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REGULARIZATION FOR A NONLINEAR BACKWARD PARABOLIC PROBLEM WITH CONTINUOUS SPECTRUM OPERATOR

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Abstract. We study the backward parabolic problem for a nonlinear parabolic equation of the form $u_t + Au(t) = f(t,u(t)), u(T) = \varphi$, where A is a positive self-adjoint unbounded operator and f is a Lipschitz function. The problem is ill-posed, in the sense that if the solution does exist, it will not depend continuously on the data. To regularize the problem, we use the quasi-boundary method and the quasi-reversibility method to establish a modified problem. We present approximated solutions that depend on a small parameter $\epsilon > 0$ and give error estimates for our regularization.

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1. INTRODUCTION

Let $K = \Re$ and let H be a Hilbert space on K with the inner product $< \cdot, \cdot >$ and the norm $\|\cdot\|$. Let $A : D(A) \to H$ be a self-adjoint operator defined on a subspace $D(A)$ of the vector space H, such that $-A$ generates a contraction semi-group on H. Consider the backward parabolic equation of finding a function $u : [0,T] \to H$, such that

$$u_t + Au(t) = f(t,u(t)), \quad 0 < t < T, \tag{1.1}$$

$$u(T) = \varphi, \tag{1.2}$$

where $\varphi \in H$ is a prescribed final value and $f : \mathbb{R} \times H \to H$ is a given Lipschitz function. We can rewrite the above problem as the following integral equation (see, e.g., [1], chapter 4)

$$u(t) = S(T-t)^{-1}\varphi - \int_t^T S(s-t)^{-1} f(s,u(s))ds, \tag{1.3}$$

where $S(t), t \geq 0$ is the semigroup (generated by $-A$) which is defined precisely later. This problem is well known to be severely ill-posed; i.e., solutions do not always exist, and when they do exist, they do not depend continuously on the given data. In addition, the ill-posed problem is very sensitive to the measurement errors (see, e.g., [2]). The final data is usually the result of discretely experimental measurements and thus patched into L^2-functions which resulted an error. With the natural error, the normal computation is intractable, that require some special regularization methods. Backward parabolic equations are very important in various practical situations and there have been many articles devoted to this problem. The linear homogeneous case $f = 0$ of this problem has been considered by many authors, using many different approaches. In the case when A has discrete spectrum, backward problems have been studied in many recent papers, such as [5–7, 9, 10, 12]. For some of works on the continuous spectrum of A, we refer the reader to N.Boussetila and F. Rebbani [3, 4], Denche and S. Djezzar [7], N.H. Tuan and D.D. Trong [13, 14]. As far as we know, we did not find any results concerned with nonlinear backward Cauchy problems with spectrum continuous operator. In this paper, we are going to modify the quasi-boundary value (QBV) and the quasi-reversibility methods to solve (1.1)–(1.2). In fact, we shall establish approximated solutions on the interval $[0, T+m]$ (instead of $[0, T]$) with $m > 0$. Moreover, we shall prove that this idea the same stability magnitude order as the one in the case of QBV method (we can see the same idea in [8]).

The paper is organized into sections as follows: In Section 3, we shall regularize the homogeneous problem. In Section 4, we describe our regularization method for the nonlinear case.

2. Auxiliary Results

In this section, we present the notation and the functional setting which will be use in this paper to prepare some material which will be used in our analysis.

Let a be a positive number. We denote by $\{E_\lambda,\ \lambda \geq a\}$ the spectral resolution of the identity associated to A.

We denote by $S(t) = e^{-tA} = \int_a^\infty e^{-t\lambda}\, dE_\lambda \in \mathcal{L}(H)$, $t \geq 0$, the C_0-semigroup generated by $-A$. Some basic properties of $S(t)$ are listed in the following theorem.

Theorem 1 (See [11], Ch.2, Theorem 6.13, p.74)**.** *For the family of operators $S(t), t \geq 0$ the following properties are valid:*

(1) $\|S(t)\| \leq 1$, *for all $t \geq 0$;*
(2) *the function $t \longmapsto S(t)$, $t > 0$, is analytic;*
(3) *for every real $r \geq 0$ and $t > 0$, the operator $S(t) \in \mathcal{L}(H, \mathcal{D}(A^r))$;*
(4) *for every $x \in \mathcal{D}(A^r)$, $r \geq 0$ we have $S(t)A^r x = A^r S(t)x$.*

Definition 1. Let $A : D(A) \subset H \to H$ be a self-adjoint operator on the Hilbert space H and let $g : \mathbb{R} \to K$ be a piecewise continuous function. We set

$$D(g(A)) = \{u \in H : \int_a^{+\infty} |g(\lambda)|^2 d\|E_\lambda u\|^2 < \infty\}$$

and define the linear operator $g(A) : D(A) \subset H \to H$ by the formula

$$g(A)u = \int_a^{+\infty} g(\lambda) dE_\lambda u,$$

for all $u \in D(g(A))$.

3. Regularization of the homogeneous problem

In this section, we shall consider the homogeneous problem

$$u_t + Au(t) = 0, \ \ 0 < t < T, \tag{3.1}$$

$$u(T) = \varphi, \tag{3.2}$$

It is useful to state that the admissible set refers to the final value φ. The following lemma gives an answer to this question.

Lemma 1. *Problem (3.1)–(3.2) has a solution if and only if*

$$\int_a^{\infty} e^{2\lambda T} d\|E_\lambda \varphi\|^2 < \infty$$

and its unique solution is represented by

$$u(t) = e^{(T-t)A}\varphi. \tag{3.3}$$

If the problem (1.1) admits a solution u then this solution can be represented by

$$u(t) = e^{(T-t)A}\varphi = \int_a^{\infty} e^{\lambda(T-t)} dE_\lambda \varphi. \tag{3.4}$$

Since $t < T$, we know from (3.4) that the terms $e^{-(t-T)\lambda}$ is the source of the instability. So, to regularize problem (3.4), we should replace it by the better terms. Let φ and φ_ϵ denote the exact and measured data at $t = T$, respectively, which satisfy

$$\|\varphi - \varphi_\epsilon\| \le \epsilon,$$

where ϵ is a noise level. In this paper, we perturbed the final condition $u(T) = \varphi$ to form an approximate nonlocal problem depending on a small parameter. We introduced the regularized problem with boundary condition containing a derivative of the same order than the equation as the following equation

$$v_t^\epsilon + Av^\epsilon = 0, \quad 0 < t < T + m$$

$$\epsilon v_t^\epsilon(0) + v^\epsilon(T + m) = \varphi_\epsilon$$

where $m > 0$ is a fixed number. From [7], the approximated solution v_ϵ corresponding to the final value φ_ϵ is given as

$$v^\epsilon(t) = \int_a^\infty \frac{e^{-\lambda t}}{\epsilon\lambda + e^{-\lambda(T+m)}} dE_\lambda \varphi_\epsilon, \quad t \in [0, T+m]. \tag{3.5}$$

To get an error estimate for $\|v^\epsilon(t+m) - u(t)\|$, we will use the function

$$u^\epsilon(t) = \int_a^\infty \frac{e^{-\lambda t}}{\epsilon\lambda + e^{-\lambda(T+m)}} dE_\lambda \varphi, \quad t \in [0, T+m]. \tag{3.6}$$

Theorem 2. *Assume that u has the eigenfunction expansion* $u(t) = \int_0^\infty dE_\lambda u(t)$. *a) Assume that there exist a positive constant* C_1 *such that* $\|Au(0)\| \le C_1$. *Then for every* $t \in [0, T]$,

$$\|v^\epsilon(t+m) - u(t)\| \le \epsilon^{\frac{t+m}{T+m}} \Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{T-t}{T+m}} + \frac{C_1}{\ln(\frac{T+m}{\epsilon})}. \tag{3.7}$$

b) Assume that there exist positive constants m and C_2 *such that*

$$\int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2 \le C_2^2. \tag{3.8}$$

Then for every $t \in [0, T]$,

$$\|v^\epsilon(t+m) - u(t)\| \le (C_2+1)\epsilon^{\frac{t+m}{T+m}} \Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{T-t}{T+m}}. \tag{3.9}$$

Proof. First, we prove the following useful lemma.

Lemma 2. *Let* $s, t, \epsilon, m, \xi, \lambda$ *be real numbers such that* $0 \le t \le s \le T$, $\lambda \in (a, \infty)$ *and* $0 < \epsilon < eT$. *Then the following estimate holds for all* $\lambda \in (a, \infty)$

$$\frac{e^{-(t+m)\lambda}}{\epsilon\lambda + e^{-(T+m)\lambda}} \le \epsilon^{\frac{t-T}{T+m}} \Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{T-t}{T+m}}. \tag{3.10}$$

Proof. For $\lambda > 0$, we define the function

$$h(\lambda) = \frac{1}{\epsilon\lambda + e^{-\lambda(T+m)}}.$$

Then

$$h(\lambda) \le h\Big(\frac{\ln(\frac{T+m}{\epsilon})}{T+m}\Big) = \frac{T+m}{\epsilon\big(1 + \ln(\frac{T+m}{\epsilon})\big)} \quad \epsilon \in (0, eT).$$

Hence

$$\frac{1}{\epsilon\lambda + e^{-(T+m)\lambda}} \le \frac{1}{\epsilon\ln(\frac{T+m}{\epsilon})}. \tag{3.11}$$

Using (3.11), we obtain

$$\begin{aligned}\frac{e^{-(t+m)\lambda}}{\epsilon\lambda+e^{-(T+m)\lambda}} &= \frac{e^{-(t+m)\lambda}}{\left(\epsilon\lambda+e^{-(T+m)\lambda}\right)^{\frac{t+m}{T+m}}\left(\epsilon+e^{-(T+m)\lambda}\right)^{\frac{T-t}{T+m}}}\\ &\le \frac{1}{(\epsilon\lambda+e^{-(T+m)\lambda})^{\frac{T-t}{T+m}}}\\ &\le \left(\frac{T+m}{\epsilon\ln(\frac{T+m}{\epsilon})}\right)^{\frac{T-t}{T+m}}\\ &= \epsilon^{\frac{t-T}{T+m}}\left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\right)^{\frac{T-t}{T+m}}.\end{aligned}$$

□

Now, we prove Theorem 2. The proof is divided into three steps.
Step 1. Estimate $\|v^\epsilon(t+m)-u^\epsilon(t+m)\|$. For every $\lambda\in(0,\infty)$,

$$v^\epsilon(t+m)-u^\epsilon(t+m)=\int_a^\infty \frac{e^{-\lambda(t+m)}}{\epsilon\lambda+e^{-\lambda(T+m)}}dE_\lambda(\varphi_\epsilon-\varphi)$$

we have

$$\begin{aligned}\|v^\epsilon(t+m)-u^\epsilon(t+m)\|^2 &= \int_a^\infty\left(\frac{e^{-\lambda(t+m)}}{\epsilon\lambda+e^{-\lambda(T+m)}}\right)^2 d\|E_\lambda(\varphi_\epsilon-\varphi)\|^2\\ &\le \epsilon^{\frac{2t-2T}{T+m}}\left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\right)^{\frac{2T-2t}{T+m}}\int_a^\infty d\|E_\lambda(\varphi_\epsilon-\varphi)\|^2\\ &\le \epsilon^{\frac{2t-2T}{T+m}}\left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\right)^{\frac{2T-2t}{T+m}}\|\varphi_\epsilon-\varphi\|^2\\ &\le \epsilon^{\frac{2t-2T}{T+m}}\left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\right)^{\frac{2T-2t}{T+m}}\epsilon^2\\ &\le \epsilon^{\frac{2t+2m}{T+m}}\left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\right)^{\frac{2T-2t}{T+m}}.\end{aligned}$$

Thus

$$\|v^\epsilon(t+m)-u^\epsilon(t+m)\|\le\epsilon^{\frac{t+m}{T+m}}\left(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\right)^{\frac{T-t}{T+m}}.$$

Step 2. We estimate $\|u^\epsilon(t+m)-u(t)\|$ if $\|Au(0)\|\le C_1$.

$$u^\epsilon(t+m)-u(t)=\int_a^\infty\left(\frac{e^{-\lambda(t+m)}}{\epsilon\lambda+e^{-\lambda(T+m)}}-e^{(T-t)\lambda}\right)dE_\lambda\varphi$$

$$= \int_a^\infty \frac{\epsilon\lambda e^{(T-t)\lambda}}{\epsilon\lambda + e^{-(T+m)\lambda}} dE_\lambda \varphi$$

Since (3.4), we have $u(0) = \int_a^\infty e^{\lambda T} dE_\lambda \varphi$. Thus $dE_\lambda \varphi = e^{-\lambda T} dE_\lambda u(0)$. This implies that

$$u^\epsilon(t+m) - u(t) = \int_a^\infty \frac{\epsilon\lambda e^{-t\lambda}}{\epsilon\lambda + e^{-(T+m)\lambda}} dE_\lambda u(0).$$

Applying the inequality (3.11), we obtain

$$\begin{aligned}
\|u^\epsilon(t+m) - u(t)\|^2 &= \int_a^\infty \Big(\frac{\epsilon\lambda e^{-t\lambda}}{\epsilon\lambda + e^{-(T+m)\lambda}}\Big)^2 d\|E_\lambda u(0)\|^2 \\
&\le \Big(\frac{\epsilon}{\epsilon \ln(\frac{T+m}{\epsilon})}\Big)^2 \int_a^\infty \lambda^2 d\|E_\lambda u(0)\|^2 \\
&\le \Big(\frac{1}{\ln(\frac{T+m}{\epsilon})}\Big)^2 \|Au(0)\|^2.
\end{aligned}$$

Therefore

$$\|u^\epsilon(t+m) - u(t)\| \le \frac{1}{\ln(\frac{T+m}{\epsilon})} \|Au(0)\|.$$

It implies that

$$\begin{aligned}
\|v^\epsilon(t+m) - u(t)\| &\le \|v^\epsilon(t+m) - u^\epsilon(t+m)\| + \|u^\epsilon(t+m) - u(t)\| \\
&\le \epsilon^{\frac{t+m}{T+m}} \Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{T-t}{T+m}} + \frac{1}{\ln(\frac{T+m}{\epsilon})} \|Au(0)\|.
\end{aligned}$$

Step 3. Estimate $\|u^\epsilon(t+m) - u(t)\|$ if $\int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2 \le C_2^2$.
Since $u^\epsilon(t+m) - u(t) = \int_a^\infty \frac{\epsilon\lambda}{\epsilon\lambda + e^{-(T+m)\lambda}} dE_\lambda u(0)$, we have

$$\begin{aligned}
u^\epsilon(t+m) - u(t) &= \int_a^\infty \frac{\epsilon\lambda}{\epsilon\lambda + e^{-(T+m)\lambda}} dE_\lambda u(0) \\
&= \int_a^\infty \frac{\epsilon\lambda e^{-(t+m)\lambda}}{\epsilon\lambda + e^{-(T+m)\lambda}} e^{(t+m)\lambda} dE_\lambda u(0).
\end{aligned}$$

Then

$$\begin{aligned}
\|u^\epsilon(t+m) - u(t)\|^2 &= \int_a^\infty \Big(\frac{\epsilon e^{-(t+m)\lambda}}{\epsilon\lambda + e^{-(T+m)\lambda}}\Big)^2 \Big(\lambda e^{(t+m)\lambda}\Big)^2 d\|E_\lambda u(0)\|^2 \\
&\le \epsilon^2 \epsilon^{\frac{2t-2T}{T+m}} \Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{2T-2t}{T+m}} \int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2 \\
&= \epsilon^{\frac{2t+2m}{T+m}} \Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{2T-2t}{T+m}} \int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2.
\end{aligned}$$

Thus

$$\|u^\epsilon(t+m)-u(t)\| \le \epsilon^{\frac{t+m}{T+m}}\Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{T-t}{T+m}}\sqrt{\int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2}.$$

Applying the triangle inequality, we obtain

$$\begin{aligned}\|v^\epsilon(t+m)-u(t)\| &\le \|v^\epsilon(t+m)-u^\epsilon(t+m)\| + \|u^\epsilon(t+m)-u(t)\| \\ &\le \epsilon^{\frac{t+m}{T+m}}\Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{T-t}{T+m}} \\ &+ \epsilon^{\frac{t+m}{T+m}}\Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{T-t}{T+m}}\sqrt{\int_a^\infty \lambda^2 e^{2(t+m)\lambda} d\|E_\lambda u(0)\|^2} \\ &\le (C_2+1)\epsilon^{\frac{t+m}{T+m}}\Big(\frac{T+m}{\ln(\frac{T+m}{\epsilon})}\Big)^{\frac{T-t}{T+m}}.\end{aligned}$$

□

4. Regularization of the nonlinear problem

In this section, we shall approximate the problem (1.1)–(1.2) by the following problem

$$\frac{d}{dt}u^\epsilon(t) + A_\epsilon u^\epsilon(t) = B(\epsilon,t) f(t,u^\epsilon(t)), \quad t \in (0,T), \tag{4.1}$$

$$u^\epsilon(T) = \varphi, \tag{4.2}$$

where $A_\epsilon, B(\epsilon,t)$ are defined in (4.3) and (4.4). For every $v \in H$ having the expansion $v = \int_a^{+\infty} dE_\lambda v$, we define

$$S(t)v = \int_a^{+\infty} e^{-t\lambda} dE_\lambda v.$$

$$A_\epsilon(v) = -\frac{1}{T+m}\int_a^{+\infty} \ln(\epsilon + e^{-(T+m)\lambda}) dE_\lambda v. \tag{4.3}$$

$$B(\epsilon,t)(v) = \int_a^{+\infty} (1+\epsilon e^{(T+m)\lambda})^{\frac{t-T}{T+m}} dE_\lambda v, \quad t \in [0,T]. \tag{4.4}$$

$$(\epsilon I + S(T+m))^{\frac{t-T}{T+m}} v = \int_a^{+\infty} \frac{dE_\lambda v}{(\epsilon + e^{-(T+m)\lambda})^{\frac{T-t}{T+m}}} \tag{4.5}$$

Notice that if $f = 0$ then the problem (4.1)–(4.2) has been studied in [4]. The main theorem is as follows

Theorem 3. *Let $\varphi \in H$. Then the problem* $(4.1)-(4.2)$ *has a unique solution $u^\epsilon \in H$. Let $m > 0$ be a positive number. Let $u \in C([0,T];H)$ be a solution of*

(1.1)–(1.3). *Assume that u has the eigenfunction expansion $u(t) = \int_a^{+\infty} dE_\lambda u(t)$ satisfying $\int_a^{+\infty} e^{2(T+m)\lambda} d\|E_\lambda u(t)\|^2 < \infty$ for every $t \in (0,T]$. Let φ_ϵ be a measured data such that $\|\varphi_\epsilon - \varphi\| \le \epsilon$ where $\epsilon \in (0, min\{T, 1-e^{-Ta}\})$. Using φ_ϵ we can construct a function $U^\epsilon : [0,T] \to H$ such that*

$$\|U^\epsilon(t) - u(t)\| \le (1+B)e^{k(T-t)}\epsilon^{\frac{t+m}{T+m}}, \quad \forall t \in (0,T], \tag{4.6}$$

where $\epsilon \in (0, min\{T, 1-e^{-Ta}\})$, U^ϵ is a solution of (4.1) with $U^\epsilon(T) = \varphi_\epsilon$ and

$$B = \sup_{t\in[0,T]} \sqrt{\int_a^{+\infty} e^{2(T+m)\lambda} d\|E_\lambda u(t)\|^2}$$

Remark 1. The estimate (4.6) for $t=0$ is $\epsilon^{\frac{m}{T+m}}$ which depends on m. To improve this, we estimate $u(0)$ by another direction. In fact, for $\epsilon > 0$, we can choose $t_\epsilon \in (0,\epsilon)$ such that

$$\|u(0) - U^\epsilon(t_\epsilon)\| \le \Big[(1+B)e^{kT} + T\|f(0)\|\Big]\frac{T+1}{T}\sqrt{\frac{T}{\ln(\frac{1}{\epsilon})}}.$$

Proof. First, we introduce some useful lemmas for the proof of main results in this paper.

Lemma 3. *Let $\epsilon > 0$ and $0 < t < s < T$. Let A_ϵ be defined in (4.3) where $\epsilon \in (0, 1-e^{-Ta})$. Let $B(\epsilon,t)$ be defined as in (4.4). Then the following inequalities hold:*

a) $\|(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\| \le \epsilon^{\frac{t-T}{T+m}}$
b) $\|S(T-s)(\epsilon I + S(T+m))^{\frac{t}{T}-1}\| \le \epsilon^{\frac{t-s}{T+m}}$.
c) $\|A_\epsilon\| \le \frac{1}{T}\ln(\frac{1}{\epsilon})$.
d) $\|B(\epsilon,t)\| \le 1$.

Proof. a) Let $v \in H$ and let $v = \int_a^{+\infty} dE_\lambda v$ be the eigenfunction expansion of v. We have

$$(\epsilon I + S(T+m))^{\frac{t-T}{T+m}} v = \int_a^{+\infty} \frac{dE_\lambda v}{(\epsilon + e^{-(T+m)\lambda})^{\frac{T-t}{T+m}}}.$$

Then

$$\begin{aligned}\|(\epsilon I + S(T+m))^{\frac{t-T}{T+m}} v\|^2 &= \int_a^{+\infty} \frac{d\|E_\lambda v\|^2}{(\epsilon + e^{-(T+m)\lambda})^{\frac{2T-2t}{T+m}}} \\ &\le \int_a^{+\infty} \frac{d\|E_\lambda v\|^2}{\epsilon^{\frac{2T-2t}{T+m}}} = \epsilon^{\frac{2t-2T}{T+m}}\|v\|^2.\end{aligned}$$

Therefore, we obtain

$$\|(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\| \le \epsilon^{\frac{t-T}{T+m}}.$$

b) First, let $v \in H$, we get

$$\begin{aligned}
&\|S(T-s)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}(v)\|^2 \\
&= \int_a^{+\infty} e^{2(s-T)\lambda}(\epsilon + e^{-(T+m)\lambda})^{\frac{2t-2T}{T+m}} d\|E_\lambda v\|^2 \\
&= \int_a^{+\infty} (\epsilon e^{(T+m)\lambda}+1)^{\frac{2s-2T}{T+m}}(\epsilon + e^{-(T+m)\lambda})^{\frac{2t-2s}{T+m}} d\|E_\lambda v\|^2 \\
&\le \int_a^{+\infty} (\epsilon + e^{-(T+m)\lambda})^{\frac{2t-2s}{T+m}} d\|E_\lambda v\|^2 \\
&\le \int_a^{+\infty} \epsilon^{\frac{2t-2s}{T+m}} d\|E_\lambda v\|^2 \\
&= \epsilon^{\frac{2t-2s}{T+m}}\|v\|^2.
\end{aligned}$$

Then, the following inequality is obtained

$$\|S(T-s)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\| \le \epsilon^{\frac{t-s}{T+m}}.$$

c) We have

$$\|A_\epsilon(v)\|^2 = \frac{1}{(T+m)^2}\int_a^{+\infty} \ln^2\Big(\frac{1}{\epsilon + e^{-(T+m)\lambda}}\Big) d\|E_\lambda v\|^2.$$

Since $\epsilon \in (0, 1-e^{-Ta})$, we obtain $\epsilon + e^{-(T+m)\lambda} < 1,\ \forall \lambda \ge a$. But

$$0 < \ln\Big(\frac{1}{\epsilon + e^{-(T+m)\lambda}}\Big) < \ln\Big(\frac{1}{\epsilon}\Big).$$

It follows that

$$\|A_\epsilon(v)\|^2 \le \frac{1}{(T+m)^2}\ln^2\Big(\frac{1}{\epsilon}\Big)\int_a^{+\infty} d\|E_\lambda v\|^2 \le \frac{1}{T^2}\ln^2\Big(\frac{1}{\epsilon}\Big)\|v\|^2.$$

d) Taking$v \in H$, we have

$$\|B(\epsilon,t)(v)\|^2 = \int_a^{+\infty}(1+\epsilon e^{(T+m)\lambda})^{\frac{2t-2T}{T+m}} d\|E_\lambda v\|^2 \le \int_a^{+\infty} d\|E_\lambda v\|^2 = \|v\|^2,$$

which finishes the proof.

□

Lemma 4. *Let $\varphi \in H$ and let $f: \mathbb{R}\times H \to H$ be a continuous operator satisfying $\|f(t,w)-f(t,v)\| \le k\|w-v\|$ for a $k>0$ independent of $w,v \in H, t \in \mathbb{R}$. Then problem (4.1)–(4.2) has a unique solution $u^\epsilon \in C([0,T];H)$ for any $0<\epsilon<1-e^{-Ta}$.*

Proof. The proof of Lemma 4 is divided into three steps.
Step 1.
For $w \in C([0,T];H)$, we define

$$F(w)(t) = (\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\Big[\varphi - \int_t^T S(T-s)f(s,w(s))ds\Big]. \tag{4.7}$$

For every $w, v \in C([0,T];H)$, we shall prove that

$$\|F^n(w)(t) - F^n(v)(t)\| \le \left(\frac{k(T-t)}{\epsilon}\right)^n \frac{C^n}{n!}|||w-v|||, \tag{4.8}$$

where $C = \max\{T,1\}$ and $|||.|||$ is the sup norm in $C([0,T];H)$. Using the induction method we can verify the latter inequality. We have

$$F(v)(t) = (\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\Big[\varphi - \int_t^T S(T-s)f(s,v(s))ds\Big]. \tag{4.9}$$

Using (4.7),(4.9), Lemma 3 and the Lipschitz property of f, we have

$$\begin{aligned}\|F(w)(t) - F(v)(t)\| &= \|\int_t^T S(T-s)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}(f(s,w(s)) - f(s,v(s))ds\| \\ &\le \frac{k}{\epsilon}\int_t^T \|w(s)-v(s)\|ds \le C\frac{k}{\epsilon}(T-t)|||w-v|||.\end{aligned}$$

Thus, (4.8) holds for $n=1$. Suppose that (4.8) holds for $n=j$. We prove that it also holds for $n=j+1$. In fact, we have

$$\begin{aligned}&\|F^{j+1}(w)(t) - F^{j+1}(v)(t)\| \\ &= \|\int_t^T S(T-s)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}(f(F^j w)(s) - f(F^j v)(s))ds\| \\ &\le \frac{1}{\epsilon}(T-t)k\int_t^T \|F^j(w)(s) - F^j(v)(s)\|^2 ds \\ &\le \left(\frac{k}{\epsilon}\right)^{(j+1)} \frac{(T-t)^{j+1}}{(j+1)!} C^{j+1}|||w-v|||.\end{aligned}$$

Therefore, by the induction principle, we have (4.8) for all $w, v \in C([0,T];H)$. We consider $F : C([0,T];H) \to C([0,T];H)$. Since $\lim_{n\to\infty}\left(\frac{kT}{\epsilon}\right)^n \frac{C^n}{n!} = 0$, there exists a positive integer n_0 such that F^{n_0} is a contraction. It follows that the equation $F^{n_0}(w) = w$ has a unique solution $u^\epsilon \in C([0,T];H)$.

We claim that $F(u^\epsilon) = u^\epsilon$. In fact, one has $F(F^{n_0}(u^\epsilon)) = F(u^\epsilon)$. Hence $F^{n_0}(F(u^\epsilon)) = F(u^\epsilon)$. By the uniqueness of the fixed point of F^{n_0}, one has $F(u^\epsilon) = u^\epsilon$, i.e., the equation $F(w) = w$ has a unique solution $u^\epsilon \in C([0,T];H)$.

Step 2.
Suppose u^ϵ is the unique solution of the integral equation (4.7), then u^ϵ is also a solution of the (4.1)–(4.2).
In fact, we have

$$u^\epsilon(t) = (\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\Big[\varphi - \int_t^T S(T-s)f(s,u^\epsilon(s))ds\Big], \quad t \in [0,T].$$

Taking the derivative of $u^\epsilon(t)$, and by direct computation, we get

$$\frac{d}{dt}u^\epsilon(t) = A_\epsilon u^\epsilon(t) + B(\epsilon,t)f(t,u^\epsilon(t)).$$

Now, we are clear to see that

$$u^\epsilon(T) = \varphi.$$

Hence, u^ϵ is a solution of problem (4.1)–(4.2).

Step 3. The problem (4.1)–(4.2) has at most one solution in $C([0,T];H)$.

Let u and v be two solutions of problem (4.1)–(4.2) such that $u, v \in C([0,T];H)$. First, we denote $g : \mathbb{R} \times H \to H$ by

$$g(t,u(t)) = B(\epsilon,t)f(t,u(t)).$$

Next, because the property of function f, we have for any $w, v \in H$

$$\|g(t,w(t)) - g(t,v(t))\| \le \|B(\epsilon,t)\|\|f(t,w) - f(t,v)\| \le k\|w - v\|.$$

Let b be a positive constant, we put

$$w(t) = e^{-b(t-T)}(u(t) - v(t)) \quad b > 0.$$

By calculating directly, we can get

$$w_t + A_\epsilon w(t) - bw(t) = e^{b(t-T)}\left(g(t,e^{-b(t-T)}u(t)) - g(t,e^{-b(t-T)}v(t))\right).$$

It follows that

$$\begin{aligned}&< w_t(t) + A_\epsilon w(t) - bw(t), w(t) > \\ &=< e^{b(t-T)}\left(g(t,e^{-b(t-T)}u(t)) - g(t,e^{-b(t-T)}v(t))\right), w(t) > .\end{aligned}$$

Using the Lipschitz property of f, we have

$$| < e^{b(t-T)}\left(g(t,e^{-b(t-T)}u(t)) - g(t,e^{-b(t-T)}v(t))\right), w(t) > | \ \le \ k\|w(t)\|^2.$$

So we obtain

$$< e^{b(t-T)}\Big(g(t,e^{-b(t-T)}u(t))-g(t,e^{-b(t-T)}v(t))\Big), w(t) > \geq\ -k\|w(t)\|^2.$$

Using Lemma 3 *c)*, we have

$$|< A_\epsilon w(t), w(t) >|\ \leq \frac{1}{T}\ln(\frac{1}{\epsilon})\|w(t)\|^2,$$

which gives

$$< A_\epsilon w(t), w(t) > \geq\ -\frac{1}{T}\ln(\frac{1}{\epsilon})\|w(t)\|^2.$$

This implies that

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2\ \geq b\|w(t)\|^2 - k\|w(t)\|^2 - \frac{1}{T}\ln(\frac{1}{\epsilon})\|w(t)\|^2.$$

Let any $t_1 \in [0,T]$. Taking the integral with respect to t from t_1 to T, we get

$$\|w(T)\|^2 - \|w(t_1)\|^2 \geq 2\int_{t_1}^{T}(b-k-\frac{1}{T}\ln(\frac{1}{\epsilon}))\|w(t)\|^2 dt.$$

Choosing $b = k + \frac{1}{T}\ln(\frac{1}{\epsilon})$ and noting that $w(T) = 0$, we get $w(t_1) = 0$. Hence, $w(t) = 0$ or $u(t) = v(t),\ \ \forall t \in [0,T]$. This completes the proof of step 3. □

Lemma 5. *The (unique) solution of problem (4.1)–(4.2) depends continuously (in $C([0,T];H)$) on φ.*

Proof. Let u and v be two solutions of problem (4.1)–(4.2) corresponding to the final values φ and ω respectively. We have

$$u(t)-v(t) = (\epsilon I + S(T+m))^{\frac{t-T}{T+m}}(\varphi-\omega)$$
$$-\int_t^T S(T-s)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}(f(u(s)-f(v(s))ds.$$

Applying Lemma 3 and the Lipchitz property of f we get

$$\|u(t)-v(t)\| \leq \|(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}(\varphi-\omega)\|$$
$$+\|\int_t^T S(T-s)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}(f(u(s)-f(v(s))ds\|$$
$$\leq \epsilon^{\frac{t-T}{T+m}}\|\varphi-\omega\| + k\int_t^T \epsilon^{\frac{t-s}{T+m}}\|u(s)-v(s)\|ds.$$

Therefore

$$\epsilon^{\frac{-t}{T+m}}\|u(t)-v(t)\| \le \epsilon^{-\frac{T}{T+m}}\|\varphi-\omega\| + k\int_t^T \epsilon^{-\frac{s}{T+m}}\|u(s)-v(s)\|ds.$$

Applying Gronwall's inequality, we obtain

$$\|u(t)-v(t)\| \le \epsilon^{\frac{t-T}{T+m}} e^{k(T-t)}\|\varphi-\omega\|.$$

So, the solution of the problem (4.1)–(4.2) is depended continuously on φ. □

Now, we turn to

Proof of Theorem 3. In view of (1.3), we have

$$u(t) = S(t-T)\varphi - \int_t^T S(t-s)f(u(s))ds.$$

It follows that

$$\begin{aligned}
&S(T-t)(\epsilon I + S(T+m))^{(t-T)/T+m}u(t)\\
&= (\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\varphi - \int_t^T S(T-s))(\epsilon I + S(T+m))^{\frac{t-T}{T+m}} f(u(s))ds.
\end{aligned}$$

Applying Lemma 3 and the inequality $1-(1+x)^{-\alpha} \le x\alpha,\ (x,\alpha>0)$ we get

$$\begin{aligned}
\|u(t)-u^{\epsilon}(t)\| &\le \int_t^T \|S(T-s)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}}\|\ \|f(u(s)) - f(u^{\epsilon}(s))\|ds\\
&+ \|(I - S(T-t)(\epsilon I + S(T+m))^{\frac{t-T}{T+m}})u(t)\|\\
&\le k\int_t^T \epsilon^{\frac{t-s}{T+m}}\|u(s)-u^{\epsilon}(s)\|ds\\
&+ \sqrt{\int_a^{+\infty}\left(1-(1+\epsilon e^{(T+m)\lambda}\right)^{\frac{t-T}{T+m}})^2 d\|E_{\lambda}u(t)\|^2}\\
&\le k\epsilon^{\frac{t+m}{T+m}}\int_t^T \epsilon^{-\frac{s+m}{T+m}}\|u(s)-u^{\epsilon}(s)\|ds\\
&+ \epsilon\frac{T-t}{T+m}\sqrt{\int_a^{+\infty} e^{2(T+m)\lambda} d\|E_{\lambda}u(t)\|^2}.
\end{aligned}$$

Noting that $0 < \epsilon < \epsilon^{\frac{t+m}{T+m}}$ $(0 < \epsilon < 1)$, we obtain

$$\epsilon^{\frac{-t-m}{T+m}}\|u(t)-u^{\epsilon}(t)\| \leq B + k\int_t^T \epsilon^{\frac{-s-m}{T+m}}\|u(s)-u^{\epsilon}(s)\|ds.$$

Using Gronwall's inequality we obtain

$$\epsilon^{\frac{-t-m}{T+m}}\|u(t)-u^{\epsilon}(t)\| \leq Be^{k(T-t)}.$$

Therefore

$$\|u(t)-u^{\epsilon}(t)\| \leq Be^{k(T-t)}\epsilon^{\frac{t+m}{T+m}}.$$

It follows from Lemma 5 that

$$\|U^{\epsilon}(t)-u^{\epsilon}(t)\| \leq \epsilon^{\frac{t-T}{T+m}}e^{k(T-t)}\|\varphi_{\epsilon}-\varphi\| \leq e^{k(T-t)}\epsilon^{\frac{t+m}{T+m}}.$$

Therefore

$$\begin{aligned}\|U^{\epsilon}(t)-u(t)\| &\leq \|U^{\epsilon}(t)-u^{\epsilon}(t)\| + \|u^{\epsilon}(t)-u(t)\| \\ &\leq (1+B)e^{k(T-t)}\epsilon^{\frac{t+m}{T+m}},\end{aligned}$$

for every $t \in (0,T)$.
This completes the proof of Theorem 3.

□

Proof of Remark 1. For $t \in (0,T)$, considering the function $h(t) = \frac{\ln t}{t} - \frac{\ln\epsilon}{T}$, we have $h(\epsilon) > 0$, $\lim_{t\to 0} h(t) = -\infty, h'(t) > 0$, $(0 < t < \epsilon)$. It follows that the equation $h(t) = 0$ has a unique solution t_ϵ in $(0,\epsilon)$. Since $\frac{\ln t_\epsilon}{t_\epsilon} = \frac{\ln\epsilon}{T}$, the inequality $\ln t > -\frac{1}{t}$ gives $t_\epsilon < \sqrt{\frac{T}{\ln\frac{1}{\epsilon}}}$.
We have $u(t_\epsilon) - u(0) = \int_0^{t_\epsilon} u'(t)dt$. Hence $\|u(0)-u(t_\epsilon)\| \leq t_\epsilon \sup_{t\in[0,T]}\|u'(t)\|$. On the other hand, one has

$$\begin{aligned}\|u'(t)\| \leq \|Au(t)\| + \|f(u(t))\| &\leq \sqrt{\int_a^{+\infty}\lambda^2 d\|E_\lambda u(t)\|^2} + k\|u(t)\| + \|f(0)\| \\ &\leq \frac{1}{T}\sqrt{\int_a^{+\infty}e^{2T\lambda}d\|E_\lambda u(t)\|^2} + k\|u(t)\| + \|f(0)\| \\ &\leq (\frac{1}{T}+k)B + \|f(0)\|.\end{aligned}$$

It follows that $\|u(0)-u(t_\epsilon)\| \leq \left[(\frac{1}{T}+k)B + \|f(0)\|\right]t_\epsilon$. From (4.3), (4.4) and the definition of t_ϵ we get

$$\|u(0)-U^{\epsilon}(t_\epsilon)\| \leq \|u(0)-u(t_\epsilon)\| + \|u(t_\epsilon)-U^{\epsilon}(t_\epsilon)\|$$

$$\leq t_\epsilon \sup_{t\in[0,T]} \|u'(t)\| + (1+B)e^{k(T-t_\epsilon)}\epsilon^{\frac{t_\epsilon+m}{T+m}}$$
$$\leq \Big[(\frac{1}{T}+k)B + \|f(0)\|\Big]t_\epsilon + (1+B)e^{kT}\epsilon^{\frac{t_\epsilon}{T}}.$$

Since $\epsilon^{\frac{t_\epsilon}{T}} = t_\epsilon$ and $(\frac{1}{T}+k)B \leq \frac{(1+B)e^{kT}}{T}$ we have

$$\|u(0) - U^\epsilon(t_\epsilon)\| \leq \Big[(1+B)e^{kT} + T\|f(0)\|\Big]\frac{T+1}{T}t_\epsilon$$
$$\leq \Big[(1+B)e^{kT} + T\|f(0)\|\Big]\frac{T+1}{T}\sqrt{\frac{T}{\ln(\frac{1}{\epsilon})}}.$$

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