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Esra Erkuş-Duman

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STATISTICAL RATES IN APPROXIMATION BY POSITIVE LINEAR OPERATORS

ESRA ERKUŞ-DUMAN

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Abstract. This study is the continuation of our former work [O. Duman and E. Erkuş, Comput. Math. Appl. 52 (2006) 967-974] in which we obtained a statistical Korovkin-type approximation theorem for a sequence of positive linear operators defined on the space of all real-valued continuous and 2π periodic functions on the real m-dimensional space. In this paper, we compute the statistical rates of this statistical approximation.

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1. INTRODUCTION

The motivation of this work are [3,4]. Let m be a positive integer, and let $C^*(\mathbb{R}^m)$ denote the space of all real-valued continuous and 2π periodic functions on the real m-dimensional space $\mathbb{R}^m$. Here, the 2π-periodicitiy of a function $f \in C^*(\mathbb{R}^m)$ is given by

$$\begin{aligned} f(u_1,u_2,...,u_m) &= f(u_1+2k\pi,u_2,....,u_m) \\ &= f(u_1,u_2+2k\pi,...,u_m) \\ &\quad ... \\ &= f(u_1,u_2,...,u_m+2k\pi) \end{aligned}$$

for every $\mathbf{u}=(u_1,u_2,...,u_m)\in\mathbb{R}^m$ and $k=0,\pm1,...$(see, for instance, [10, p. 126]). Consider the usual supremum norm on $C^*(\mathbb{R}^m)$ defined by

$$\|f\|_* := \sup_{(u_1,u_2,...,u_m)\in\mathbb{R}^m} |f(u_1,u_2,...,u_m)|, \quad f\in C^*(\mathbb{R}^m).$$

Assume that

$$A := [a_{jn}] \quad (j,n\in\mathbb{N} := \{1,2,...\})$$

is a non-negative summability matrix. Recently, in [4], it has been proved that, for any sequence $\{L_n\}$ of positive linear operators mapping $C^*(\mathbb{R}^m)$ into itself,

$$st_A - \lim_n \|L_n(f) - f\|_* = 0 \quad \text{for all} \quad f \in C^*(\mathbb{R}^m) \tag{1.1}$$

if and only if

$$st_A - \lim_n \|L_n(f_i) - f_i\|_* = 0 \quad \text{for each } i = 0, 1, ..., 2m+1, \tag{1.2}$$

where

$$\begin{aligned} f_0(u_1, ..., u_m) &= 1, \\ f_i(u_1, ..., u_m) &= \cos u_i \;\; \text{for} \;\; i = 1, 2, ..., m, \\ f_j(u_1, ..., u_m) &= \sin u_j \;\; \text{for} \;\; j = m+1, m+2, ..., 2m. \end{aligned}$$

Notice that by $st_A - \lim_n x_n = L$ we mean that the sequence $x := \{x_n\}$ is A-statistically convergent to L (see [8]), i.e., for every $\varepsilon > 0$,

$$\lim_j \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0.$$

It is well-known that every convergent sequence is A-statistically convergent to the same value, however the converse is not always true. Also, taking special regular matrices, one can obtain many convergence methods from the A-statistical convergence. For example, if $A = C_1$, the Cesáro matrix, then it reduces to the concept of statistical convergence (see [7, 9]), and if $A = I$, the identity matrix, then it coincides with the ordinary convergence. Hence, with such properties, the usage of A-statistical convergence in approximation theory provides us more powerful results than the classical theory does. Observe that the above statistical approximation theorem in the space $C^*(\mathbb{R}^m)$ contains the classical uniform convergence. However, it is also possible to construct a sequence of positive linear operators satisfying (1.1) or (1.2) but not the corresponding classical case (see [4]).

In this paper, we mainly discuss the following problem: how can we compute the statistical rates of the A-statistical convergence to zero of the difference sequence $\{\|L_n(f) - f\|_*\}$ in (1.1)? Such a problem has recently been investigated for the one dimensional case by Duman [3]. In order to compute the statistical rates we use the following two definitions (see, for instance, [5, 6]):

Let $\{p_n\}$ be a positive non-increasing sequence of real numbers. For a given sequence $\{x_n\}$, we say that

- $\{x_n\}$ is A-statistically convergent to the number L with rate $o(p_n)$, denoted by $x_n - L = st_A - o(p_n)$ as $n \to \infty$, if, for every $\varepsilon > 0$,

$$\lim_j \frac{1}{p_j} \sum_{n:|x_n - L| \geq \varepsilon} a_{jn} = 0,$$

- $\{x_n\}$ is A-statistically convergent to the number L with rate $o_m(p_n)$, denoted by $x_n - L = st_A - o_m(p_n)$ as $n \to \infty$, if, for every $\varepsilon > 0$,

$$\lim_j \sum_{n:|x_n - L| \geq \varepsilon p_n} a_{jn} = 0.$$

Notice that the *rate of convergence* given by the little o is influenced more strongly by the summability method then by the terms of the sequence $\{x_n\}$. For instance, when one takes the identity matrix I, if $\{p_n\}$ is any positive non-increasing sequence satisfying the following inequality:

$$\frac{1}{p_n} \leq M \quad (\text{for } M > 0 \text{ and } n \in \mathbb{N}),$$

then we have $x_n - L = st_A - o(p_n)$ as $n \to \infty$ for any convergent sequence $\{x_n - L\}$ regardless of how slowly it goes to zero. To avoid such an unfortunate situation, one may borrow the concept of *convergence in measure* from the measure theory to define the rate of convergence as in the notion of the little o_m.

2. Statistical rates of the approximation

In this section, we compute the statistical rates of the statistical approximation in (1.1). To see this we first recall the concept of modulus of continuity in the space $C^*(\mathbb{R}^m)$. The modulus of continuity of a function f belonging to $C^*(\mathbb{R}^m)$ is given by, for any $\delta > 0$,

$$w(f,\delta) = \sup_{\sqrt{(u_1-x_1)^2+...+(u_m-x_m)^2} \leq \delta} |f(u_1,...,u_m) - f(x_1,...,x_m)|. \tag{2.1}$$

It is well-known that a necessary and sufficient condition for a function f to belong to $C^*(\mathbb{R}^m)$ is $\lim_{\delta \to 0} w(f,\delta) = 0$ (see [1, p. 80]). Furthermore, it follows from (2.1) that, for every $f \in C^*(\mathbb{R}^m)$ and $(u_1,...,u_m), (x_1,...,x_m) \in \mathbb{R}^m$,

$$|f(u_1,...,u_m) - f(x_1,...,x_m)| \leq w\left(f, \sqrt{(u_1-x_1)^2 + ... + (u_m-x_m)^2}\right), \tag{2.2}$$

and

$$w(f,c\delta) \leq (1+c)w(f,\delta) \quad \text{for any} \quad c,\delta > 0. \tag{2.3}$$

Now we are ready to state our main results.

Theorem 1. *Let $A = [a_{jn}]$ be a non-negative regular summability matrix and let $\{L_n\}$ be a sequence of positive linear operators mapping $C^*(\mathbb{R}^m)$ into itself. For each $(x_1,...,x_m) \in \mathbb{R}^m$, define the function $\varphi_{x_1,...,x_m}$ by*

$$\varphi_{x_1,...,x_m}(u_1,...,u_m) = \sin^2\left(\frac{u_1-x_1}{2}\right) + ... + \sin^2\left(\frac{u_m-x_m}{2}\right). \tag{2.4}$$

Assume that $\{p_n\}$ and $\{q_n\}$ are positive non-increasing sequences of real numbers. If, for every $f \in C^*(\mathbb{R}^m)$, the conditions

(i) $\|L_n(f_0) - f_0\|_* = st_A - o(p_n)$ *as* $n \to \infty$ *with* $f_0(u_1, ..., u_m) = 1$,
(ii) $w(f, \delta) = st_A - o(q_n)$ *as* $n \to \infty$ *with* $\delta_n := \sqrt{\|L_n(\varphi_{x_1,...,x_m})\|_*}$

hold, then we have

$$\|L_n(f) - f\|_* = st_A - o(r_n) \ \ as \ \ n \to \infty$$

with $r_n := \max\{p_n, q_n\}$ *for each* $n \in \mathbb{N}$.

Proof. Let $f \in C^*(\mathbb{R}^m)$ and $(x_1, ..., x_m) \in \mathbb{K} := [-\pi, \pi] \times ... \times [-\pi, \pi]$ be fixed. Since

$$\begin{aligned} & |L_n(f; x_1, ..., x_m) - f(x_1, ..., x_m)| \\ \leq \ & L_n(|f(u_1, ..., u_m) - f(x_1, ..., x_m)|; x_1, ..., x_m) \\ & + |f(x_1, ..., x_m)| \, |L_n(f_0; x_1, ..., x_m) - f_0(x_1, ..., x_m)|, \end{aligned}$$

it follows from (2.2) that

$$\begin{aligned} & |L_n(f; x_1, ..., x_m) - f(x_1, ..., x_m)| \\ & \leq L_n\left(w\left(f, \sqrt{(u_1 - x_1)^2 + ... + (u_m - x_m)^2}\right); x_1, ..., x_m\right) \\ & + M\,|L_n(f_0; x_1, ..., x_m) - f_0(x_1, ..., x_m)|, \end{aligned} \tag{2.5}$$

where $M := \|f\|_*$. Since, for all $t \in [-\pi, \pi]$,

$$|t| \leq \pi \left|\sin \frac{t}{2}\right|$$

we have

$$\sqrt{(u_1 - x_1)^2 + ... + (u_m - x_m)^2} \leq \pi \sqrt{\varphi_{x_1,...,x_m}(u_1, ..., u_m)},$$

where $\varphi_{x_1,...,x_m}$ is given by (2.4). Combining this with (2.5), we obtain that

$$\begin{aligned} |L_n(f; x_1, ..., x_m) - f(x_1, ..., x_m)| & \leq L_n\left(w\left(f, \pi\sqrt{\varphi_{x_1,...,x_m}}\right); x_1, ..., x_m\right) \\ & + M\,|L_n(f_0; x_1, ..., x_m) - f_0(x_1, ..., x_m)|. \end{aligned}$$

Also, using (2.3), we may write that, for any $\delta > 0$,

$$\begin{aligned} |L_n(f; x_1, ..., x_m) - f(x_1, ..., x_m)| & \leq w(f, \delta) L_n\left(1 + \frac{\pi}{\delta}\sqrt{\varphi_{x_1,...,x_m}}; x_1, ..., x_m\right) \\ & + M\,|L_n(f_0; x_1, ..., x_m) - f_0(x_1, ..., x_m)| \\ & \leq w(f, \delta) L_n(f_0; x_1, ..., x_m) \\ & + \frac{\pi\, w(f, \delta)}{\delta} L_n\left(\sqrt{\varphi_{x_1,...,x_m}}; x_1, ..., x_m\right) \\ & + M\,|L_n(f_0; x_1, ..., x_m) - f_0(x_1, ..., x_m)|. \end{aligned}$$

From the Cauchy-Schwarz inequality for positive linear operators (see [2]), we have

$$|L_n(f; x_1, ..., x_m) - f(x_1, ..., x_m)|$$

$$\leq w(f,\delta)+w(f,\delta)\left|L_n(f_0;x_1,\ldots,x_m)-f_0(x_1,\ldots,x_m)\right|$$
$$+\frac{\pi w(f,\delta)}{\delta}\sqrt{L_n(f_0;x_1,\ldots,x_m)}\sqrt{L_n\left(\varphi_{x_1,\ldots,x_m};x_1,\ldots,x_m\right)}$$
$$+M\left|L_n(f_0;x_1,\ldots,x_m)-f_0(x_1,\ldots,x_m)\right|.$$

Now taking supremum over $(x_1,\ldots,x_m)\in\mathbb{K}$ and choosing $\delta:=\delta_n=\sqrt{\left\|L_n\left(\varphi_{x_1,\ldots,x_m}\right)\right\|_*}$, we have

$$\begin{aligned}\|L_n(f)-f\|_* \quad\leq\quad & (1+\pi)\,w(f,\delta_n)+w(f,\delta_n)\,\|L_n(f_0)-f_0\|_* \\ & +\pi\,w(f,\delta_n)\sqrt{\|L_n(f_0)-f_0\|_*} \\ & +M\,\|L_n(f_0)-f_0\|_*.\end{aligned}$$

Letting $C:=\max\{1+\pi,\ M\}$, the last inequality gives that

$$\begin{aligned}\|L_n(f)-f\|_* \quad & \leq C\left\{w(f,\delta_n)+w(f,\delta_n)\,\|L_n(f_0)-f_0\|_*\right. \\ & \left.+w(f,\delta_n)\sqrt{\|L_n(f_0)-f_0\|_*}+\|L_n(f_0)-f_0\|_*\right\}.\end{aligned}\tag{2.6}$$

Now, for a given $\varepsilon>0$, define the following sets:

$$\begin{aligned}D\quad &:\quad =\{n\in\mathbb{N}:\|L_n(f)-f\|_*\geq\varepsilon\},\\ D_1\quad &:\quad =\left\{n\in\mathbb{N}:\|L_n(f_0)-f_0\|_*\geq\frac{\varepsilon}{4C}\right\},\\ D_2\quad &:\quad =\left\{n\in\mathbb{N}:w(f,\delta_n)\geq\frac{\varepsilon}{4C}\right\},\\ D_3\quad &:\quad =\left\{n\in\mathbb{N}:w(f,\delta_n)\,\|L_n(f_0)-f_0\|_*\geq\frac{\varepsilon}{4C}\right\},\\ D_4\quad &:\quad =\left\{n\in\mathbb{N}:w(f,\delta_n)\sqrt{\|L_n(f_0)-f_0\|_*}\geq\frac{\varepsilon}{4C}\right\}.\end{aligned}$$

By (2.6) it is easy to that $D\subseteq\bigcup_{i=0}^{4}D_i$. Furthermore, consider the sets

$$D_3':=\left\{n\in\mathbb{N}:w(f,\delta_n)\geq\sqrt{\frac{\varepsilon}{4C}}\right\},$$

$$D_3'':=\left\{n\in\mathbb{N}:\|L_n(f_0)-f_0\|_*\geq\sqrt{\frac{\varepsilon}{4C}}\right\}.$$

Then observe that $D_3\subseteq D_3'\cup D_3''$ and $D_4\subseteq D_1\cup D_3''$, so we have

$$D\subseteq D_1\cup D_2\cup D_3'\cup D_3''.$$

This inclusion implies, for all $j\in\mathbb{N}$, that

$$\frac{1}{r_j}\sum_{n\in D}a_{jn}\leq\frac{1}{p_j}\sum_{n\in D_1}a_{jn}+\frac{1}{q_j}\sum_{n\in D_2}a_{jn}+\frac{1}{q_j}\sum_{n\in D_3'}a_{jn}+\frac{1}{p_j}\sum_{n\in D_3''}a_{jn}.\tag{2.7}$$

Now taking limit as $j \to \infty$ in (2.7) and using the hypotheses (i) and (ii), we conclude that

$$\lim_j \frac{1}{r_j} \sum_{n \in D} a_{jn} = 0,$$

which means

$$\|L_n(f) - f\|_* = st_A - o(r_n) \text{ as } n \to \infty,$$

so the theorem is proved. □

In a similar manner, we get the next result, which involves the statistical rate with the little o_m.

Theorem 2. *Let $A = [a_{jn}]$, $\{L_n\}$, $\{\delta_n\}$, $\{p_n\}$ and $\{q_n\}$ be the same as in Theorem 1. Assume that the conditions (i) and (ii) of Theorem 1 hold for the little o_m instead of the little o. Then, for any $f \in C^*(\mathbb{R}^m)$, we have*

$$\|L_n(f) - f\|_* = st_A - o_m(s_n) \text{ as } n \to \infty$$

with $s_n := \max\left\{p_n, q_n, \sqrt{p_n}, p_n q_n\right\}$ for each $n \in \mathbb{N}$.

Proof. Since $s_n = \max\left\{p_n, q_n, \sqrt{p_n}, p_n q_n\right\}$, from (2.6), we immediately get that

$$\begin{aligned} \frac{1}{s_n}\|L_n(f) - f\|_* \leq\ & C\left\{\frac{1}{q_n} w(f, \delta_n) + \frac{1}{p_n q_n} w(f, \delta_n) \|L_n(f_0) - f_0\|_* \right. \\ & \left. + \frac{1}{q_n} w(f, \delta_n) \sqrt{\frac{1}{p_n}\|L_n(f_0) - f_0\|_*} + \frac{1}{p_n}\|L_n(f_0) - f_0\|_* \right\}. \end{aligned} \tag{2.8}$$

Then, for a given $\varepsilon > 0$, consider the following sets:

$$\begin{aligned} E &: = \{n \in \mathbb{N} : \|L_n(f) - f\|_* \geq \varepsilon s_n\}, \\ E_1 &: = \left\{n \in \mathbb{N} : \|L_n(f_0) - f_0\|_* \geq \frac{\varepsilon p_n}{4C}\right\}, \\ E_2 &: = \left\{n \in \mathbb{N} : w(f, \delta_n) \geq \frac{\varepsilon q_n}{4C}\right\}, \\ E_3 &: = \left\{n \in \mathbb{N} : \frac{w(f, \delta_n)}{q_n} \frac{\|L_n(f_0) - f_0\|_*}{p_n} \geq \frac{\varepsilon}{4C}\right\}, \\ E_4 &: = \left\{n \in \mathbb{N} : \frac{w(f, \delta_n)}{q_n} \sqrt{\frac{\|L_n(f_0) - f_0\|_*}{p_n}} \geq \frac{\varepsilon}{4C}\right\}. \end{aligned}$$

In this case, it follows from (2.8) that

$$E \subseteq \bigcup_{i=1}^{4} E_i.$$

Also letting

$$E_3' \; : \; = \left\{ n \in \mathbb{N} : w(f, \delta_n) \geq q_n \sqrt{\frac{\varepsilon}{4C}} \right\},$$

$$E_3'' \; : \; = \left\{ n \in \mathbb{N} : \| L_n(f_0) - f_0 \|_* \geq p_n \sqrt{\frac{\varepsilon}{4C}} \right\},$$

we get $E_3 \subseteq E_3' \cup E_3''$ and $E_4 \subseteq E_1 \cup E_3''$, which implies

$$E \subseteq E_1 \cup E_2 \cup E_3' \cup E_3''.$$

Hence, the last inclusion yields that, for every $j \in \mathbb{N}$,

$$\sum_{n \in E} a_{jn} \leq \sum_{n \in E_1} a_{jn} + \sum_{n \in E_2} a_{jn} + \sum_{n \in E_3'} a_{jn} + \sum_{n \in E_3''} a_{jn}.$$

Now taking limit as $j \to \infty$ and using the hypotheses, we immediately see that

$$\lim_j \sum_{n \in E} a_{jn} = 0,$$

whence the result is proved. □

Remark 1. Specializing the sequences $\{p_n\}$ and $\{q_n\}$ as $p_n = q_n = 1$ for each $n \in \mathbb{N}$, Theorem 2.3 of [4] is a special case of our Theorem 1 or Theorem 2. So, our results give us the statistical rates of approximation to a function $f \in C^*(\mathbb{R}^m)$ by positive linear operators mapping $C^*(\mathbb{R}^m)$ into itself.

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Author's address

Esra Erkuş-Duman
Gazi University, Faculty of Arts and Sciences, Department of Mathematics, Teknikokullar TR-06500, Ankara, Turkey
E-mail address: eduman@gazi.edu.tr