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## Relations between hypergeometric function of Appel $F_3$ and Kampé de Fériet functions

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## RELATIONS BETWEEN THE HYPERGEOMETRIC FUNCTION OF APPELL $F_3$ AND KAMPÉ DE FÉRIET FUNCTIONS

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*Abstract.* Carlson [Some extensions of Lardner’s relations between  ${}_0F_3$  and Bessel functions, *SIAM J. Math. Anal.* **1**(2) (1970), 232–242] presented certain connections between Bessel and generalized hypergeometric functions, which generalizes some earlier results. Here, by simply splitting the hypergeometric Appell series  $F_3$  into four parts, we show how some useful and generalized relations between  $F_3$  and Kampé de Fériet function  $F_{2;1;1}^{0;4;4}$  can be obtained. These main results are shown to be specialized to yield certain relations between functions  ${}_0F_1$ ,  ${}_0F_3$ ,  $F_{2;1;1}^{0;4;4}$ , and the Humbert function  $\Psi_2$  some of which are still reduced to produce the Carlson’s relations and some other interesting relations between the exponential function, the hyperbolic functions, and modified Bessel functions. Furthermore, decomposition formulas and integral representations of Euler type with hypergeometric function in the kernel for the function  $F_{2;1;1}^{0;4;4}$  are derived by means of Burchnell-Chaudy operator method which has recently been revived.

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### 1. INTRODUCTION

Lardner [14] presented certain interesting relations between Bessel functions and the confluent hypergeometric series  ${}_0F_3$ , for example,

$${}_0F_3\left(\frac{1}{2}, \frac{1}{2}, 1; z\right) = \frac{1}{2} \left[ J_0\left(4z^{\frac{1}{4}}\right) + I_0\left(4z^{\frac{1}{4}}\right) \right]$$

and

$$\text{ber}(x) = {}_0F_3\left(\frac{1}{2}, \frac{1}{2}, 1; -\frac{x^4}{256}\right), \quad \text{and} \quad \text{bei}(x) = \frac{x^2}{4} {}_0F_3\left(\frac{3}{2}, \frac{3}{2}, 1; -\frac{x^4}{256}\right),$$

where  $J_\nu$  and  $I_\nu$  denote a Bessel function and a modified Bessel function of order  $\nu$  (see [1]; also [19]) defined by

$$J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} {}_0F_1\left(-; \nu+1; -\frac{z^2}{4}\right),$$

$$I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} {}_0F_1\left(-; \nu+1; \frac{z^2}{4}\right),$$

and  $\text{ber}(x)$  and  $\text{bei}(x)$  ( $x$  real) denote Kelvin's functions (see [7, p. 6]) defined by

$$\text{ber}(x) + i \text{bei}(x) = J_0\left(x e^{i\frac{3}{4}\pi}\right) = I_0\left(x e^{i\frac{1}{4}\pi}\right).$$

Carlson [5] generalized these results for arbitrary parameters to yield the following results

$${}_0F_3\left(\frac{1}{2}, c, c + \frac{1}{2}; z\right) = \frac{1}{2} \Gamma(2c) \left(2z^{\frac{1}{4}}\right)^{1-2c} \left[ I_{2c-1}\left(4z^{\frac{1}{4}}\right) + J_{2c-1}\left(4z^{\frac{1}{4}}\right) \right] \quad (1.1)$$

and

$${}_0F_3\left(\frac{3}{2}, c, c + \frac{1}{2}; z\right) = \frac{1}{2} \Gamma(2c) \left(2z^{\frac{1}{4}}\right)^{-2c} \left[ I_{2c-2}\left(4z^{\frac{1}{4}}\right) - J_{2c-2}\left(4z^{\frac{1}{4}}\right) \right]. \quad (1.2)$$

Here, by simply splitting the hypergeometric Appell series  $F_3$  into four parts, we can investigate relations between  $F_3$  and the Kampé de Fériet function  $F_{2;1;1}^{0;4;4}$ . Our main results can be specialized to yield certain relations between functions  ${}_0F_1$ ,  ${}_0F_3$ ,  $F_{2;1;1}^{0;4;4}$ , and  $\Psi_2$ . Some of these can be reduced to Carlson's relations (1.1) and (1.2) (see Corollary 1), while some of them yield interesting relations between the exponential function, the hyperbolic functions, and the modified Bessel functions (see Corollary 2). Furthermore, decomposition formulas and integral representations of Euler type with hypergeometric function in the kernel for the function  $F_{2;1;1}^{0;4;4}$  are derived by means of Burchnell-Chaudy operator method (see [3, 4, 6]), which has recently been revived (see, for example, [11–13]).

It may be remarked before passing to the next section, that certain special functions have still played important roles in theories and applications (see, for example, [9–13]). In fact, multiple hypergeometric functions arise mostly during the solution of certain differential equations. Yet, there are other important cases in which these special functions are involved. For example, many auxiliary algebraic and integral transformations in various physical models have been found to be connected with these functions. Especially, many problems in gas dynamics lead to those second-order partial differential equations which can be solvable in terms of multiple hypergeometric functions. In the investigation of the boundary value problems for these partial differential equations, it might be necessary to decompose the hypergeometric functions of several variables in terms of simpler hypergeometric functions of the Gauss, Appell, and Kampé de Fériet types, and so on (see, for example, [15, 16, 18]). More specially, the hypergeometric function  $F_2$  determines the fundamental solutions of the generalized bi-axially symmetric Helmholtz equation [9] and of certain three-dimensional elliptic equations with singular coefficients [10].

2. RELATIONSHIPS BETWEEN THE APPELL FUNCTION  $F_3$  AND KAMPÉ DE FÉRIET FUNCTION  $F_{2;1;1}^{0;4;4}$

We use the notations as in [2, 8]. The hypergeometric function of Appell  $F_3$  is defined by

$$F_3(a_1, a_2, b_1, b_2; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a_1)_m (a_2)_n (b_1)_m (b_2)_n}{(c)_{m+n} m! n!} x^m y^n, |x| < 1, |y| < 1, \tag{2.1}$$

where  $(\lambda)_n$  denotes the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-) \tag{2.2}$$

in terms of the Gamma function  $\Gamma$ ,  $\mathbb{C}$  and  $\mathbb{Z}_0^-$  are the sets of complex numbers and nonpositive integers,  $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ , and  $\mathbb{N}$  is the set of positive integers. Since the series  $F_3$  converges absolutely for any  $c \in \mathbb{C} \setminus \mathbb{Z}_0^-$  and any finite  $x, y \in \mathbb{C}$ , it can be rearranged into the following four series:

$$\begin{aligned} & F_3(a_1, a_2, b_1, b_2; c; x, y) \\ &= \sum_{p,q=0}^{\infty} \frac{(a_1)_{2p} (a_2)_{2q} (b_1)_{2p} (b_2)_{2q}}{(c)_{2p+2q} (2p)! (2q)!} x^{2p} y^{2q} \\ &+ xy \sum_{p,q=0}^{\infty} \frac{(a_1)_{2p+1} (a_2)_{2q+1} (b_1)_{2p+1} (b_2)_{2q+1}}{(c)_{2p+1+2q+1} (2p+1)! (2q+1)!} x^{2p} y^{2q} \\ &+ y \sum_{p,q=0}^{\infty} \frac{(a_1)_{2p} (a_2)_{2q+1} (b_1)_{2p} (b_2)_{2q+1}}{(c)_{2p+2q+1} (2p)! (2q+1)!} x^{2p} y^{2q} \\ &+ x \sum_{p,q=0}^{\infty} \frac{(a_1)_{2p+1} (a_2)_{2q} (b_1)_{2p+1} (b_2)_{2q}}{(c)_{2p+1+2q} (2p+1)! (2q)!} x^{2p} y^{2q}. \end{aligned} \tag{2.3}$$

Applying some known (or easily-derivable) identities for Pochhammer symbols (see [8]) such as

$$(\alpha)_{2m+2n+2} = \alpha(\alpha+1) 4^{m+n} \left(\frac{\alpha+2}{2}\right)_{m+n} \left(\frac{\alpha+3}{2}\right)_{m+n} \quad (m, n \in \mathbb{N}_0)$$

to (2.3) and simplifying each of the four resulting series, we can express the Appell function  $F_3$  in terms of Kampé de Fériet function  $F_{2;1;1}^{0;4;4}$  (see [17, p. 27]) as in the following theorem.

**Theorem 1.** *The following relationship between the Appell function  $F_3$  and Kampé de Fériet function  $F_{2;1;1}^{0;4;4}$  holds true.*

$$\begin{aligned}
 F_3(a_1, a_2, b_1, b_2; c; x, y) &= F_{2;1;1}^{0;4;4} \left[ \begin{array}{c} - : \frac{a_1}{2}, \frac{a_1+1}{2}, \frac{b_1}{2}, \frac{b_1+1}{2}; \\ \frac{c}{2}, \frac{c+1}{2} : \end{array} \right. \\
 &\quad \left. \frac{a_2}{2}, \frac{a_2+1}{2}, \frac{b_2}{2}, \frac{b_2+1}{2}; x^2, y^2 \right] \\
 &+ \frac{a_1 a_2 b_1 b_2 x y}{c(c+1)} F_{2;1;1}^{0;4;4} \left[ \begin{array}{c} - : \frac{a_1+1}{2}, \frac{a_1+2}{2}, \frac{b_1+1}{2}, \frac{b_1+2}{2}; \\ \frac{c+2}{2}, \frac{c+3}{2} : \end{array} \right. \\
 &\quad \left. \frac{a_2+1}{2}, \frac{a_2+2}{2}, \frac{b_2+1}{2}, \frac{b_2+2}{2}; x^2, y^2 \right] \\
 &+ \frac{a_2 b_2 y}{c} F_{2;1;1}^{0;4;4} \left[ \begin{array}{c} - : \frac{a_1}{2}, \frac{a_1+1}{2}, \frac{b_1}{2}, \frac{b_1+1}{2}; \\ \frac{c+1}{2}, \frac{c+2}{2} : \end{array} \right. \\
 &\quad \left. \frac{a_2+1}{2}, \frac{a_2+2}{2}, \frac{b_2+1}{2}, \frac{b_2+2}{2}; x^2, y^2 \right] \\
 &+ \frac{a_1 b_1 x}{c} F_{2;1;1}^{0;4;4} \left[ \begin{array}{c} - : \frac{a_1+1}{2}, \frac{a_1+2}{2}, \frac{b_1+1}{2}, \frac{b_1+2}{2}; \\ \frac{c+1}{2}, \frac{c+2}{2} : \end{array} \right. \\
 &\quad \left. \frac{a_2}{2}, \frac{a_2+1}{2}, \frac{b_2}{2}, \frac{b_2+1}{2}; x^2, y^2 \right], \tag{2.4}
 \end{aligned}$$

where the Kampé de Fériet function  $F_{2;1;1}^{0;4;4}$  is defined by

$$\begin{aligned}
 &F_{2;1;1}^{0;4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; x, y \\ \alpha_1, \alpha_2 : \quad \quad \quad \beta; \quad \quad \quad \gamma; \end{array} \right] \tag{2.5} \\
 &= \sum_{m,n=0}^{\infty} \frac{(b_1)_m (b_2)_m (b_3)_m (b_4)_m (c_1)_n (c_2)_n (c_3)_n (c_4)_n}{(\alpha_1)_{m+n} (\alpha_2)_{m+n} (\beta)_m (\gamma)_n m! n!} x^m y^n \quad (|x| < 1; |y| < 1).
 \end{aligned}$$

In view of the relation in (2.4), the four Kampé de Fériet functions in Theorem 1 can be expressed in terms of the Appell functions  $F_3$  as in the following theorem.

**Theorem 2.** *Each of the following relationship between Kampé de Fériet function  $F_{2;1;1}^{0;4;4}$  and the Appell function  $F_3$  holds true.*

$$\begin{aligned}
 &4F_{2;1;1}^{0;4;4} \left[ \begin{array}{c} - : \frac{a_1}{2}, \frac{a_1+1}{2}, \frac{b_1}{2}, \frac{b_1+1}{2}; \frac{a_2}{2}, \frac{a_2+1}{2}, \frac{b_2}{2}, \frac{b_2+1}{2}; \\ \frac{c}{2}, \frac{c+1}{2} : \end{array} \right. \\
 &\quad \left. \frac{1}{2}; x^2, y^2 \right] \tag{2.6} \\
 &= F_3(a_1, a_2, b_1, b_2; c; x, y) + F_3(a_1, a_2, b_1, b_2; c; -x, y) \\
 &\quad + F_3(a_1, a_2, b_1, b_2; c; x, -y) + F_3(a_1, a_2, b_1, b_2; c; -x, -y);
 \end{aligned}$$

$$4 \frac{a_1 a_2 b_1 b_2 x y}{c(c+1)} F_{2;1;1}^{0;4;4} \left[ \begin{array}{c} - : \frac{a_1+1}{2}, \frac{a_1+2}{2}, \frac{b_1+1}{2}, \frac{b_1+2}{2}; \\ \frac{c+2}{2}, \frac{c+3}{2} : \frac{3}{2}; \\ \frac{a_2+1}{2}, \frac{a_2+2}{2}, \frac{b_2+1}{2}, \frac{b_2+2}{2}; \end{array} ; x^2, y^2 \right] \quad (2.7)$$

$$= F_3(a_1, a_2, b_1, b_2; c; x, y) + F_3(a_1, a_2, b_1, b_2; c; -x, y) - F_3(a_1, a_2, b_1, b_2; c; x, -y) + F_3(a_1, a_2, b_1, b_2; c; -x, -y);$$

$$4 \frac{a_2 b_2 y}{c} F_{2;1;1}^{0;4;4} \left[ \begin{array}{c} - : \frac{a_1}{2}, \frac{a_1+1}{2}, \frac{b_1}{2}, \frac{b_1+1}{2}; \frac{a_2+1}{2}, \frac{a_2+2}{2}, \frac{b_2+1}{2}, \frac{b_2+2}{2}; \\ \frac{c+1}{2}, \frac{c+2}{2} : \frac{1}{2}; \frac{3}{2}; \end{array} ; x^2, y^2 \right]$$

$$= F_3(a_1, a_2, b_1, b_2; c; x, y) + F_3(a_1, a_2, b_1, b_2; c; -x, y) - F_3(a_1, a_2, b_1, b_2; c; x, -y) - F_3(a_1, a_2, b_1, b_2; c; -x, -y); \quad (2.8)$$

$$4 \frac{a_1 b_1 x}{c} F_{2;1;1}^{0;4;4} \left[ \begin{array}{c} - : \frac{a_1+1}{2}, \frac{a_1+2}{2}, \frac{b_1+1}{2}, \frac{b_1+2}{2}; \\ \frac{c+1}{2}, \frac{c+2}{2} : \frac{3}{2}; \\ \frac{a_2}{2}, \frac{a_2+1}{2}, \frac{b_2}{2}, \frac{b_2+1}{2}; \end{array} ; x^2, y^2 \right] \quad (2.9)$$

$$= F_3(a_1, a_2, b_1, b_2; c; x, y) - F_3(a_1, a_2, b_1, b_2; c; -x, y) + F_3(a_1, a_2, b_1, b_2; c; x, -y) - F_3(a_1, a_2, b_1, b_2; c; -x, -y).$$

### 3. INTEGRAL REPRESENTATIONS OF EULER TYPE FOR $F_{2;1;1}^{0;4;4}$

The Kampé de Fériet hypergeometric function  $F_{2;1;1}^{0;4;4}$  has the following Euler type integral representation:

$$F_{2;1;1}^{0;4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \beta; \gamma; \end{array} ; x, y \right]$$

$$= \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(b_1)\Gamma(c_2)\Gamma(\alpha_1-b_1)\Gamma(\alpha_2-c_2)}$$

$$\cdot \int_0^1 \int_0^1 \xi^{b_1-1} \eta^{c_2-1} (1-\xi)^{\alpha_1-b_1-1} (1-\eta)^{\alpha_2-c_2-1}$$

$$\cdot {}_3F_2 \left( \begin{array}{c} b_2, b_3, b_4 \\ \alpha_2 - c_2, \beta \end{array} ; x \xi (1-\eta) \right) {}_3F_2 \left( \begin{array}{c} c_1, c_3, c_4 \\ \alpha_1 - b_1, \gamma \end{array} ; y (1-\xi) \eta \right) d\xi d\eta$$

$$(\Re(\alpha_1) > \Re(b_1) > 0, \Re(\alpha_2) > \Re(c_2) > 0);$$

$$\begin{aligned}
& F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \quad \quad \quad \beta; \quad \quad \quad \gamma; \end{array} x, y \right] \\
&= \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(b_1)\Gamma(c_1)\Gamma(\beta-b_1)\Gamma(\gamma-c_1)} \\
&\cdot \int_0^1 \int_0^1 \xi^{b_1-1} \eta^{c_1-1} (1-\xi)^{\beta-b_1-1} (1-\eta)^{\gamma-c_1-1} \\
&\cdot F_{2:0;0}^{0:3;3} \left[ \begin{array}{c} - : b_2, b_3, b_4; c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \quad \quad \quad -; \quad \quad \quad -; \end{array} x\xi, y\eta \right] d\xi d\eta \\
&\quad (\Re(\beta) > \Re(b_1) > 0, \Re(\gamma) > \Re(c_1) > 0);
\end{aligned}$$

$$\begin{aligned}
& F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; \alpha_1 - b_1, c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \quad \quad \quad \beta; \quad \quad \quad \gamma; \end{array} x, y \right] = \frac{\Gamma(\alpha_1)}{\Gamma(b_1)\Gamma(\alpha_1 - b_1)} \\
&\cdot \int_0^1 \xi^{b_1-1} (1-\xi)^{\alpha_1-b_1-1} F_{1:1;1}^{0:3;3} \left[ \begin{array}{c} - : b_2, b_3, b_4; c_2, c_3, c_4; \\ \alpha_2 : \quad \quad \quad \beta; \quad \quad \quad \gamma; \end{array} x\xi, y(1-\xi) \right] d\xi \\
&\quad (\Re(\alpha_1) > \Re(b_1) > 0);
\end{aligned}$$

$$\begin{aligned}
& F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \quad \quad \quad \beta; \quad \quad \quad \gamma; \end{array} x, y \right] = \frac{\Gamma(b_1 + c_1)}{\Gamma(b_1)\Gamma(c_1)} \\
&\cdot \int_0^1 \xi^{b_1-1} (1-\xi)^{c_1-1} F_{2:1;1}^{1:3;3} \left[ \begin{array}{c} b_1 + c_1 : b_2, b_3, b_4; c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \quad \quad \quad \beta; \quad \quad \quad \gamma; \end{array} x\xi, y(1-\xi) \right] d\xi \\
&\quad (\Re(b_1) > 0, \Re(c_1) > 0);
\end{aligned}$$

$$\begin{aligned}
& F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \quad \quad \quad \beta; \quad \quad \quad \gamma; \end{array} x, y \right] = \frac{\Gamma(b_1 + c_1)\Gamma(b_2 + c_2)}{\Gamma(b_1)\Gamma(b_2)\Gamma(c_1)\Gamma(c_2)} \\
&\cdot \int_0^1 \int_0^1 \xi^{b_1-1} \eta^{b_2-1} (1-\xi)^{c_1-1} (1-\eta)^{c_2-1} \\
&F_{2:1;1}^{2:2;2} \left[ \begin{array}{c} b_1 + c_1, b_2 + c_2 : b_3, b_4; c_3, c_4; \\ \alpha_1, \alpha_2 : \quad \quad \quad \beta; \quad \quad \quad \gamma; \end{array} x\xi\eta, y(1-\xi)(1-\eta) \right] d\xi d\eta \\
&\quad (\Re(b_i) > 0, \Re(c_i) > 0) (i = 1, 2);
\end{aligned}$$

$$\begin{aligned}
 & F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \beta; \gamma; \end{array} x, y \right] \\
 &= \frac{\Gamma(b_1 + c_1) \Gamma(b_2 + c_2) \Gamma(b_3 + c_3)}{\Gamma(b_1) \Gamma(b_2) \Gamma(b_3) \Gamma(c_1) \Gamma(c_2) \Gamma(c_3)} \\
 &\cdot \int_0^1 \int_0^1 \int_0^1 \xi^{b_1-1} \eta^{b_2-1} \zeta^{b_3-1} (1-\xi)^{c_1-1} (1-\eta)^{c_2-1} (1-\zeta)^{c_3-1} \\
 &\cdot F_{2:1;1}^{3:1;1} \left[ \begin{array}{c} b_1 + c_1, b_2 + c_2, b_3 + c_3 : b_4; c_4; \\ \alpha_1, \alpha_2 : \beta; \gamma; \end{array} x\xi\eta\zeta, y(1-\xi)(1-\eta)(1-\zeta) \right] \\
 &\quad d\xi d\eta d\zeta \quad (\Re(b_i) > 0, \Re(c_i) > 0) \quad (i = 1, 2, 3); \\
 & F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \beta; \gamma; \end{array} x, y \right] = \prod_{i=1}^4 \frac{\Gamma(b_i + c_i)}{\Gamma(b_i) \Gamma(c_i)} \\
 &\cdot \int_0^1 \dots \int_0^1 \xi^{b_1-1} \eta^{b_2-1} \zeta^{b_3-1} \omega^{b_4-1} (1-\xi)^{c_1-1} (1-\eta)^{c_2-1} (1-\zeta)^{c_3-1} (1-\omega)^{c_4-1} \\
 &\cdot F_{2:1;1}^{3:1;1} \left[ \begin{array}{c} b_1 + c_1, b_2 + c_2, b_3 + c_3, b_4 + c_4 : -; -; \\ \alpha_1, \alpha_2 : \beta; \gamma; \end{array} \right. \\
 &\quad \left. x\xi\eta\zeta\omega, y(1-\xi)(1-\eta)(1-\zeta)(1-\omega) \right] d\xi d\eta d\zeta d\omega \\
 &\quad (\Re(b_i) > 0, \Re(c_i) > 0) \quad (i = 1, 2, 3).
 \end{aligned}$$

*Proof.* Note that each of the integral representations in Section 3 can be proved by expressing the series definition of the involved special function in each integrand and changing the order of the integration and the summation, and finally using the following well-known relationship between the Beta function  $B(\alpha, \beta)$  and the Gamma function  $\Gamma$ :

$$B(\alpha, \beta) := \begin{cases} \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt & (\Re(\alpha) > 0; \Re(\beta) > 0) \\ \frac{\Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha + \beta)} & (\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-). \end{cases}$$

□

#### 4. RELATIONS DEDUCED FROM THE MAIN RESULTS

In this section we deduce certain interesting relations from the main results in Section 2. Transforming  $b_2$  by  $1/\varepsilon$  and  $y$  by  $\varepsilon y$  in the formulas (2.4), (2.6), (2.7),



(2.8) and (2.9), and taking the limit of each of the obtained identities as  $\varepsilon \rightarrow 0$ , we get some identities as in the following corollary (for a review of this method, see [2]).

**Corollary 1.** *Each of the following identities holds true.*

$$\begin{aligned} \mathcal{E}_1(a_1, a_2, b_1; c; x, y) &= F_{2;1;1}^{0;4;2} \left[ \begin{array}{c} - : \frac{a_1}{2}, \frac{a_1+1}{2}, \frac{b_1}{2}, \frac{b_1+1}{2}; \frac{a_2}{2}, \frac{a_2+1}{2}; x^2, \frac{y^2}{4} \\ \frac{c}{2}, \frac{c+1}{2} : \frac{1}{2} \end{array} \right] \\ &+ \frac{a_1 a_2 b_1 x y}{c(c+1)} F_{2;1;1}^{0;4;2} \left[ \begin{array}{c} - : \frac{a_1+1}{2}, \frac{a_1+2}{2}, \frac{b_1+1}{2}, \frac{b_1+2}{2}; \frac{a_2+1}{2}, \frac{a_2+2}{2}; x^2, \frac{y^2}{4} \\ \frac{c+2}{2}, \frac{c+3}{2} : \frac{3}{2} \end{array} \right] \\ &+ \frac{a_2 y}{c} F_{2;1;1}^{0;4;2} \left[ \begin{array}{c} - : \frac{a_1}{2}, \frac{a_1+1}{2}, \frac{b_1}{2}, \frac{b_1+1}{2}; \frac{a_2+1}{2}, \frac{a_2+2}{2}; x^2, \frac{y^2}{4} \\ \frac{c+1}{2}, \frac{c+2}{2} : \frac{1}{2} \end{array} \right] \\ &+ \frac{a_1 b_1 x}{c} F_{2;1;1}^{0;4;2} \left[ \begin{array}{c} - : \frac{a_1+1}{2}, \frac{a_1+2}{2}, \frac{b_1+1}{2}, \frac{b_1+2}{2}; \frac{a_2}{2}, \frac{a_2+1}{2}; x^2, \frac{y^2}{4} \\ \frac{c+1}{2}, \frac{c+2}{2} : \frac{3}{2} \end{array} \right]; \end{aligned} \quad (4.1)$$

$$\begin{aligned} 4F_{2;1;1}^{0;4;2} \left[ \begin{array}{c} - : \frac{a_1}{2}, \frac{a_1+1}{2}, \frac{b_1}{2}, \frac{b_1+1}{2}; \frac{a_2}{2}, \frac{a_2+1}{2}; x^2, \frac{y^2}{4} \\ \frac{c}{2}, \frac{c+1}{2} : \frac{1}{2} \end{array} \right] &= \mathcal{E}_1(a_1, a_2, b_1; c; x, y) \\ &+ \mathcal{E}_1(a_1, a_2, b_1; c; -x, y) + \mathcal{E}_1(a_1, a_2, b_1; c; x, -y) + \mathcal{E}_1(a_1, a_2, b_1; c; -x, -y); \end{aligned} \quad (4.2)$$

$$\begin{aligned} 4 \frac{a_1 a_2 b_1 x y}{c(c+1)} F_{2;1;1}^{0;4;2} \left[ \begin{array}{c} - : \frac{a_1+1}{2}, \frac{a_1+2}{2}, \frac{b_1+1}{2}, \frac{b_1+2}{2}; \frac{a_2+1}{2}, \frac{a_2+2}{2}; x^2, \frac{y^2}{4} \\ \frac{c+2}{2}, \frac{c+3}{2} : \frac{3}{2} \end{array} \right] \\ = \mathcal{E}_1(a_1, a_2, b_1; c; x, y) - \mathcal{E}_1(a_1, a_2, b_1; c; -x, y) \\ - \mathcal{E}_1(a_1, a_2, b_1; c; x, -y) + \mathcal{E}_1(a_1, a_2, b_1; c; -x, -y); \end{aligned} \quad (4.3)$$

$$\begin{aligned} 4 \frac{a_2 y}{c} F_{2;1;1}^{0;4;2} \left[ \begin{array}{c} - : \frac{a_1}{2}, \frac{a_1+1}{2}, \frac{b_1}{2}, \frac{b_1+1}{2}; \frac{a_2+1}{2}, \frac{a_2+2}{2}; x^2, \frac{y^2}{4} \\ \frac{c+1}{2}, \frac{c+2}{2} : \frac{1}{2} \end{array} \right] \\ = \mathcal{E}_1(a_1, a_2, b_1; c; x, y) + \mathcal{E}_1(a_1, a_2, b_1; c; -x, y) \\ - \mathcal{E}_1(a_1, a_2, b_1; c; x, -y) - \mathcal{E}_1(a_1, a_2, b_1; c; -x, -y); \end{aligned} \quad (4.4)$$

$$\begin{aligned} 4 \frac{a_1 b_1 x}{c} F_{2;1;1}^{0;4;2} \left[ \begin{array}{c} - : \frac{a_1+1}{2}, \frac{a_1+2}{2}, \frac{b_1+1}{2}, \frac{b_1+2}{2}; \frac{a_2}{2}, \frac{a_2+1}{2}; x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2} : \frac{3}{2} \end{array} \right] \\ = \mathcal{E}_1(a_1, a_2, b_1; c; x, y) - \mathcal{E}_1(a_1, a_2, b_1; c; -x, y) \\ + \mathcal{E}_1(a_1, a_2, b_1; c; x, -y) - \mathcal{E}_1(a_1, a_2, b_1; c; -x, -y) \end{aligned} \quad (4.5)$$

where  $\mathcal{E}_1$  denotes the Humbert hypergeometric function (see [2, 8, 17]) defined by

$$\mathcal{E}_1(a_1, a_2, b_1; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a_1)_m (a_2)_n (b_1)_m}{(c)_{m+n} m! n!} x^m y^n \quad (|x| < 1; |y| < \infty). \quad (4.6)$$

Transforming  $a_2$  by  $1/\varepsilon$  and  $y$  by  $\varepsilon y$  in the formulas (4.1), (4.2), (4.3), (4.4) and (4.5), and taking the limit of each of the obtained identities as  $\varepsilon \rightarrow 0$ , we deduce some identities as in the following corollary (again, see [2]).

**Corollary 2.** *Each of the following functional relations holds true.*

$$\begin{aligned} \mathcal{E}_2(a_1, b_1; c; x, y) &= F_{2;1;1}^{0:4;0} \left[ \begin{matrix} - : & a_1, a_1+1, b_1, b_1+1; & -; & x^2, y^2 \\ \frac{c}{2}, \frac{c+1}{2} : & & \frac{1}{2}; & \frac{1}{2}; \end{matrix} \frac{y^2}{16} \right] \\ &+ \frac{a_1 b_1 x y}{c(c+1)} F_{2;1;1}^{0:4;0} \left[ \begin{matrix} - : & a_1+1, a_1+2, b_1+1, b_1+2; & & x^2, y^2 \\ \frac{c+2}{2}, \frac{c+3}{2} : & & \frac{3}{2}; & \frac{3}{2}; \end{matrix} \frac{y^2}{16} \right] \\ &+ \frac{y}{c} F_{2;1;1}^{0:4;0} \left[ \begin{matrix} - : & a_1, a_1+1, b_1, b_1+1; & -; & x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2} : & & \frac{1}{2}; & \frac{3}{2}; \end{matrix} \frac{y^2}{16} \right] \\ &+ \frac{a_1 b_1 x}{c} F_{2;1;1}^{0:4;0} \left[ \begin{matrix} - : & a_1+1, a_1+2, b_1+1, b_1+2; & -; & x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2} : & & \frac{3}{2}; & \frac{1}{2}; \end{matrix} \frac{y^2}{16} \right]; \end{aligned} \tag{4.7}$$

$$\begin{aligned} &4 F_{2;1;1}^{0:4;0} \left[ \begin{matrix} - : & a_1, a_1+1, b_1, b_1+1; & -; & x^2, y^2 \\ \frac{c}{2}, \frac{c+1}{2} : & & \frac{1}{2}; & \frac{1}{2}; \end{matrix} \frac{y^2}{16} \right] \\ &= \mathcal{E}_2(a_1, b_1; c; x, y) + \mathcal{E}_2(a_1, b_1; c; -x, y) \\ &\quad + \mathcal{E}_2(a_1, b_1; c; x, -y) + \mathcal{E}_2(a_1, b_1; c; -x, -y); \end{aligned} \tag{4.8}$$

$$\begin{aligned} &4 \frac{a_1 b_1 x y}{c(c+1)} F_{2;1;1}^{0:4;0} \left[ \begin{matrix} - : & a_1+1, a_1+2, b_1+1, b_1+2; & -; & x^2, y^2 \\ \frac{c+2}{2}, \frac{c+3}{2} : & & \frac{3}{2}; & \frac{3}{2}; \end{matrix} \frac{y^2}{16} \right] \\ &= \mathcal{E}_2(a_1, b_1; c; x, y) - \mathcal{E}_2(a_1, b_1; c; -x, y) \\ &\quad - \mathcal{E}_2(a_1, b_1; c; x, -y) + \mathcal{E}_2(a_1, b_1; c; -x, -y); \end{aligned} \tag{4.9}$$

$$\begin{aligned} &4 \frac{y}{c} F_{2;1;1}^{0:4;0} \left[ \begin{matrix} - : & a_1, a_1+1, b_1, b_1+1; & -; & x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2} : & & \frac{1}{2}; & \frac{3}{2}; \end{matrix} \frac{y^2}{16} \right] \\ &= \mathcal{E}_2(a_1, b_1; c; x, y) + \mathcal{E}_2(a_1, b_1; c; -x, y) \\ &\quad - \mathcal{E}_2(a_1, b_1; c; x, -y) - \mathcal{E}_2(a_1, b_1; c; -x, -y); \end{aligned} \tag{4.10}$$

$$\begin{aligned} &4 \frac{a_1 b_1 x}{c} F_{2;1;1}^{0:4;0} \left[ \begin{matrix} - : & a_1+1, a_1+2, b_1+1, b_1+2; & -; & x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2} : & & \frac{3}{2}; & \frac{1}{2}; \end{matrix} \frac{y^2}{16} \right] \\ &= \mathcal{E}_2(a_1, b_1; c; x, y) - \mathcal{E}_2(a_1, b_1; c; -x, y) \\ &\quad + \mathcal{E}_2(a_1, b_1; c; x, -y) - \mathcal{E}_2(a_1, b_1; c; -x, -y), \end{aligned} \tag{4.11}$$

where  $\mathcal{E}_2$  denotes the Humbert hypergeometric function (see [2, 8, 17]) defined by

$$\mathcal{E}_2(a_1, b_1; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a_1)_m (b_1)_m}{(c)_{m+n} m! n!} x^m y^n \quad (|x| < 1; |y| < \infty). \tag{4.12}$$

Transforming  $b_1$  by  $1/\varepsilon$  and  $x$  by  $\varepsilon x$  in the formulas (4.1), (4.2), (4.3), (4.4) and (4.5), and taking the limit of each of the resulting identities as  $\varepsilon \rightarrow 0$ , we obtain some identities as in the following corollary (see [2]).

**Corollary 3.** *We deduce the following identities.*

$$\begin{aligned} \Phi_2(a_1, a_2; c; x, y) &= F_{2:1;1}^{0:2;2} \left[ \begin{matrix} - : & a_1, a_1+1, -; & a_2, a_2+1; & x^2, y^2 \\ \frac{c}{2}, \frac{c+1}{2} : & & \frac{1}{2}; & \frac{1}{2}; & \frac{1}{4}, \frac{1}{4} \end{matrix} \right] \\ &+ \frac{a_1 a_2 x y}{c(c+1)} F_{2:1;1}^{0:2;2} \left[ \begin{matrix} - : & a_1+1, a_1+2, -; & a_2+1, a_2+2; & x^2, y^2 \\ \frac{c+2}{2}, \frac{c+3}{2} : & & \frac{3}{2}; & \frac{3}{2}; & \frac{1}{4}, \frac{1}{4} \end{matrix} \right] \\ &+ \frac{a_2 y}{c} F_{2:1;1}^{0:2;2} \left[ \begin{matrix} - : & a_1, a_1+1, -; & a_2+1, a_2+2; & x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2} : & & \frac{1}{2}; & \frac{3}{2}; & \frac{1}{4}, \frac{1}{4} \end{matrix} \right] \\ &+ \frac{a_1 x}{c} F_{2:1;1}^{0:2;2} \left[ \begin{matrix} - : & a_1+1, a_1+2, -; & a_2, a_2+1; & x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2} : & & \frac{3}{2}; & \frac{1}{2}; & \frac{1}{4}, \frac{1}{4} \end{matrix} \right]; \end{aligned} \quad (4.13)$$

$$\begin{aligned} &4 F_{2:1;1}^{0:2;2} \left[ \begin{matrix} - : & a_1, a_1+1; & a_2, a_2+1; & x^2, y^2 \\ \frac{c}{2}, \frac{c+1}{2} : & & \frac{1}{2}; & \frac{1}{2}; & \frac{1}{4}, \frac{1}{4} \end{matrix} \right] \\ &= \Phi_2(a_1, a_2; c; x, y) + \Phi_2(a_1, a_2; c; -x, y) \\ &\quad + \Phi_2(a_1, a_2; c; x, -y) + \Phi_2(a_1, a_2; c; -x, -y); \end{aligned} \quad (4.14)$$

$$\begin{aligned} &4 \frac{a_1 a_2 x y}{c(c+1)} F_{2:1;1}^{0:2;2} \left[ \begin{matrix} - : & a_1+1, a_1+2; & a_2+1, a_2+2, -; & x^2, y^2 \\ \frac{c+2}{2}, \frac{c+3}{2} : & & \frac{3}{2}; & \frac{3}{2}; & \frac{1}{4}, \frac{1}{4} \end{matrix} \right] \\ &= \Phi_2(a_1, a_2; c; x, y) - \Phi_2(a_1, a_2; c; -x, y) \\ &\quad - \Phi_2(a_1, a_2; c; x, -y) + \Phi_2(a_1, a_2; c; -x, -y); \end{aligned} \quad (4.15)$$

$$\begin{aligned} &4 \frac{a_2 y}{c} F_{2:1;1}^{0:2;2} \left[ \begin{matrix} - : & a_1, a_1+1; & a_2+1, a_2+2, -; & x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2} : & & \frac{1}{2}; & \frac{3}{2}; & \frac{1}{4}, \frac{1}{4} \end{matrix} \right] \\ &= \Phi_2(a_1, a_2; c; x, y) + \Phi_2(a_1, a_2; c; -x, y) \\ &\quad - \Phi_2(a_1, a_2; c; x, -y) - \Phi_2(a_1, a_2; c; -x, -y); \end{aligned} \quad (4.16)$$

$$\begin{aligned} &4 \frac{a_1 x}{c} F_{2:1;1}^{0:2;2} \left[ \begin{matrix} - : & a_1+1, a_1+2; & a_2, a_2+1, -; & x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2} : & & \frac{3}{2}; & \frac{1}{2}; & \frac{1}{4}, \frac{1}{4} \end{matrix} \right] \\ &= \Phi_2(a_1, a_2; c; x, y) - \Phi_2(a_1, a_2; c; -x, y) \\ &\quad + \Phi_2(a_1, a_2; c; x, -y) - \Phi_2(a_1, a_2; c; -x, -y), \end{aligned} \quad (4.17)$$

where  $\Phi_2$  denotes the Humbert hypergeometric function (see [2, 8, 17]) defined by

$$\Phi_2(a_1, a_2; c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a_1)_m (a_2)_n}{(c)_{m+n} m! n!} x^m y^n \quad (|x| < \infty; |y| < \infty). \quad (4.18)$$

Transforming  $a_2$  by  $1/\varepsilon$  and  $y$  by  $\varepsilon y$  in the formulas (4.13), (4.14), (4.15), (4.16) and (4.17), and taking the limit of each of the resulting identities as  $\varepsilon \rightarrow 0$ , we obtain some identities as in the following corollary (see [2]).

**Corollary 4.** *We find that the following functional relations hold true.*

$$\begin{aligned} \Phi_3(a_1; c; x, y) &= F_{2:1;1}^{0:2;0} \left[ \begin{matrix} - : & \frac{a_1}{2}, \frac{a_1+1}{2}, -; & -; & x^2, y^2 \\ \frac{c}{2}, \frac{c+1}{2} : & & \frac{1}{2}; & \frac{1}{2}; & 4, 16 \end{matrix} \right] \\ &+ \frac{a_1xy}{c(c+1)} F_{2:1;1}^{0:2;0} \left[ \begin{matrix} - : & \frac{a_1+1}{2}, \frac{a_1+2}{2}, -; & -; & x^2, y^2 \\ \frac{c+2}{2}, \frac{c+3}{2} : & & \frac{3}{2}; & \frac{3}{2}; & 4, 16 \end{matrix} \right] \\ &+ \frac{y}{c} F_{2:1;1}^{0:2;0} \left[ \begin{matrix} - : & \frac{a_1}{2}, \frac{a_1+1}{2}, -; & -; & x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2} : & & \frac{1}{2}; & \frac{3}{2}; & 4, 16 \end{matrix} \right] \\ &+ \frac{a_1x}{c} F_{2:1;1}^{0:2;0} \left[ \begin{matrix} - : & \frac{a_1+1}{2}, \frac{a_1+2}{2}, -; & -; & x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2} : & & \frac{3}{2}; & \frac{1}{2}; & 4, 16 \end{matrix} \right]; \end{aligned} \tag{4.19}$$

$$\begin{aligned} &4F_{2:1;1}^{0:2;0} \left[ \begin{matrix} - : & \frac{a_1}{2}, \frac{a_1+1}{2}; & -; & x^2, y^2 \\ \frac{c}{2}, \frac{c+1}{2} : & & \frac{1}{2}; & \frac{1}{2}; & 4, 16 \end{matrix} \right] \\ &= \Phi_3(a_1; c; x, y) + \Phi_3(a_1; c; -x, y) + \Phi_3(a_1; c; x, -y) + \Phi_3(a_1; c; -x, -y); \end{aligned} \tag{4.20}$$

$$\begin{aligned} &4 \frac{a_1xy}{c(c+1)} F_{2:1;1}^{0:2;0} \left[ \begin{matrix} -; & \frac{a_1+1}{2}, \frac{a_1+2}{2}; & -; & x^2, y^2 \\ \frac{c+2}{2}, \frac{c+3}{2}; & & \frac{3}{2}; & \frac{3}{2}; & 4, 16 \end{matrix} \right] \\ &= \Phi_3(a_1; c; x, y) - \Phi_3(a_1; c; -x, y) - \Phi_3(a_1; c; x, -y) + \Phi_3(a_1; c; -x, -y); \end{aligned} \tag{4.21}$$

$$\begin{aligned} &4 \frac{y}{c} F_{2:1;1}^{0:2;0} \left[ \begin{matrix} -; & \frac{a_1}{2}, \frac{a_1+1}{2}; & -; & x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2}; & & \frac{1}{2}; & \frac{3}{2}; & 4, 16 \end{matrix} \right] \\ &= \Phi_3(a_1; c; x, y) + \Phi_3(a_1; c; -x, y) - \Phi_3(a_1; c; x, -y) - \Phi_3(a_1; c; -x, -y); \end{aligned} \tag{4.22}$$

$$\begin{aligned} &4 \frac{a_1x}{c} F_{2:1;1}^{0:2;0} \left[ \begin{matrix} -; & \frac{a_1+1}{2}, \frac{a_1+2}{2}; & -; & x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2}; & & \frac{3}{2}; & \frac{1}{2}; & 4, 16 \end{matrix} \right] \\ &= \Phi_3(a_1; c; x, y) - \Phi_3(a_1; c; -x, y) + \Phi_3(a_1; c; x, -y) - \Phi_3(a_1; c; -x, -y), \end{aligned} \tag{4.23}$$

where  $\Phi_3$  denotes the Humbert hypergeometric function (see [2, 8, 17]) defined by

$$\Phi_3(a_1, c; x, y) = \sum_{m,n=0}^{\infty} \frac{(a_1)_m}{(c)_{m+n} m! n!} x^m y^n \quad (|x| < \infty; |y| < \infty). \tag{4.24}$$

Transforming  $a_1$  by  $1/\varepsilon$  and  $x$  by  $\varepsilon x$  in the formulas (4.19), (4.20), (4.21), (4.22) and (4.23), and taking the limit of each of the obtained identities as  $\varepsilon \rightarrow 0$ , we deduce some identities as in the following corollary (see [2]).

**Corollary 5.** *Each of the following identities holds true.*

$$\begin{aligned} {}_0F_1(c; x+y) &= F_{2;1;1}^{0;0;0} \left[ \begin{matrix} - : - ; - ; x^2, y^2 \\ \frac{c}{2}, \frac{c+1}{2} : \frac{1}{2}; \frac{1}{2}; \frac{1}{16}, \frac{1}{16} \end{matrix} \right] \\ &\quad + \frac{xy}{c(c+1)} F_{2;1;1}^{0;0;0} \left[ \begin{matrix} - : - ; - ; x^2, y^2 \\ \frac{c+2}{2}, \frac{c+3}{2} : \frac{3}{2}; \frac{3}{2}; \frac{1}{16}, \frac{1}{16} \end{matrix} \right] \\ &\quad + \frac{y}{c} F_{2;1;1}^{0;0;0} \left[ \begin{matrix} - : - ; - ; x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2} : \frac{1}{2}; \frac{3}{2}; \frac{1}{16}, \frac{1}{16} \end{matrix} \right] \\ &\quad + \frac{x}{c} F_{2;1;1}^{0;0;0} \left[ \begin{matrix} - : - ; - ; x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2} : \frac{3}{2}; \frac{1}{2}; \frac{1}{16}, \frac{1}{16} \end{matrix} \right]; \end{aligned} \quad (4.25)$$

$$\begin{aligned} 4F_{2;1;1}^{0;0;0} \left[ \begin{matrix} - : - ; - ; x^2, y^2 \\ \frac{c}{2}, \frac{c+1}{2} : \frac{1}{2}; \frac{1}{2}; \frac{1}{16}, \frac{1}{16} \end{matrix} \right] &= {}_0F_1(c; x+y) \\ &\quad + {}_0F_1(a_1; c; -x+y) + {}_0F_1(a_1; c; x-y) + {}_0F_1(a_1; c; -x-y); \end{aligned} \quad (4.26)$$

$$\begin{aligned} \frac{4xy}{c(c+1)} F_{2;1;1}^{0;0;0} \left[ \begin{matrix} - : - ; - ; x^2, y^2 \\ \frac{c+2}{2}, \frac{c+3}{2}; \frac{3}{2}; \frac{3}{2}; \frac{1}{16}, \frac{1}{16} \end{matrix} \right] &= {}_0F_1(c; x+y) \\ &\quad - {}_0F_1(c; -x+y) - {}_0F_1(c; x-y) + {}_0F_1(c; -x-y); \end{aligned} \quad (4.27)$$

$$\begin{aligned} \frac{4y}{c} F_{2;1;1}^{0;0;0} \left[ \begin{matrix} - : - ; - ; x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2}; \frac{1}{2}; \frac{3}{2}; \frac{1}{16}, \frac{1}{16} \end{matrix} \right] &= {}_0F_1(c; x+y) \\ &\quad + {}_0F_1(c; -x+y) - {}_0F_1(c; x-y) - {}_0F_1(c; -x-y); \end{aligned} \quad (4.28)$$

$$\begin{aligned} \frac{4x}{c} F_{2;1;1}^{0;0;0} \left[ \begin{matrix} - : - ; - ; x^2, y^2 \\ \frac{c+1}{2}, \frac{c+2}{2}; \frac{3}{2}; \frac{1}{2}; \frac{1}{16}, \frac{1}{16} \end{matrix} \right] &= {}_0F_1(c; x+y) \\ &\quad - {}_0F_1(c; -x+y) + {}_0F_1(c; x-y) - {}_0F_1(c; -x-y), \end{aligned} \quad (4.29)$$

where  ${}_0F_1(c; x)$  denotes the Bessel function (see [2, 8, 17]).

## 5. DECOMPOSITION FORMULAS FOR THE KAMPÉ DE FÉRIET FUNCTION $F_{2;1;1}^{0;4;4}$

Now we need some decomposition formulas for the function  $F_{2;1;1}^{0;4;4}$  in order to investigate properties of the Appell function  $F_3$ . For this purpose we recall the definition of the Burchnall-Chaundy operators (see [3, 4, 6])

$$\nabla_{x,y}(h) := \sum_{k=0}^{\infty} \frac{(-\delta_1)_k (-\delta_2)_k}{(h)_k k!}, \quad (5.1)$$

$$\begin{aligned} \Delta_{x,y}(h) &:= \sum_{k=0}^{\infty} \frac{(-\delta_1)_k (-\delta_2)_k}{(1-h-\delta_1-\delta_2)_k k!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (h)_{2k} (-\delta_1)_k (-\delta_2)_k}{(h+k-1)_k (h+\delta_1)_k (h+\delta_2)_k k!} \end{aligned} \quad (5.2)$$

$$\left( \delta_1 := x \frac{\partial}{\partial x}; \delta_2 := y \frac{\partial}{\partial y} \right)$$

and its one-dimensional analogue of (5.1), (5.2)

$$H_x(\alpha, \beta) := \sum_{k=0}^{\infty} \frac{(\beta - \alpha)_k (-\delta_1)_k}{(\beta)_k k!}, \tag{5.3}$$

and

$$\bar{H}_x(\alpha, \beta) := \sum_{k=0}^{\infty} \frac{(\beta - \alpha)_k (-\delta_1)_k}{(1 - \alpha - \delta_1)_k k!}. \tag{5.4}$$

By making use of the operators (5.1), (5.2), (5.3) and (5.4), we find the following operator identities:

**Lemma 1.** *Each of the following operator identities holds true.*

$$\begin{aligned} & F(b_1, b_2; \beta; x) F(c_1, c_2; \gamma; y) \\ &= \nabla_{x,y}(\alpha_1) \nabla_{x,y}(\alpha_2) F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, \alpha_1, \alpha_2; \quad c_1, c_2, \alpha_1, \alpha_2; \\ \alpha_1, \alpha_2 : \quad \beta; \quad \gamma; \end{array} x, y \right]; \end{aligned} \tag{5.5}$$

$$\begin{aligned} & {}_3F_2(b_1, b_2, b_3; \alpha_1, \beta_1; x) {}_3F_2(c_1, c_2, c_3; \alpha_2, \gamma_1; y) \\ &= \nabla_{x,y}(\alpha_1) \nabla_{x,y}(\alpha_2) F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, \alpha_2; \quad c_1, c_2, c_3, \alpha_1; \\ \alpha_1, \alpha_2 : \quad \beta_1; \quad \gamma_1; \end{array} x, y \right]; \end{aligned} \tag{5.6}$$

$$\begin{aligned} & {}_4F_3(b_1, b_2, b_3, b_4; \alpha_1, \alpha_2, \beta_1; x) {}_4F_3(c_1, c_2, c_3, c_4; \alpha_1, \alpha_2, \gamma_1; y) \\ &= \nabla_{x,y}(\alpha_1) \nabla_{x,y}(\alpha_2) F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; \quad c_1, c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \quad \beta_1; \quad \gamma_1; \end{array} x, y \right]; \end{aligned} \tag{5.7}$$

$$\begin{aligned} & F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; \quad c_1, c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \quad \beta_1; \quad \gamma_1; \end{array} x, y \right] = \Delta_{x,y}(\alpha_1) \Delta_{x,y}(\alpha_2) \\ & \quad \cdot {}_4F_3(b_1, b_2, b_3, b_4; \alpha_1, \alpha_2, \beta_1; x) {}_4F_3(c_1, c_2, c_3, c_4; \alpha_1, \alpha_2, \gamma_1; y); \end{aligned} \tag{5.8}$$

$$\begin{aligned} & F_{1:1;1}^{0:3;3} \left[ \begin{array}{c} - : b_2, b_3, b_4; \quad c_2, c_3, c_4; \\ \alpha_2 : \quad \beta; \quad \gamma; \end{array} x, y \right] \\ &= \nabla_{x,y}(\alpha_1) F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : \alpha_1, b_2, b_3, b_4; \quad \alpha_1, c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \quad \beta; \quad \gamma; \end{array} x, y \right]; \end{aligned} \tag{5.9}$$

$$\begin{aligned} & F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; \quad c_1, c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \quad \beta; \quad \gamma; \end{array} x, y \right] \\ &= H_x(b_1, \beta) F_{2:0;1}^{0:3;3} \left[ \begin{array}{c} - : b_2, b_3, b_4; \quad c_1, c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \quad -; \quad \gamma; \end{array} x, y \right]; \end{aligned} \tag{5.10}$$

$$\begin{aligned}
& F_{2:0;1}^{0:3;3} \left[ \begin{array}{c} - : b_2, b_3, b_4; c_1, c_2, c_3, c_4; x, y \\ \alpha_1, \alpha_2 : \quad \quad \quad -; \quad \quad \quad \gamma; \end{array} \right] \\
&= \bar{H}_x(b_1, \beta) F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; x, y \\ \alpha_1, \alpha_2 : \quad \quad \quad \beta; \quad \quad \quad \gamma; \end{array} \right]; \tag{5.11}
\end{aligned}$$

$$\begin{aligned}
& F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; x, y \\ \alpha_1, \alpha_2 : \quad \quad \quad \beta; \quad \quad \quad \gamma; \end{array} \right] \\
&= H_x(b_1, \beta) H_y(c_1, \gamma) F_{2:0;0}^{0:3;3} \left[ \begin{array}{c} - : b_2, b_3, b_4; c_2, c_3, c_4; x, y \\ \alpha_1, \alpha_2 : \quad \quad \quad -; \quad \quad \quad -; \end{array} \right]; \tag{5.12}
\end{aligned}$$

$$\begin{aligned}
& F_{2:0;0}^{0:3;3} \left[ \begin{array}{c} - : b_2, b_3, b_4; c_2, c_3, c_4; x, y \\ \alpha_1, \alpha_2 : \quad \quad \quad -; \quad \quad \quad -; \end{array} \right] \\
&= \bar{H}_x(b_1, \beta) \bar{H}_y(c_1, \gamma) F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; x, y \\ \alpha_1, \alpha_2 : \quad \quad \quad \beta; \quad \quad \quad \gamma; \end{array} \right]. \tag{5.13}
\end{aligned}$$

The operator identities in Lemma 1 can be proved by using the known Mellin transformations (see [14]). Furthermore, by carrying out the operator identities (5.5) to (5.13) respectively, we find some decomposition formulas as in the following theorem.

**Theorem 3.** *Each of the following decomposition formulas holds true.*

$$\begin{aligned}
& F(b_1, b_2; \beta; x) F(c_1, c_2; \gamma; y) \\
&= \sum_{i,j=0}^{\infty} \frac{[(\alpha_1)_{i+j}]^2 (\alpha_2)_{i+j} (b_1)_{i+j} (b_2)_{i+j} (c_1)_{i+j} (c_2)_{i+j}}{(\alpha_1)_i (\alpha_1)_{2i+2j} (\alpha_2)_{i+2j} (\beta)_{i+j} (\gamma)_{i+j} i! j!} x^{i+j} y^{i+j} \\
&\cdot F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1 + i + j, b_2 + i + j, \alpha_1 + i + j, \\ \alpha_1 + 2i + 2j, \alpha_2 + i + 2j : \\ \alpha_2 + i + j; c_1 + i + j, c_2 + i + j, \alpha_1 + i + j, \alpha_2 + i + j; \\ \beta + i + j; \quad \quad \quad \gamma + i + j; \end{array} x, y \right]; \tag{5.14}
\end{aligned}$$

$$\begin{aligned}
& {}_3F_2(b_1, b_2, b_3; \alpha_1, \beta_1; x) {}_3F_2(c_1, c_2, c_3; \alpha_2, \gamma_1; y) \\
&= \sum_{i,j=0}^{\infty} \frac{(\alpha_1)_{i+j} (b_1)_{i+j} (b_2)_{i+j} (b_3)_{i+j} (c_1)_{i+j} (c_2)_{i+j} (c_3)_{i+j}}{(\alpha_1)_i (\alpha_1)_{2i+2j} (\alpha_2)_{i+2j} (\beta_1)_{i+j} (\gamma_1)_{i+j} i! j!} x^{i+j} y^{i+j} \\
&\cdot F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1 + i + j, b_2 + i + j, b_3 + i + j, \\ \alpha_1 + 2i + 2j, \alpha_2 + i + 2j : \\ \alpha_2 + i + j; c_1 + i + j, c_2 + i + j, c_3 + i + j, \alpha_1 + i + j; \\ \beta_1 + i + j; \quad \quad \quad \gamma_1 + i + j; \end{array} x, y \right]; \tag{5.15}
\end{aligned}$$

$$\begin{aligned}
 & {}_4F_3(b_1, b_2, b_3, b_4; \alpha_1, \alpha_2, \beta_1; x) {}_4F_3(c_1, c_2, c_3, c_4; \alpha_1, \alpha_2, \gamma_1; y) \\
 &= \sum_{i,j=0}^{\infty} \frac{(b_1)_{i+j} (b_2)_{i+j} (b_3)_{i+j} (b_4)_{i+j} (c_1)_{i+j} (c_2)_{i+j} (c_3)_{i+j} (c_4)_{i+j} x^{i+j} y^{i+j}}{(\alpha_1)_i (\alpha_1)_{2i+2j} (\alpha_2)_{i+j} (\alpha_2)_{i+2j} (\beta_1)_{i+j} (\gamma_1)_{i+j} i! j!} \\
 &\cdot F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1 + i + j, b_2 + i + j, b_3 + i + j, \\ \alpha_1 + 2i + 2j, \alpha_2 + i + 2j : \\ b_4 + i + j; c_1 + i + j, c_2 + i + j, c_3 + i + j, c_4 + i + j; \\ \beta_1 + i + j; \gamma_1 + i + j; \end{array} x, y \right]; \tag{5.16}
 \end{aligned}$$

$$\begin{aligned}
 & F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \beta_1; \gamma_1; \end{array} x, y \right] \\
 &= \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} (\alpha_1)_{2i} (\alpha_2)_{2i+2j} (b_1)_{i+j} (b_2)_{i+j} (b_3)_{i+j}}{(\alpha_1 + i - 1)_i (\alpha_2 + 2i + j - 1)_j [(\alpha_1)_{2i+2j}]^2} \\
 &\cdot \frac{(b_4)_{i+j} (c_1)_{i+j} (c_2)_{i+j} (c_3)_{i+j} (c_4)_{i+j} x^{i+j} y^{i+j}}{[(\alpha_2)_{2i+2j}]^2 (\beta_1)_{i+j} (\gamma_1)_{i+j} i! j!} \\
 &\cdot {}_4F_3 \left[ \begin{array}{c} b_1 + i + j, b_2 + i + j, b_3 + i + j, b_4 + i + j; \\ \alpha_1 + 2i + j, \alpha_2 + 2i + 2j, \beta_1 + i + j; \end{array} x \right] \\
 &\cdot {}_4F_3 \left[ \begin{array}{c} c_1 + i + j, c_2 + i + j, c_3 + i + j, c_4 + i + j; \\ \alpha_1 + 2i + j, \alpha_2 + 2i + 2j, \gamma_1 + i + j; \end{array} y \right]; \tag{5.17}
 \end{aligned}$$

$$\begin{aligned}
 & F_{1:1;1}^{0:3;3} \left[ \begin{array}{c} - : b_2, b_3, b_4; c_2, c_3, c_4; \\ \alpha_2 : \beta; \gamma; \end{array} x, y \right] \\
 &= \sum_{i=0}^{\infty} \frac{(\alpha_1)_i (b_2)_i (b_3)_i (b_4)_i (c_2)_i (c_3)_i (c_4)_i x^i y^i}{(\alpha_1)_{2i} (\alpha_2)_{2i} (\beta)_i (\gamma)_i i!} \\
 &\cdot F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : \alpha_1 + i, b_2 + i, b_3 + i, b_4 + i; \\ \alpha_1 + 2i, \alpha_2 + 2i : \beta + i; \\ \alpha_1 + i, c_2 + i, c_3 + i, c_4 + i; \\ \gamma_1 + i; \end{array} x, y \right]; \tag{5.18}
 \end{aligned}$$

$$\begin{aligned}
 & F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \beta; \gamma; \end{array} x, y \right] \\
 &= \sum_{i=0}^{\infty} \frac{(-1)^i (\beta - b_1)_i (b_2)_i (b_3)_i (b_4)_i x^i}{(\alpha_1)_i (\alpha_2)_i (\beta)_i i!} \\
 &\cdot F_{2:0;1}^{0:3;4} \left[ \begin{array}{c} - : b_2 + i, b_3 + i, b_4 + i; c_1, c_2, c_3, c_4; \\ \alpha_1 + i, \alpha_2 + i : -; \gamma; \end{array} x, y \right]; \tag{5.19}
 \end{aligned}$$



$$\begin{aligned}
& F_{2:0;1}^{0:3;4} \left[ \begin{array}{c} - : b_2, b_3, b_4; c_1, c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \quad \quad \quad -; \quad \quad \quad \gamma; \end{array} x, y \right] \\
&= \sum_{i=0}^{\infty} \frac{(\beta - b_1)_i (b_2)_i (b_3)_i (b_4)_i}{(\alpha_1)_i (\alpha_2)_i (\beta)_i i!} x^i \quad (5.20)
\end{aligned}$$

$$\begin{aligned}
& \cdot F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2 + i, b_3 + i, b_4 + i; c_1, c_2, c_3, c_4; \\ \alpha_1 + i, \alpha_2 + i : \quad \quad \quad \beta + i; \quad \quad \quad \gamma; \end{array} x, y \right]; \\
& F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \quad \quad \quad \beta; \quad \quad \quad \gamma; \end{array} x, y \right] \\
&= \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} (\beta - b_1)_i (\gamma - c_1)_j (b_2)_i (b_3)_i (b_4)_i (c_2)_j (c_3)_j (c_4)_j}{(\alpha_1)_{i+j} (\alpha_2)_{i+j} (\beta)_i (\gamma)_j i! j!} x^i y^j \\
& \cdot F_{2:0;0}^{0:3;3} \left[ \begin{array}{c} - : b_2 + i, b_3 + i, b_4 + i; \\ \alpha_1 + i + j, \alpha_2 + i + j : \quad \quad \quad -; \\ \quad \quad \quad \quad \quad \quad \quad c_2 + j, c_3 + j, c_4 + j; \\ \quad \quad \quad \quad \quad \quad \quad -; \end{array} x, y \right]; \quad (5.21)
\end{aligned}$$

$$\begin{aligned}
& F_{2:0;0}^{0:3;3} \left[ \begin{array}{c} - : b_2, b_3, b_4; c_2, c_3, c_4; \\ \alpha_1, \alpha_2 : \quad \quad \quad -; \quad \quad \quad -; \end{array} x, y \right] \\
&= \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j} (\beta - b_1)_i (\gamma - c_1)_j (b_2)_i (b_3)_i (b_4)_i (c_2)_j (c_3)_j (c_4)_j}{(\alpha_1)_{i+j} (\alpha_2)_{i+j} (\beta)_i (\gamma)_j i! j!} x^i y^j \\
& \cdot F_{2:1;1}^{0:4;4} \left[ \begin{array}{c} - : b_1, b_2 + i, b_3 + i, b_4 + i; \\ \alpha_1 + i + j, \alpha_2 + i + j : \quad \quad \quad \beta + i; \\ \quad \quad \quad \quad \quad \quad \quad c_1, c_2 + j, c_3 + j, c_4 + j; \\ \quad \quad \quad \quad \quad \quad \quad \gamma + j; \end{array} x, y \right]. \quad (5.22)
\end{aligned}$$

Note that certain decomposition formulas for multiple hypergeometric functions have been presented in [11–13].

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