binom{n}{h}h^m U_{hk}=D\left[\sum_{h=0}^{n}\binom{n}{h}h^{m-1}U_{hk}\right].$$

Thus

$$\begin{aligned}\sum_{r=0}^{m}a_{m,r}(n)\,X_{k(n-r)} &= D\left[\sum_{r=0}^{m-1}a_{m-1,r}(n)\,X_{k(n-r)}\right]\\ &=\sum_{r=0}^{m-1}a_{m-1,r}(n)(n-r)\left(X_{k(n-r)}-X_{k(n-r-1)}\right)\\ &=a_{m-1,0}(n)\,X_{kn}\\ &\quad+\sum_{r=1}^{m-1}(n-r)\,a_{m-1,r}(n)-(n-r+1)\,a_{m-1,r}(n)\,X_{k(n-r)}\\ &\quad-(n-m+1)\,a_{m-1,m-1}(n)\,X_{k(n-m)}.\end{aligned}$$

Since $a_{m,r}(n)=0$ if $r<0$ or $r>m$, we write

$$\sum_{r=0}^{m}a_{m,r}(n)\,X_{k(n-r)}=\sum_{r=0}^{m}(n-r)\,a_{m-1,r}(n)-(n-r+1)\,a_{m-1,r}(n)\,X_{k(n-r)}.$$

The recurrence for $a_{m,r}(n)$ follows directly by comparing coefficients. The rest of the claimed identities could be proved similarl. □

For example, when $p=2$ in (2.8), we have that $\alpha=\beta=1$, so $X_n=n2^{n-1}$, which was also given in [3]. In order to get different examples of our results, suppose that $k=m=2$, then by using the results above, we obtain

$$\sum_{h=0}^{n}\binom{n}{h}h^2U_{2h}=\sum_{r=0}^{2}a_{2,r}(n)\,X_{2(n-r)}=2^{n-2}n^2(n+3).$$

Now let $p=3$, $k=2$ and $m=1$. Then $U_{2h}=F_{4h}$, so we have

$$Y_{2n}=\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}F_{4h}=\begin{cases}5^{(n-1)/2}L_{2n}, & \text{if } n \text{ is odd,}\\ 5^{n/2}F_{2n}, & \text{if } n \text{ is even.}\end{cases} \tag{2.12}$$

By (2.12), we also get

$$\begin{aligned}\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}hF_{4h} &= \sum_{r=0}^{1}(-1)^{r}a_{1,r}(n)Y_{2(n-r)} \\ &= a_{1,0}(n)Y_{2n}-a_{1,1}(n)Y_{2(n-1)} \\ &= \begin{cases} n5^{(n-1)/2}(L_{2n}+F_{2n-2}), & \text{if } n \text{ is odd,} \\ n5^{(n-2)/2}(5F_{2n}+L_{2n-2}), & \text{if } n \text{ is even.}\end{cases}\end{aligned}$$

Similar to the above examples, one can obtain various results for different values of k and p from the results above.

3. The main results

In this section, we give general formulas for the sums:

$$\sum_{h=0}^{n}\binom{n}{h}h^{m}U_{ht}^{2m+\varepsilon}, \quad \sum_{h=0}^{n}\binom{n}{h}h^{m}V_{ht}^{2m+\varepsilon},$$

$$\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^{m}U_{ht}^{2m+\varepsilon} \text{ and } \sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^{m}V_{ht}^{2m+\varepsilon},$$

where t is an positive integer and $\varepsilon\in\{0,1\}$.

We shall assume that p in the definition of the sequence $\{U_n\}$ is a positive integer different from 2 throughout in this section.

For the readers convenience and for later use, it would be convenient to recall some facts from [8]: For any real numbers m and n,

$$(m+n)^{t} = \begin{cases} \sum\limits_{i=0}^{(t-1)/2}\binom{t}{i}(mn)^{i}\left(m^{t-2i}+n^{t-2i}\right) & \text{if } t \text{ is odd,} \\ \sum\limits_{i=0}^{t/2-1}\binom{t}{i}(mn)^{i}\left(m^{t-2i}+n^{t-2i}\right)+\binom{t}{t/2}(mn)^{t/2} & \text{if } t \text{ is even,}\end{cases} \tag{3.1}$$

and

$$(m-n)^{t} = \begin{cases} \sum\limits_{i=0}^{(t-1)/2}\binom{t}{i}(mn)^{i}(-1)^{i}\left(m^{t-2i}-n^{t-2i}\right) & \text{if } t \text{ is odd,} \\ \sum\limits_{i=0}^{t/2-1}\binom{t}{i}(mn)^{i}(-1)^{i}\left(m^{t-2i}+n^{t-2i}\right) & \\ +\binom{t}{t/2}(mn)^{t/2}(-1)^{t/2} & \text{if } t \text{ is even.}\end{cases} \tag{3.2}$$

Theorem 2. *i) For $t,m>0$,*

$$\sum_{h=0}^{n}\binom{n}{h}h^{m}U_{ht}^{2m} = \frac{U_{t}^{2m}}{(V_{t}^{2}-4)^{m}}\sum_{i=0}^{m-1}(-1)^{i}\binom{2m}{i}\sum_{r=0}^{m}a_{m,r}(n)W_{t(2m-2i)(n-r)}$$

$$+\frac{U_t^{2m}}{\left(V_t^2-4\right)^m}\binom{2m}{m}(-1)^m 2^{n-m}P(n),$$

where $P(n)$ *is a monic polynomial of degree* m *satisfying* $\sum_{h=0}^{n}\binom{n}{h}h^m = 2^{n-m}P(n)$.

ii) For $t,m>0$,

$$\sum_{h=0}^{n}\binom{n}{h}h^m U_{ht}^{2m+1} = \frac{U_t^{2m}}{\left(V_t^2-4\right)^m}\sum_{i=0}^{m}(-1)^i\binom{2m}{i}\sum_{r=0}^{m}a_{m,r}(n)\,X_{t(2m+1-2i)(n-r)}.$$

Proof. i) For $t>0$, by the Binet formula of $\{U_n\}$, we have

$$\sum_{h=0}^{n}\binom{n}{h}h^m U_{ht}^{2m} = \sum_{h=0}^{n}\binom{n}{h}h^m\left(\frac{\alpha^{ht}-\beta^{ht}}{\alpha-\beta}\right)^{2m}.$$

Using (3.2), we write

$$\begin{aligned}
&\sum_{h=0}^{n}\binom{n}{h}h^m U_{ht}^{2m}\\
&=\frac{1}{(\alpha-\beta)^{2m}}\sum_{h=0}^{n}\binom{n}{h}h^m\left[\sum_{i=0}^{m-1}\binom{2m}{i}(-1)^i\left(\alpha^{ht(2m-2i)}+\beta^{ht(2m-2i)}\right)\right.\\
&\qquad\left.+\binom{2m}{m}(-1)^m\right]\\
&=\frac{1}{\left(p^2-4\right)^m}\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}\sum_{h=0}^{n}\binom{n}{h}h^m V_{ht(2m-2i)}\\
&\qquad+\binom{2m}{m}(-1)^m\sum_{h=0}^{n}\binom{n}{h}h^m.
\end{aligned}$$

By taking $k=2t(m-i)$ in (2.10) and $\sum_{h=0}^{n}\binom{n}{h}h^m = 2^{n-m}P(n)$, where $P(n)$ is a monic polynomial of degree m (for the coefficients of this polynomials, see the sequence *A102573* in the OEIS or see [1], p. 135), we write

$$\begin{aligned}
&\sum_{h=0}^{n}\binom{n}{h}h^m U_{ht}^{2m}\\
&=\frac{U_t^{2m}}{\left(V_t^2-4\right)^m}\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}\sum_{r=0}^{m}a_{m,r}(n)\,W_{t(2m-2i)(n-r)}
\end{aligned}$$

$$+\frac{U_t^{2m}}{(V_t^2-4)^m}\binom{2m}{m}(-1)^m 2^{n-m} P(n).$$

ii) The proof is similar to the proof of i). □

For example, when $p=6$, $m=3$ and $t=2$, we derive

$$\sum_{h=0}^{n}\binom{n}{h}h^3U_{2h}^7 = \frac{1}{32}\sum_{i=0}^{3}(-1)^i\binom{6}{i}\sum_{r=0}^{3}a_{3,r}(n)X_{(7-2i)(n-r)}$$
$$=\frac{1}{32}\left(\sum_{r=0}^{3}a_{3,r}(n)X_{7(n-r)}-6\sum_{r=0}^{3}a_{3,r}(n)X_{5(n-r)}\right.$$
$$\left.+15\sum_{r=0}^{3}a_{3,r}(n)X_{3(n-r)}-20\sum_{r=0}^{3}a_{3,r}(n)X_{(n-r)}\right).$$

For $p=6$ in the definition of sequence $\{U_n\}$, we have $U_n=\frac{1}{2}P_{2n}$, where P_n is the n^{th} Pell number. Thus

$$X_n=\frac{1}{2}\sum_{h=0}^{n}\binom{n}{h}P_{2n}=U_n(\sqrt{2}), \tag{3.3}$$

where $U_n(x)$ is the Chebyshev's polynomials of the second kind. We have also another formula for X_n as shown:

$$X_n=\frac{1}{2}\sum_{h=0}^{n}\binom{n}{h}P_{2n}=W_{n+1}(0,1;8,-8),$$

where $W_n(a,b;p,q)$ is the Horadam sequence (see [2]).

Using (3.3), we get

$$\sum_{h=0}^{n}\binom{n}{h}h^3U_{2h}^7$$
$$=\frac{1}{32}\Big[n^3U_{7n}(\sqrt{2})-n\left(2n^2+2n-1\right)U_{7(n-1)}(\sqrt{2})$$
$$+3n(n-1)^2U_{7(n-2)}(\sqrt{2})-n(n-1)(n-2)U_{7(n-3)}(\sqrt{2})$$
$$-6\Big(n^3U_{5n}(\sqrt{2})-n\left(2n^2+2n-1\right)U_{5(n-1)}(\sqrt{2})$$
$$+3n(n-1)^2U_{5(n-2)}(\sqrt{2})-n(n-1)(n-2)U_{5(n-3)}(\sqrt{2})\Big)$$
$$+15\Big(n^3U_{3n}(\sqrt{2})-n\left(2n^2+2n-1\right)U_{3(n-1)}(\sqrt{2})$$
$$+3n(n-1)^2U_{3(n-2)}(\sqrt{2})-n(n-1)(n-2)U_{3(n-3)}(\sqrt{2})\Big)$$

$$-20\left(n^3U_n(\sqrt{2})-n\left(2n^2+2n-1\right)U_{(n-1)}(\sqrt{2})\right.$$
$$\left.\left.+3n\,(n-1)^2\,U_{(n-2)}(\sqrt{2})-n\,(n-1)\,(n-2)\,U_{(n-3)}(\sqrt{2})\right)\right].$$

Theorem 3. *i) For $t,m>0$,*

$$\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^mU_{ht}^{2m}$$
$$=\frac{U_t^{2m}}{\left(V_t^2-4\right)^m}\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}\sum_{r=0}^{m}(-1)^r a_{m,r}(n)\,Z_{2t(m-i)(n-r)}$$
$$+\frac{U_t^{2m}}{\left(V_t^2-4\right)^m}\binom{2m}{m}(-1)^m\,n!S(m,n),$$

where $S(m,n)$ is the Stirling numbers of the second kind.

ii) For $t,m>0$,

$$\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^mU_{ht}^{2m+1}$$
$$=\frac{U_t^{2m}}{\left(V_t^2-4\right)^m}\sum_{i=0}^{m}(-1)^i\binom{2m+1}{i}\sum_{r=0}^{m}(-1)^r a_{m,r}(n)\,Y_{t(2m+1-2i)(n-r)}.$$

Proof. i) For $t>0$, consider

$$\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^mU_{ht}^{2m}=\sum_{h=0}^{n}\binom{n}{h}(-1)^h h^m\left(\frac{\alpha^{ht}-\beta^{ht}}{\alpha-\beta}\right)^{2m}.$$

By (3.2), we write

$$\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^mU_{ht}^{2m}$$
$$=\frac{1}{(\alpha-\beta)^{2m}}\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^m$$
$$\left[\sum_{i=0}^{m-1}\binom{2m}{i}(-1)^i\left(\alpha^{ht(2m-2i)}+\beta^{ht(2m-2i)}\right).+\binom{2m}{m}(-1)^m\right]$$
$$=\frac{1}{(\alpha-\beta)^{2m}}\sum_{i=0}^{m-1}(-1)^i\binom{2m}{i}\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^mV_{ht(2m-2i)}$$

$$+\frac{1}{(\alpha-\beta)^{2m}}\binom{2m}{m}(-1)^m\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^m.$$

By taking $k=2t(m-i)$ in (2.11) and since

$$\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^m=n!S(m,n)$$

where $S(m,n)$ is the Stirling numbers of the second kind, the claim is obtained.

ii) The proof is similar to the proof of i). □

For example, when $p=3$, we get $U_n=F_{2n}$. For $m=t=2$, we have

$$\begin{aligned}&\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^2U_{2h}^5\\&=\frac{1}{5^2}\sum_{i=0}^{2}(-1)^i\binom{5}{i}\sum_{r=0}^{2}(-1)^r a_{2,r}(n)Y_{2(5-2i)(n-r)}\\&=\frac{1}{5^2}\left(\sum_{r=0}^{2}(-1)^r a_{2,r}(n)Y_{10(n-r)}-5\sum_{r=0}^{2}(-1)^r a_{2,r}(n)Y_{6(n-r)}\right.\\&\quad\left.+10\sum_{r=0}^{2}(-1)^r a_{2,r}(n)Y_{2(n-r)}\right).\end{aligned}$$

Since

$$Y_n=\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}F_{2h}=F_n,$$

we get

$$\begin{aligned}\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^2U_{2h}^5=&\frac{1}{5^2}\left[nF_{10n}+n(2n-1)F_{10(n-1)}+n(n-1)F_{10(n-2)}\right.\\&-5\left(nF_{6n}+n(2n-1)F_{6(n-1)}+n(n-1)F_{6(n-2)}\right)\\&\left.+10\left(nF_{2n}+n(2n-1)F_{2(n-1)}+n(n-1)F_{2(n-2)}\right)\right].\end{aligned}$$

Theorem 4. *i) For $t,m>0$,*

$$\sum_{h=0}^{n}\binom{n}{h}h^mV_{ht}^{2m}=\sum_{i=0}^{m-1}\binom{2m}{i}\sum_{r=0}^{m}a_{m,r}(n)W_{t(2m-2i)(n-r)}+\binom{2m}{m}2^{n-m}P(n),$$

where the polynomial $P(n)$ is defined as before.

$ii)$ For $t,\, m > 0$,

$$\sum_{h=0}^{n}\binom{n}{h}h^{m}V_{ht}^{2m+1}=\sum_{i=0}^{m}\binom{2m+1}{i}\sum_{r=0}^{m}a_{m,r}\left(n\right)W_{t(2m+1-2i)(n-r)}.$$

Proof. $i)$ For $t > 0$, by the Binet formula of $\{V_n\}$, we write

$$\sum_{h=0}^{n}\binom{n}{k}h^{m}V_{ht}^{2m}=\sum_{h=0}^{n}\binom{n}{h}h^{m}\left(\alpha^{ht}+\beta^{ht}\right)^{2m},$$

which, by (3.1) and since $\alpha\beta = 1$, satisfies

$$\begin{aligned}\sum_{h=0}^{n}\binom{n}{h}h^{m}V_{ht}^{2m}&=\sum_{h=0}^{n}\binom{n}{h}h^{m}\left[\sum_{i=0}^{m-1}\binom{2m}{i}\left(\alpha^{ht(2m-2i)}+\beta^{ht(2m-2i)}\right)\right.\\&\qquad\left.+\binom{2m}{m}(\alpha\beta)^{thm}\right]\\&=\sum_{i=0}^{m-1}\binom{2m}{i}\sum_{h=0}^{n}\binom{n}{h}h^{m}V_{ht(2m-2i)}+\binom{2m}{m}\sum_{h=0}^{n}\binom{n}{h}h^{m}.\end{aligned}$$

By taking $k = 2t\,(m-i)$ in (2.10) and since $\sum\limits_{h=0}^{n}\binom{n}{h}h^{m} = 2^{n-m}P\left(n\right)$, where $P\left(n\right)$ is defined as before, we get

$$\sum_{h=0}^{n}\binom{n}{h}h^{m}V_{ht}^{2m}=\sum_{i=0}^{m-1}\binom{2m}{i}\sum_{r=0}^{m}a_{m,r}\left(n\right)W_{t(2m-2i)(n-r)}+\binom{2m}{m}2^{n-m}P\left(n\right).$$

$ii)$ The proof is similar to the proof of $i)$. □

Theorem 5. $i)$ *For* $t,m > 0$,

$$\begin{aligned}&\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^{m}V_{ht}^{2m}\\&=\sum_{i=0}^{m-1}\binom{2m}{i}\sum_{r=0}^{m}(-1)^{r}a_{m,r}\left(n\right)Z_{t(2m-2i)(n-r)}+\binom{2m}{m}n!S\left(m,n\right),\end{aligned}$$

where $S\left(m,n\right)$ *is the Stirling numbers of the second kind.*

$ii)$ For $t,\, m > 0$,

$$\sum_{h=0}^{n}\binom{n}{h}(-1)^{n+h}h^{m}V_{ht}^{2m+1}$$

$$= \sum_{i=0}^{m} \binom{2m+1}{i} \sum_{r=0}^{m} (-1)^r a_{m,r}(n) Z_{t(2m+1-2i)(n-r)}.$$

Proof. i) From the Binet formula of $\{V_n\}$, we write

$$\sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h} h^m V_{ht}^{2m} = \sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h} h^m \left(\alpha^{ht} + \beta^{ht}\right)^{2m},$$

which, by (3.1) and since $\alpha\beta = 1$, is equivalent to

$$\begin{aligned} &\sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h} h^m V_{ht}^{2m} \\ &= \sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h} h^m \left(\sum_{i=0}^{m-1} \binom{2m}{i} \left(\alpha^{ht(2m-2i)} + \beta^{ht(2m-2i)}\right) \right. \\ &\quad \left. + \binom{2m}{m} (\alpha\beta)^{thm} \right) \\ &= \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h} h^m V_{ht(2m-2i)} \\ &\quad + \binom{2m}{m} \sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h} h^m. \end{aligned}$$

By taking $k = 2t(m-i)$ in (2.11) and since

$$\sum_{h=0}^{n} \binom{n}{h} (-1)^{n+h} h^m = n! S(m,n),$$

where $S(m,n)$ is defined as before, the claim is obtained.

ii) The proof is similar to the proof of i). □

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