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Endpoint boundedness for multilinear commutators of Littlewood-Paley operator

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ENDPOINT BOUNDEDNESS FOR MULTILINEAR COMMUTATORS OF LITTLEWOOD-PALEY OPERATORS

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Abstract. In this paper, we prove endpoint boundedness of multilinear commutators of Littlewood-Paley operators.

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1. INTRODUCTION AND NOTATIONS

Let $b \in BMO(R^n)$ and let T be the Calderón-Zygmund operator. The commutator $[b, T]$ of b and T is defined by

$$[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberb and Weiss (see [3]) states that the commutator $[b, T]$ is bounded on $L^p(R^n)$, ($1 < p < \infty$). In [2] and [5], boundedness properties of the commutators for the extreme values of p are obtained. In this paper, we will introduce the multilinear commutators of Littlewood-Paley operators and prove boundedness properties of the operators in the extreme cases.

First let us introduce some notations (see [1],[4],[8],[9],[10]). Throughout this paper, Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and for a locally integrable function f , let $f_Q = |Q|^{-1} \int_Q f(x)dx$ and $f^\#(x) = \sup_{y \in Q} |Q|^{-1} \int_Q |f(y) - f_Q|dy$. Moreover, f is said to belong to $BMO(R^n)$ if $f^\# \in L^\infty$ and define $\|f\|_{BMO} = \|f^\#\|_{L^\infty}$. We also define the central BMO space by $CMO(R^n)$, which is the space of those functions $f \in L_{loc}(R^n)$ such that

$$\|f\|_{CMO} = \sup_{r>1} |Q(0,r)|^{-1} \int_Q |f(y) - f_Q|dy < \infty.$$

It has been known that(see [9])

$$\|f\|_{CMO} = \sup_{r>1} |Q(0,r)|^{-1} \int_Q |f(y) - f_Q|dy < \infty.$$

Also, we give the concepts of the atom and H^1 space. A function a is called as a H^1 atom if there exists a cube Q such that a is supported on Q , $\|a\|_{L^\infty} \leq |Q|^{-1}$ and $\int a(x)dx = 0$. It is well known that the Hardy space $H^1(\mathbb{R}^n)$ can be characterized in terms of the atomic decomposition (see [4], [9]).

Definition 1. Let $0 < \delta < n$ and $1 < p < n/\delta$. We shall call $B_p^\delta(\mathbb{R}^n)$ the space of those functions f on \mathbb{R}^n such that

$$\|f\|_{B_p^\delta} = \sup_{r>1} r^{-n(1/p-\delta/n)} \|f\chi_{Q(0,r)}\|_{L^p} < \infty.$$

Definition 2. Let $\varepsilon > 0$, $n > \delta > 0$ and let ψ be a fixed function that satisfies the following properties:

- 1) $\int \psi(x)dx = 0$,
- 2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$,
- 3) $|\psi(x + y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$;

We denote $\Gamma(x) = \{(y, t) \in \mathbb{R}_+^{n+1} : |x - y| < t\}$ and the characteristic function of $\Gamma(x)$ is written as $\chi_{\Gamma(x)}$. Let $\vec{b} = (b_1, \dots, b_m)$ with $b_j \in BMO(\mathbb{R}^n)$ for $1 \leq j \leq m$. Then the multilinear commutator of our Littlewood-Paley operator is defined by

$$S_\delta^{\vec{b}}(f)(x) = \left[\int \int_{\Gamma(x)} |F_t^{\vec{b}}(f)(x, y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x, y) = \int_{\mathbb{R}^n} \prod_{j=1}^m (b_j(x) - b_j(z)) \psi_t(y - z) f(z) dz;$$

When $m = 1$, set

$$S_\delta^b(f)(x) = \left[\int \int_{\Gamma(x)} |F_t^b(f)(x, y)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2},$$

where

$$F_t^b(f)(x, y) = \int_{\mathbb{R}^n} (b(x) - b(z)) \psi_t(y - z) f(z) dz$$

and $\psi_t(x) = t^{-n+\delta} \psi(x/t)$ for $t > 0$. Set $F_t(f)(x) = f * \psi_t(x)$, we also define

$$S_\delta(f)(x) = \left[\int \int_{\Gamma(x)} |f * \psi_t(x)|^2 \frac{dydt}{t^{n+1}} \right]^{1/2},$$

which is the Littlewood-Paley operator (see [1][6, 7][10]).

Let H be the Hilbert space $H = \{h : \|h\| = (\int_{\mathbb{R}_+^{n+1}} |h(y, t)|^2 dydt / t^{n+1})^{1/2} < \infty\}$. Then for each fixed $x \in \mathbb{R}^n$, $F_t(f)(x, y)$ may be viewed as a mapping from $[0, +\infty)$ to H . It is clear that

$$S_\delta(f)(x) = \|\chi_{\Gamma(x)} F_t(f)(y)\| \quad \text{and} \quad S_\delta^{\vec{b}}(f)(x) = \|\chi_{\Gamma(x)} F_t^{\vec{b}}(f)(y)\|.$$

Given a positive integer m and $1 \leq j \leq m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \dots, \sigma(j)\}$ of $\{1, \dots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \dots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \dots, b_m)$ and $\sigma = \{\sigma(1), \dots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \dots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and set $\|\vec{b}_\sigma\|_{BMO} = \|b_{\sigma(1)}\|_{BMO} \cdots \|b_{\sigma(j)}\|_{BMO}$.

2. THEOREMS AND PROOFS

We begin with a few preliminary lemmas.

Lemma 1. *Let $1 < r < \infty$, $b_j \in BMO(\mathbb{R}^n)$ for $j = 1, \dots, k$ and $k \in \mathbb{N}$. Then we have*

$$\frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy \leq C \prod_{j=1}^k \|b_j\|_{BMO}$$

and

$$\left(\frac{1}{|Q|} \int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^r dy \right)^{1/r} \leq C \prod_{j=1}^k \|b_j\|_{BMO}.$$

Lemma 2. *Let $0 < \delta < n$, $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then S_δ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.*

Theorem 1. *Let $0 < \delta < n$ and $\vec{b} = (b_1, \dots, b_m)$ with $b_j \in BMO(\mathbb{R}^n)$ for $1 \leq j \leq m$. Then $S_\delta^{\vec{b}}$ is bounded from $L^{n/\delta}(\mathbb{R}^n)$ to $BMO(\mathbb{R}^n)$.*

Proof of Theorem 1. It is necessary only to prove that there exist a constant C_Q such that

$$\frac{1}{|Q|} \int_Q |S_\delta^{\vec{b}}(f)(x) - C_Q| dx \leq C \|f\|_{L^{n/\delta}}.$$

Fix a cube $Q = Q(x_0, r)$, we decompose f into $f = f_1 + f_2$ with $f_1 = f\chi_Q$, $f_2 = f\chi_{(\mathbb{R}^n \setminus Q)}$.

When $m = 1$, set $(b_1)_Q = |Q|^{-1} \int_Q b_1(y) dy$, we have

$$\begin{aligned} & F_t^{b_1}(f)(x, y) \\ &= (b_1(x) - (b_1)_Q) F_t(f)(y) - F_t((b_1 - (b_1)_Q) f_1)(y) - F_t((b_1 - (b_1)_Q) f_2)(y), \end{aligned}$$

so

$$\begin{aligned} & |S_\delta^{b_1}(f)(x) - S_\delta(((b_1)_Q - b_1) f_2)(x_0)| \\ &= \left| \|\chi_{\Gamma(x)} F_t^{b_1}(f)(x, y)\| - \|\chi_{\Gamma(x_0)} F_t(((b_1)_Q - b_1) f_2)(y)\| \right| \\ &\leq \|\chi_{\Gamma(x)} F_t^{b_1}(f)(x, y) - \chi_{\Gamma(x_0)} F_t(((b_1)_Q - b_1) f_2)(y)\| \\ &\leq \|\chi_{\Gamma(x)} (b_1(x) - (b_1)_Q) F_t(f)(y)\| + \|\chi_{\Gamma(x)} F_t(((b_1)_Q - b_1) f_1)(y)\| \end{aligned}$$

$$\begin{aligned}
& + \|\chi_{\Gamma(x)} F_t((b_1 - (b_1)_Q) f_2)(y) - \chi_{\Gamma(x_0)} F_t((b_1 - (b_1)_Q) f_2)(y)\| \\
& = A(x) + B(x) + C(x).
\end{aligned}$$

For $A(x)$, set $1 < p < n/\delta$, $1/q = 1/p - \delta/n$ and $1/q + 1/q' = 1$. By Hölder's inequality and Lemma 1.2, we have

$$\begin{aligned}
\frac{1}{|Q|} \int_Q |A(x)| dx & \leq \left(\frac{1}{|Q|} \int_Q |b_1(x) - (b_1)_Q|^{q'} dx \right)^{1/q'} \\
& \quad \left(\frac{1}{|Q|} \int_{R^n} |S_\delta(f)(x)|^q \chi_Q(x) dx \right)^{1/q} \\
& \leq C \|b_1\|_{BMO} \frac{1}{|Q|^q} \left(\int_{R^n} |f(x)|^p \chi_Q(x) dx \right)^{1/p} \\
& \leq C \|b_1\|_{BMO} \frac{1}{|Q|^q} \|f\|_{L^{n/\delta}} |Q|^{(1-\delta p/n)p} \\
& \leq C \|b_1\|_{BMO} \|f\|_{L^{n/\delta}}.
\end{aligned}$$

For $B(x)$, take $1 < r < n/\delta$ and $1/s = 1/r - \delta/n$, then by Hölder's inequality we have

$$\begin{aligned}
\frac{1}{|Q|} \int_Q |B(x)| dx & \leq \left(\frac{1}{|Q|} \int_{R^n} (S_\delta((b_1(x) - (b_1)_Q) f_1)(x))^s dx \right)^{1/s} \\
& \leq C |Q|^{-1/s} \|(b_1(x) - (b_1)_Q) f \chi_Q\|_{L^r} \\
& \leq C \left(\frac{1}{|Q|} \int_Q |b_1(x) - (b_1)_Q|^s dx \right)^{1/s} \|f\|_{L^{n/\delta}} \\
& \leq C \|b_1\|_{BMO} \|f\|_{L^{n/\delta}}.
\end{aligned}$$

For $C(x)$, we have

$$\begin{aligned}
& C(x) \\
& \leq \left[\int \int_{R_+^{n+1}} \left(\int_{Q^c} |\chi_{\Gamma(x)} - \chi_{\Gamma(x_0)}| |b_1(z) - (b_1)_Q| |\psi_t(y-z)| |f(z)| dz \right)^2 \frac{dy dt}{t^{n+1}} \right]^{1/2} \\
& \leq C \int_{Q^c} |b_1(z) - (b_1)_Q| |f(z)| \int \int_{|x-y| \leq t} \frac{t^{1-n} dy dt}{(t + |y-z|)^{2n+2-2\delta}} \\
& \quad - \int \int_{|x_0-y| \leq t} \frac{t^{1-n} dy dt}{(t + |y-z|)^{2n+2-2\delta}} |f(z)|^{1/2} dz \\
& \leq C \int_{Q^c} |b_1(z) - (b_1)_Q| |f(z)|
\end{aligned}$$

$$\begin{aligned} & \times \left(\iint_{|y| \leq t, |x+y-z| \leq t} \left| \frac{1}{(t+|x+y-z|)^{2n+2-2\delta}} \right. \right. \\ & \quad \left. \left. - \frac{1}{(t+|x_0+y-z|)^{2n+2-2\delta}} \left| \frac{dydt}{t^{n-1}} \right| \right)^{1/2} dz \\ & \leq \int_{Q^c} |b_1(z) - (b_1)_Q| |f(z)| \end{aligned}$$

$$\left(\iint_{|y| \leq t, |x+y-z| \leq t} \frac{|x-x_0|t^{1-n}}{(t+|x+y-z|)^{2n+3-2\delta}} dydt \right)^{1/2} dz,$$

note that $2t + |x + y - z| \geq 2t + |x - z| - |y| \geq t + |x - z|$ when $|y| \leq t$ and

$$\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+3-2\delta}} = C|x-z|^{-2n-1+2\delta}.$$

Then, for $x \in Q$,

$$\begin{aligned} & C(x) \\ & \leq C \int_{Q^c} |b_1(z) - (b_1)_Q| |f(z)| |x-x_0|^{1/2} \\ & \quad \left(\iint_{|y| \leq t} \frac{t^{1-n} dydt}{(t+|x-z|)^{2n+3-2\delta}} \right)^{1/2} dz \\ & \leq C \int_{Q^c} |b_1(z) - (b_1)_Q| |f(z)| |x-x_0|^{1/2} \\ & \quad \left(\int_0^\infty \frac{tdt}{(t+|x-z|)^{2n+3-2\delta}} \right)^{1/2} dz \\ & \leq C \int_{Q^c} |b_1(z) - (b_1)_Q| |f(z)| \frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2-\delta}} dz \\ & \leq C \sum_{k=0}^\infty \int_{2^{k+1}Q \setminus 2^kQ} |b_1(z) - (b_1)_Q| |f(z)| \frac{|x_0-x|^{1/2}}{|x_0-z|^{n+1/2-\delta}} dz \\ & \leq C \sum_{k=1}^\infty 2^{-k/2} \left(\frac{1}{|2^kQ|} \int_{2^kQ} |b_1(z) - (b_1)_Q|^{n/(n-\delta)} dz \right)^{(n-\delta)/n} \\ & \quad \left(\int_{2^kQ} |f(z)|^{n/\delta} dz \right)^{\delta/n} \\ & \leq C \sum_{k=1}^\infty k 2^{-k/2} \|b_1\|_{BMO} \|f\|_{L^{n/\delta}} \\ & \leq C \|b_1\|_{BMO} \|f\|_{L^{n/\delta}}. \end{aligned}$$

When $m > 1$, set $\vec{b}_Q = ((b_1)_Q, \dots, (b_m)_Q) \in R^n$, where $(b_j)_Q = |Q|^{-1} \int_Q b_j(y) dy$, $1 \leq j \leq m$. Then we have

$$\begin{aligned} F_t^{\vec{b}}(f)(x, y) &= (b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q) F_t(f)(y) \\ &\quad + (-1)^m F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q) f)(y) \\ &\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (\vec{b}(x) - \vec{b}_Q)_\sigma \int_{R^n} (\vec{b}(z) - \vec{b}_Q)_{\sigma^c} \psi_t(y-z) f(z) dz \\ &= (b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q) F_t(f)(y) \\ &\quad + (-1)^m F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q) f_1)(y) \\ &\quad + (-1)^m F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q) f_2)(y) \\ &\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (\vec{b}(x) - \vec{b}_Q)_\sigma F_t((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x, y), \end{aligned}$$

thus

$$\begin{aligned} &|S_\delta^{\vec{b}}(f)(x) - S_\delta(((b_1)_Q - b_1) \cdots ((b_m)_Q - b_m) f_2)(x_0)| \\ &\leq \|\chi_{\Gamma(x)} F_t^{\vec{b}}(f)(x, y) - \chi_{\Gamma(x_0)} F_t(((b_1)_Q - b_1) \cdots ((b_m)_Q - b_m) f_2)(y)\| \\ &\leq \|\chi_{\Gamma(x)} (b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q) F_t(f)(y)\| \\ &\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\chi_{\Gamma(x)} (\vec{b}(x) - \vec{b}_Q)_\sigma F_t((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x, y)\| \\ &\quad + \|\chi_{\Gamma(x)} F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q) f_1)(y)\| \\ &\quad + \|\chi_{\Gamma(x)} F_t(\prod_{j=1}^m (b_j - (b_j)_Q) f_2)(y) - \chi_{\Gamma(x_0)} F_t(\prod_{j=1}^m (b_j - (b_j)_Q) f_2)(y)\| \\ &= I_1(x) + I_2(x) + I_3(x) + I_4(x). \end{aligned}$$

For $I_1(x)$, by taking $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$, and by Hölder's inequality and Lemma 1.2, we have

$$\begin{aligned} &\frac{1}{|Q|} \int_Q I_1(x) dx \\ &\leq \left(\frac{1}{|Q|} \int_Q \left| \prod_{j=1}^m (b_j(x) - (b_j)_Q) \right|^{q'} dx \right)^{1/q'} \left(\frac{1}{|Q|} \int_Q |S_\delta(f)(x)|^q dx \right)^{1/q} \\ &\leq C \|\vec{b}\|_{BMO} |Q|^{-1/q} \left(\int_Q |f(x)|^p dx \right)^{1/p} \leq C \|\vec{b}\|_{BMO} \|f\|_{L^{n/\delta}}. \end{aligned}$$

For $I_2(x)$, if we take $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$, then

$$\begin{aligned} & \frac{1}{|Q|} \int_Q I_2(x) dx \\ & \leq \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \frac{1}{|Q|} \int_Q |(b(\vec{x}) - \vec{b}_Q)_\sigma| |S_\delta((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)| dx \\ & \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left(\frac{1}{|Q|} \int_Q |(b(\vec{x}) - \vec{b}_Q)_\sigma|^{q'} dx \right)^{1/q'} \\ & \quad \left(\frac{1}{|Q|} \int_Q |S_\delta((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)|^q dx \right)^{1/q} \\ & \leq C \sum_{j=1}^{m-1} \|\vec{b}_\sigma\|_{BMO} |Q|^{1/q} \left(\int_{R^n} |(b(\vec{x}) - \vec{b}_Q)_{\sigma^c} f(x)|^p \chi_Q dx \right)^{1/q} \\ & \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} \left(\frac{1}{|Q|} \int_Q |(b(\vec{x}) - \vec{b}_Q)_\sigma|^q dx \right)^{1/q} \|f\|_{L^{n/\delta}} \\ & \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} \|\vec{b}_{\sigma^c}\|_{BMO} \|f\|_{L^{n/\delta}} \\ & \leq C \|\vec{b}\|_{BMO} \|f\|_{L^{n/\delta}}. \end{aligned}$$

For $I_3(x)$, take $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$, so we get

$$\begin{aligned} & \frac{1}{|Q|} \int_Q I_3(x) dx \\ & \leq \left(\frac{1}{|Q|} \int_Q |S_\delta((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q) f_1)(x)|^q dx \right)^{1/q} \\ & \leq C |Q|^{-1/q} \|((b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q) f_1(x)\|_{L^p} \\ & \leq C \left(\frac{1}{|Q|} \int_Q |(b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)|^q dx \right)^{1/q} \|f\|_{L^{n/\delta}} \\ & \leq C \|\vec{b}\|_{BMO} \|f\|_{L^{n/\delta}}. \end{aligned}$$

For $I_4(x)$, similiary as in the proof of $C(x)$ in Case $m = 1$, we have

$$\begin{aligned} I_4(x) &\leq C \int_{Q^c} |x_0 - x|^{1/2} |x_0 - z|^{-(n+1/2-\delta)} \left| \prod_{j=1}^m (b_j(z) - (b_j)_Q) \right| |f(z)| dz \\ &\leq C \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} |x_0 - x|^{1/2} |x_0 - z|^{-(n+1/2-\delta)} \left| \prod_{j=1}^m (b_j(z) - (b_j)_Q) \right| |f(z)| dz \\ &\leq C \sum_{k=0}^{\infty} 2^{-k/2} \frac{1}{|2^{k+1}Q|^{1-\delta/n}} \int_{2^{k+1}Q} \left| \prod_{j=1}^m (b_j(z) - (b_j)_Q) \right| |f(z)| dz \\ &\leq C \sum_{k=1}^{\infty} 2^{-k/2} \left(\frac{1}{|2^kQ|} \int_{2^kQ} \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) \right|^{n/(n-\delta)} dy \right)^{(n-\delta)/n} \|f\|_{L^{n/\delta}} \\ &\leq C \|\vec{b}\|_{BMO} \|f\|_{L^{n/\delta}}. \end{aligned}$$

This completes the proof of Theorem 1. □

Theorem 2. Let $0 < \delta < n$, $1 < p < n/\delta$ and $\vec{b} = (b_1, \dots, b_m)$ with $b_j \in BMO(R^n)$ for $1 \leq j \leq m$. Then $S_\delta^{\vec{b}}$ is bounded from $B_p^\delta(R^n)$ to $CMO(R^n)$.

Proof of Theorem 2. It suffices to prove that there exist a constant C_Q , such that

$$\frac{1}{|Q|} \int_Q |S_\delta^{\vec{b}}(f)(x) - C_Q| dx \leq C \|f\|_{B_p^\delta}$$

holds for any cube $Q = Q(0, r)$ with $r > 1$. Fix a cube $Q = Q(0, r)$ with $r > 1$. Set $f_1 = f \chi_Q$, $f_2 = f \chi_{R^n \setminus Q}$ and $\vec{b}_Q = ((b_1)_Q, \dots, (b_m)_Q) \in R^n$, where $(b_j)_Q = |Q|^{-1} \int_Q |b_j(y)| dy$, $1 \leq j \leq m$. Then we have

$$\begin{aligned} &|S_\delta^{\vec{b}}(f)(x) - S_\delta^{\vec{b}}(((b_1)_Q - b_1) \cdots ((b_m)_Q - b_m) f_2)(x_0)| \\ &\leq \|\chi_{\Gamma(x)}(b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q) F_t(f)(y)\| \\ &+ \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^n} \|\chi_{\Gamma(x)}(\vec{b}(x) - \vec{b}_Q)_\sigma F_t((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x, y)\| \\ &+ \|\chi_{\Gamma(x)} F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q) f_1)(y)\| \\ &+ \|\chi_{\Gamma(x)} F_t(\prod_{j=1}^m (b_j - (b_j)_Q) f_2)(y) - \chi_{\Gamma(x_0)} F_t(\prod_{j=1}^m (b_j - (b_j)_Q) f_2)(y)\| \\ &= H_1(x) + H_2(x) + H_3(x) + H_4(x). \end{aligned}$$

For $H_1(x)$, we take $1 < p < n/\delta$, $1/s = 1/r - \delta/n$. By Hölder's inequality and Lemma 1,2, we have

$$\begin{aligned} & \frac{1}{|Q|} \int_Q H_1(x) dx \\ & \leq \left(\frac{1}{|Q|} \int_Q \left| \prod_{j=1}^m (b_j(x) - (b_j)_Q) \right|^{q'} dx \right)^{1/q'} \left(\frac{1}{|Q|} \int_Q |S_\delta(f)(x)|^q dx \right)^{1/q} \\ & \leq C \|\vec{b}\|_{BMO} |Q|^{-1/q} \left(\int_Q |f(x)|^p dx \right)^{1/p} \\ & \leq C \|\vec{b}\|_{BMO} d^{-n(1/p - \delta/n)} \|f\chi_Q\|_{L^p} \\ & \leq C \|\vec{b}\|_{BMO} \|f\|_{B_p^\delta}. \end{aligned}$$

For $H_2(x)$, taking $1 < p < n/\delta$, $1/s = 1/r - \delta/n$, and $1/s' + 1/s = 1$, we obtain that

$$\begin{aligned} & \frac{1}{|Q|} \int_Q H_2(x) dx \\ & \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \left(\frac{1}{|Q|} \int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{s'} dx \right)^{1/s'} \\ & \quad \left(\frac{1}{|Q|} \int_Q |S_\delta((\vec{b} - \vec{b}_Q)_{\sigma^c}) f(x)|^s dx \right)^{1/s} \\ & \leq C \sum_{j=1}^{m-1} \|\vec{b}_\sigma\|_{BMO} |Q|^{-1/s} \left(\int_{R^n} |(b(x) - \vec{b}_Q)_{\sigma^c}|^r f(x)^r \chi_Q dx \right)^{1/r} \\ & \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} \left(\frac{1}{|Q|} \int_Q |(\vec{b}(x) - \vec{b}_Q)_{\sigma^c}|^{pr/(p-r)} dx \right)^{(p-r)/pr} \\ & \quad |Q|^{(\delta/n - 1/p)} \|f\chi_Q\|_{L^p} \\ & \leq C \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} \|\vec{b}_\sigma\|_{BMO} \|\vec{b}_{\sigma^c}\|_{BMO} d^{-n(1/p - \delta/n)} \|f\chi_Q\|_{L^p} \\ & \leq C \|\vec{b}\|_{BMO} \|f\|_{B_p^\delta}. \end{aligned}$$

For $H_3(x)$, taking $1 < p < n/\delta$, $1/s = 1/r - \delta/n$ and $1/s' + 1/s = 1$, we get

$$\frac{1}{|Q|} \int_Q H_3(x) dx$$

$$\begin{aligned}
&\leq \left(\frac{1}{|Q|} \int_Q |\mu_\delta((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q) f_1)(x)|^s dx \right)^{1/s} \\
&\leq C |Q|^{-1/s} \|((b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q) f \chi_Q)\|_{L^r} \\
&\leq C \left(\frac{1}{|Q|} \int_Q |(b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)|^{pr/(p-r)} dx \right)^{(p-r)/pr} \\
&\quad d^{-n(1/p-\delta/n)} \|f \chi_Q\|_{L^p} \\
&\leq C \|\vec{b}\|_{BMO} \|f\|_{B_p^\delta}.
\end{aligned}$$

For $H_4(x)$, we have

$$\begin{aligned}
&H_4(x) \\
&\leq \left[\int \int_{R_+^{n+1}} \left(\int_{Q^c} |\chi_{\Gamma(x)} - \chi_{\Gamma(x_0)}| \right. \right. \\
&\quad \left. \left. \prod_{j=1}^m |b_j(z) - (b_j)_Q| |\psi_t(y-z)| |f(z)| dz \right)^2 \frac{dy dt}{t^{n+1}} \right]^{1/2} \\
&\leq C \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} |x_0 - x|^{1/2} |x_0 - z|^{-(n+1/2-\delta)} \left| \prod_{j=1}^m (b_j(z) - (b_j)_Q) \right| |f(z)| dz \\
&\leq C \sum_{k=1}^{\infty} 2^{-k/2} \frac{1}{|2^k Q|^{1-\delta/n}} \int_{2^k Q} \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) \right| |f(y)| dy \\
&\leq C \sum_{k=1}^{\infty} 2^{-k/2} \frac{1}{|2^k Q|^{1-\delta/n}} \left(\int_{2^k Q} \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) \right|^{p/(p-1)} dy \right)^{(p-1)/p} \\
&\quad \times \left(\int_{2^k Q} |f(y)|^p dy \right)^{1/p} \\
&\leq C \sum_{k=1}^{\infty} 2^{-k/2} \left(\frac{1}{|2^k Q|} \int_{2^k Q} \left| \prod_{j=1}^m (b_j(y) - (b_j)_{2Q}) \right|^{p/(p-1)} dy \right)^{(p-1)/p} \\
&\quad \times |2^k Q|^{-(1/p-\delta/n)} \|f \chi_{2^k Q}\|_{L^p} \\
&\leq C \|\vec{b}\|_{BMO} \|f\|_{B_p^\delta}.
\end{aligned}$$

This completes the proof of Theorem 2. \square

Theorem 3. Let $0 < \delta < n$ and $\vec{b} = (b_1, \dots, b_m)$ with $b_j \in BMO(\mathbb{R}^n)$ for $1 \leq j \leq m$. Assume that the following inequality holds for any $H^1(\mathbb{R}^n)$ -atom a supported on a certain cube Q and for $u \in Q$.

$$\sum_{j=1}^m \sum_{\sigma \in C_j^m} \int_{(2Q)^c} \left[|(b(x) - b_Q)_{\sigma^c}| \left(\int \int_{\Gamma(x)} \left(\int_Q |(b(z) - b_Q)_{\sigma} \psi_t(y-u)a(z)| dz \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \right]^{n/(n-\delta)} dx \leq C$$

Then $S_{\delta}^{\vec{b}}$ is bounded from $H^1(\mathbb{R}^n)$ to $L^{n/(n-\delta)}(\mathbb{R}^n)$.

Proof of Theorem 3. Let a be an atom supported in some cube Q and $u \in Q$. We write

$$\int_{\mathbb{R}^n} |S_{\delta}^{\vec{b}}(a)(x)|^{n/(n-\delta)} dx = \int_{2Q} |S_{\delta}^{\vec{b}}(a)(x)|^{n/(n-\delta)} dx + \int_{(2Q)^c} |S_{\delta}^{\vec{b}}(a)(x)|^{n/(n-\delta)} dx = I + II.$$

For I , take $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then we have

$$I \leq \| \mu_{\delta}^{\vec{b}}(a) \|_{L^q}^{n/(n-\delta)} |2Q|^{1-n/((n-\delta)q)} \leq C \| a \|_{L^p}^{n/(n-\delta)} |Q|^{1-n/((n-\delta)q)} \leq C.$$

For II , we first calculate $F_t^{\vec{b}}(a)(x)$. When $m = 1$, we have

$$\begin{aligned} |F_t^{b_1}(a)(x, y)| &\leq \left| \int_Q \psi_t(y-z)a(z)(b_1(x) - (b_1)_Q) dz \right| \\ &\quad + \left| \int_Q (\psi_t(y-z) - \psi_t(y-u))a(z)(b_1(z) - (b_1)_Q) dz \right| \\ &\quad + \left| \int_Q \psi_t(y-u)(b_1(z) - (b_1)_Q)a(z) dz \right| \\ &= v'_1 + v'_2 + v'_3, \end{aligned}$$

so

$$\begin{aligned} S_{\delta}^{b_1}(a)(x) &= \| \chi_{\Gamma(x)} F_t^{b_1}(a)(x, y) \| \\ &\leq \left(\int \int_{\Gamma(x)} |v'_1|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} + \left(\int \int_{\Gamma(x)} |v'_2|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \\ &\quad + \left(\int \int_{\Gamma(x)} |v'_3|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \end{aligned}$$

$$= A'(x) + B'(x) + C'(x).$$

For $A'(x)$, we have

$$A'(x) \leq S_\delta(a)(x)|b_1(x) - (b_1)_Q|,$$

thus

$$\begin{aligned} & \left(\int_{(2Q)^c} (A'(x))^{n/(n-\delta)} dx \right)^{(n-\delta)/n} \\ &= \left(\int_{(2Q)^c} [|b_1(x) - (b_1)_Q| \right. \\ & \times \left. \left(\int_{\Gamma(x)} \int_Q |\int_Q (\psi_t(y-z) - \psi_t(y-u)) a(z) dz|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2}]^{n/(n-\delta)} dx \right)^{(n-\delta)/n} \\ & \leq C |Q|^{1+\varepsilon/n} \|a\|_{L^\infty} \\ & \sum_{k=1}^{\infty} \left(\int_{2^{k+1}Q} \left(\frac{|2^k Q|^{\delta/n}}{|2^k Q|^{(n+\varepsilon)/n}} |b_1(x) - (b_1)_Q| \right)^{n/(n-\delta)} dx \right)^{(n-\delta)/n} \\ & \leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |b_1(x) - (b_1)_Q|^{n/(n-\delta)} dx \right)^{(n-\delta)/n} \\ & \leq C \|b_1\|_{BMO}. \end{aligned}$$

For $B'(x)$, we have

$$\begin{aligned} & B'(x) \\ & \leq C \left(\int_{\Gamma(x)} \int_Q \left(\int_Q \frac{t|u-z|^\varepsilon}{(t+|y-u|)^{n+1+\varepsilon-\delta}} |a(z)||b(z) - (b_1)_Q| dz \right)^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \\ & \leq C \left(\int_0^\infty \frac{t dt}{(t+|x-u|)^{2(n+1+\varepsilon-\delta)}} \right)^{1/2} \int_Q |u-z|^\varepsilon |a(z)||b_1(z) - (b_1)_Q| dz \\ & \leq C \|b_1\|_{BMO} |x-u|^{-(n+\varepsilon-\delta)} |Q|^{1+\varepsilon/n} \|a\|_{L^\infty}, \end{aligned}$$

thus

$$\begin{aligned} & \left(\int_{(2Q)^c} (B'(x))^{n/(n-\delta)} dx \right)^{(n-\delta)/n} \\ & \leq C \|b_1\|_{BMO} \|a\|_{L^\infty} \left[\sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} \left(\frac{|Q|^{1+\varepsilon/n}}{|x-u|^{n+\varepsilon-\delta}} \right)^{n/(n-\delta)} dx \right]^{(n-\delta)/n} \end{aligned}$$

$$\begin{aligned} &\leq C \|b_1\|_{BMO} \sum_{k=1}^{\infty} 2^{-k\varepsilon} \\ &\leq C \|b_1\|_{BMO}. \end{aligned}$$

From that we know, if

$$\begin{aligned} &\int_{(2Q)^c} (C'(x))^{n/(n-\delta)} dx \\ &= \int_{(2Q)^c} \left[\left(\int_{\Gamma(x)} \left(\int_Q |(b_1(z) - (b_1)_Q) \right. \right. \right. \\ &\quad \left. \left. \left. \psi_t(y-u)a(z)|dz \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \right]^{n/(n-\delta)} dx \leq C, \end{aligned}$$

then

$$\int_{R^n} |S_\delta^{b_1}(a)(x)|^{n/(n-\delta)} dx \leq C.$$

When $m > 1$, we have

$$\begin{aligned} |F_t^{\vec{b}}(a)(x, y)| &\leq \left| \prod_{j=1}^m (b_j(x) - (b_j)_Q) \int_Q \psi_t(y-z)a(z)dz \right| \\ &+ \sum_{j=1}^m \sum_{\sigma \in C_j^m} \left| (\vec{b}(x) - \vec{b}_Q)_{\sigma^c} \int_Q (\psi_t(y-z) - \psi_t(y-u)) (\vec{b}(z) - \vec{b}_Q)_\sigma a(z)dz \right| \\ &+ \sum_{j=1}^m \sum_{\sigma \in C_j^m} \left| (\vec{b}(x) - \vec{b}_Q)_{\sigma^c} \int_Q \psi_t(y-u) (\vec{b}(z) - \vec{b}_Q)_\sigma a(z)dz \right| \\ &= v_1 + v_2 + v_3, \end{aligned}$$

so

$$\begin{aligned} S_\delta^{\vec{b}}(a)(x) &= \|\chi_{\Gamma(x)} F_t^{\vec{b}}(a)(x, y)\| \\ &\leq \left(\int \int_{\Gamma(x)} |v_1|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} + \left(\int \int_{\Gamma(x)} |v_2|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \\ &\quad + \left(\int \int_{\Gamma(x)} |v_3|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \\ &= A(x) + B(x) + C(x). \end{aligned}$$

For $A(x)$, we have

$$\begin{aligned} A(x) &= \left(\int \int_{\Gamma(x)} \prod_{j=1}^m |b_j(x) - (b_j)_Q|^2 \left| \int_Q \psi_t(y-z)a(z)dz \right|^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \\ &= \prod_{j=1}^m |b_j(x) - (b_j)_Q| S_\delta(a)(x), \end{aligned}$$

thus

$$\begin{aligned} & \left(\int_{(2Q)^c} (A(x))^{n/(n-\delta)} dx \right)^{(n-\delta)/n} \\ & \leq C \|a\|_{L^\infty} \\ & \left[\sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} \left(\frac{|Q|^{1+\varepsilon/n}}{|x-u|^{n+\varepsilon-\delta}} \prod_{j=1}^m |b_j(x) - (b_j)_Q| \right)^{n/(n-\delta)} dx \right]^{(n-\delta)/n} \\ & \leq C |Q|^{1+\varepsilon/n} \|a\|_{L^\infty} \\ & \sum_{k=1}^{\infty} \left(\int_{2^{k+1}Q} \left(\frac{|2^kQ|^{\delta/n}}{|2^kQ|^{(n+\varepsilon)/n}} \prod_{j=1}^m |b_j(x) - (b_j)_Q| \right)^{n/(n-\delta)} dx \right)^{(n-\delta)/n} \\ & \leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} \left(\prod_{j=1}^m |b_j(x) - (b_j)_Q| \right)^{n/(n-\delta)} dx \right)^{(n-\delta)/n} \\ & \leq C \|\vec{b}\|_{BMO}. \end{aligned}$$

For $B(x)$, we have

$$\begin{aligned} B(x) &= \left(\int \int_{\Gamma(x)} \left| \sum_{j=1}^m \sum_{\sigma \in C_j^m} (\vec{b}(x) - \vec{b}_Q)_\sigma \right. \right. \\ & \quad \times \left. \int_Q (\psi_t(y-z) - \psi_t(y-u)) (\vec{b}(z) - \vec{b}_Q)_\sigma a(z) dz \right|^2 \frac{dydt}{t^{n+1}} \Big)^{1/2} \\ & \leq C \sum_{j=1}^m \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}_Q)_\sigma| \\ & \quad \times \left(\int \int_{\Gamma(x)} \left(\int_Q \frac{t|u-z|^\varepsilon}{(t+|y-u|)^{n+1+\varepsilon-\delta}} |(\vec{b}(z) - \vec{b}_Q)_\sigma| |a(z)| dz \right)^2 \frac{dydt}{t^{n+1}} \right)^{1/2} \end{aligned}$$

$$\begin{aligned} &\leq C \sum_{j=1}^m \sum_{\sigma \in C_j^m} |(\vec{b}(x) - b_Q)_{\sigma^c}| \left(\int \int_{\Gamma(x)} \frac{t^{1-n} dy dt}{(t + |y - u|)^{2(n+1+\varepsilon-\delta)}} \right)^{1/2} \\ &\times \int_Q |u - z|^\varepsilon |a(z)| |(\vec{b}(z) - \vec{b}_Q)_\sigma| dz \\ &\leq C \sum_{j=1}^m \sum_{\sigma \in C_j^m} |(\vec{b}(x) - \vec{b}_Q)_{\sigma^c}| |x - u|^{-(n+\varepsilon)} |Q|^{1+\varepsilon/n-\delta} \|a\|_{L^\infty} \|\vec{b}_\sigma\|_{BMO}, \end{aligned}$$

thus

$$\begin{aligned} &\left(\int_{(2Q)^c} (B(x))^{n/(n-\delta)} dx \right)^{(n-\delta)/n} \\ &\leq C \sum_{j=1}^m \sum_{\sigma \in C_j^m} \sum_{k=1}^\infty 2^{-k\varepsilon} \\ &\left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |(\vec{b}(x) - \vec{b}_Q)_{\sigma^c}|^{n/(n-\delta)} dx \right)^{(n-\delta)/n} \|\vec{b}_\sigma\|_{BMO} \\ &\leq C \|\vec{b}\|_{BMO}. \end{aligned}$$

So, if

$$\begin{aligned} &\int_{(2Q)^c} (C(x))^{n/(n-\delta)} dx \\ &= \sum_{j=1}^m \sum_{\sigma \in C_j^m} \int_{(2Q)^c} \left[|(\vec{b}(x) - b_Q)_{\sigma^c}| \left(\int \int_{\Gamma(x)} \left(\int_Q |(b(z) - b_Q)_\sigma \right. \right. \right. \\ &\quad \left. \left. \left. \psi_t(y - u) a(z) |dz\right)^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} \right]^{n/(n-\delta)} dx \\ &\leq C, \end{aligned}$$

then

$$\int_{R^n} |S_\delta^\vec{b}(a)(x)|^{n/(n-\delta)} dx \leq C.$$

This completes the proof of Theorem 3. □

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