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## Alternating sums of the powers of Fibonacci and Lucas numbers

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## ALTERNATING SUMS OF THE POWERS OF FIBONACCI AND LUCAS NUMBERS

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*Abstract.* We shall consider alternating Melham's sums for Fibonacci and Lucas numbers of the form  $\sum_{k=1}^n (-1)^k F_{2k+\delta}^{2m+\varepsilon}$  and  $\sum_{k=1}^n (-1)^k L_{2k+\delta}^{2m+\varepsilon}$ , where  $\varepsilon, \delta \in \{0, 1\}$ .

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### 1. INTRODUCTION

The Fibonacci  $F_n$  and Lucas numbers  $L_n$  are defined by the following recursions: for  $n > 0$ ,

$$F_{n+1} = F_n + F_{n-1} \quad \text{and} \quad L_{n+1} = L_n + L_{n-1},$$

where  $F_0 = 0$ ,  $F_1 = 1$  and  $L_0 = 2$ ,  $L_1 = 1$ , respectively.

If the roots of the characteristic equation  $x^2 - x - 1 = 0$  are  $\alpha$  and  $\beta$ , then the Binet formulas are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n.$$

Wiemann and Cooper [6] raised certain conjectures for the Melham sum:

$$\sum_{k=1}^n F_{2k}^{2m+1}.$$

Ozeki [2] considered Melham's sum and he gave an explicit expansion for it as a polynomial in  $F_{2n+1}$ .

More generally, Prodinger [3] derived a formula for the sum:

$$\sum_{k=0}^n F_{2k+\delta}^{2m+\varepsilon},$$

where  $\varepsilon, \delta \in \{0, 1\}$ . He also evaluated the corresponding sums for the Lucas numbers.

In this paper, we consider the alternating analogs of Melham's sums. We derive explicit formulas for the sums:

$$\sum_{k=1}^n (-1)^k F_{2k+\delta}^{2m+\varepsilon} \text{ and } \sum_{k=1}^n (-1)^k L_{2k+\delta}^{2m+\varepsilon},$$

where  $\varepsilon, \delta \in \{0, 1\}$ .

## 2. ALTERNATING MELHAM'S SUMS FOR FIBONACCI NUMBERS

In this section we will start with some lemmas and then we shall derive our results about the alternating Melham's sum.

**Lemma 1.** *For positive integers  $n, m$  and  $t$  such that  $m \geq t$ ,*

$$\begin{aligned} i) \quad & (-1)^{t+1} F_{(2m-2t+1)n} + F_{(2m-2t+1)(n+1)} \\ &= \sum_{j=0}^{2m-2t} (-1)^{j(t-2)} F_{(2m-2t+1)n+j+(1-3(-1)^t)/2}, \end{aligned}$$

and

$$ii) \quad F_{2(m-t)(n+1)} - (-1)^t F_{2(m-t)n} = \begin{cases} F_{2(m-t)n+m-t} L_{m-t} & \text{if } m \text{ is odd,} \\ L_{2(m-t)n+m-t} F_{m-t} & \text{if } m \text{ is even.} \end{cases}$$

*Proof.* i) We can write

$$\begin{aligned} & (-1)^{t+1} F_{(2m-2t+1)n} + F_{(2m-2t+1)(n+1)} \\ &= (-1)^{t+1} \left( \frac{\alpha^{(2m-2t+1)n} - \beta^{(2m-2t+1)n}}{\alpha - \beta} \right) \\ & \quad + \left( \frac{\alpha^{(2m-2t+1)(n+1)} - \beta^{(2m-2t+1)(n+1)}}{\alpha - \beta} \right) \\ &= \frac{\alpha^{(2m-2t+1)n} \left( (-1)^{t+1} + \alpha^{2m-2t+1} \right)}{\alpha - \beta} \\ & \quad - \frac{\beta^{(2m-2t+1)n} \left( (-1)^{t+1} + \beta^{2m-2t+1} \right)}{\alpha - \beta} \\ &= \begin{cases} \sum_{j=0}^{2m-2t} (-1)^j \left( F_{(2m-2t+1)n+j} + F_{(2m-2t+1)n+j+1} \right) & \text{if } t \text{ is odd,} \\ - \sum_{j=0}^{2m-2t} F_{(2m-2t+1)n+j} - F_{(2m-2t+1)n+j+1} & \text{if } t \text{ is even,} \end{cases} \end{aligned}$$

$$\begin{aligned}
&= \begin{cases} \sum_{j=0}^{2m-2t} (-1)^j F_{(2m-2t+1)n+j+2} & \text{if } t \text{ is odd,} \\ \sum_{j=0}^{2m-2t} F_{(2m-2t+1)n+j-1} & \text{if } t \text{ is even,} \end{cases} \\
&= \sum_{j=0}^{2m-2t} (-1)^{j(t-2)} F_{(2m-2t+1)n+j+(1-3(-1)^t)/2}.
\end{aligned}$$

ii) The proof is similar to the ones for the Binet formulas of  $\{F_n\}$  and  $\{L_n\}$ .  $\square$

From [4], we have the following result for the Gibonacci sequence  $\{G_n\}$ , defined by for  $n > 0$

$$G_{n+1} = G_n + G_{n-1},$$

with arbitrary initial values  $G_0$  and  $G_1$ .

**Lemma 2.** *Let  $a, p \neq 0, q$  be arbitrary integers. Then for  $n > 0$ ,*

$$\sum_{i=a}^n G_{pi+q} = \frac{G_{p(n+1)+q} + (-1)^{p+1} G_{pn+q} + (-1)^p G_{p(a-1)+q} - G_{pa+q}}{L_p - 1 - (-1)^p},$$

and

$$\begin{aligned}
&\sum_{i=a}^n (-1)^i G_{pi+q} \\
&= \frac{(-1)^n G_{p(n+1)+q} + (-1)^{p+n} G_{pn+q} + (-1)^a G_{pa+q} + (-1)^{a+p} G_{p(a-1)+q}}{1 + (-1)^p + L_p}.
\end{aligned}$$

As a consequence of Lemma 2, for further use we state the following result:

**Corollary 1.** *For any integer  $r$  and positive even integer  $t$ ,*

$$\sum_{k=1}^n (-1)^{kr} L_{kt} = \frac{(-1)^{nr} F_{t(n+1)} + (-1)^{nr-r} F_{tn} - F_t}{F_t}$$

and

$$\sum_{k=1}^n (-1)^{kr} F_{k(t+1)} = \frac{(-1)^{(n+1)r} F_{(t+1)n} + (-1)^{nr} F_{(t+1)(n+1)} - F_{t+1}}{L_{t+1}}.$$

*Proof.* Clearly

$$\sum_{k=1}^n (-1)^{kr} L_{kt} = (-1)^r L_t + L_{2t} + \dots + (-1)^{nr} L_{nt}.$$

For the first claim, we consider two cases: the first case is when  $n$  is an odd integer. Here

$$\sum_{k=1}^n (-1)^{kr} L_{kt} = (-1)^r \sum_{j=1}^{(n+1)/2} L_{(2j-1)t} + \sum_{j=1}^{(n-1)/2} L_{2jt}. \quad (2.1)$$

If we take  $a = 1$ ,  $p = 2t$ ,  $q = -t$  and  $n \rightarrow \frac{n+1}{2}$  in Lemma 2, then we get

$$\sum_{j=1}^{(n+1)/2} L_{(2j-1)t} = \frac{L_{t(n+2)} - L_{tn} + L_{-t} - L_t}{L_{2t} - 2}.$$

The following identities are well known [1, 5]:

$$L_{c+t} - L_{c-t} = 5F_c F_t \quad (2.2)$$

for even  $t$ , and

$$L_{2c} - (-1)^c 2 = 5F_c^2 \text{ and } L_{-c} = (-1)^c L_c \quad (2.3)$$

for any integer  $c$ . Thus we have

$$\sum_{j=1}^{(n+1)/2} L_{(2j-1)t} = \frac{5F_{t(n+1)}F_t}{5F_t^2} = \frac{F_{t(n+1)}}{F_t}. \quad (2.4)$$

Meanwhile, if we take  $a = 1$ ,  $p = 2t$ ,  $q = 0$  and  $n \rightarrow \frac{n-1}{2}$  in Lemma 2, then we get

$$\sum_{j=1}^{(n-1)/2} L_{2jt} = \frac{L_{t(n+1)} - L_{t(n-1)} + L_0 - L_{2t}}{L_{2t} - 2}.$$

Since  $t$  is even, by (2.2) and (2.3), we rewrite the last equation as

$$\sum_{j=1}^{(n-1)/2} L_{2jt} = \frac{5F_{tn}F_t}{5F_t^2} - 1 = \frac{F_{tn}}{F_t} - 1. \quad (2.5)$$

If we substitute (2.4) and (2.5) in (2.1), then we obtain

$$\begin{aligned} \sum_{k=1}^n (-1)^{kr} L_{kt} &= (-1)^r \left( \frac{F_{t(n+1)}}{F_t} \right) + \left( \frac{F_{tn}}{F_t} - 1 \right) \\ &= \frac{(-1)^r F_{t(n+1)} + F_{tn} - F_t}{F_t}. \end{aligned} \quad (2.6)$$

For the second case, let  $n$  be an even integer, thus

$$\sum_{k=1}^n (-1)^{kr} L_{kt} = (-1)^r \sum_{j=1}^{n/2} L_{(2j-1)t} + \sum_{j=1}^{n/2} L_{2jt}. \quad (2.7)$$

By taking  $a = 1, p = 2t, q = -t$  and  $n \rightarrow \frac{n}{2}$  and  $a = 1, p = 2t, q = 0$  and  $n \rightarrow \frac{n}{2}$  in Lemma 2, respectively, we obtain the following result by (2.2) and (2.3), for even  $t$ ,

$$\sum_{j=1}^{n/2} L_{(2j-1)t} = \frac{F_{tn}}{F_t}, \quad (2.8)$$

$$\sum_{j=1}^{n/2} L_{2jt} = \frac{F_{t(n+1)}}{F_t} - 1. \quad (2.9)$$

If we put (2.8) and (2.9) in (2.7), we get

$$\begin{aligned} \sum_{k=1}^n (-1)^{kr} L_{kt} &= (-1)^r \left( \frac{F_{tn}}{F_t} \right) + \left( \frac{F_{t(n+1)}}{F_t} - 1 \right) \\ &= \frac{(-1)^r F_{tn} + F_{t(n+1)} - F_t}{F_t}. \end{aligned} \quad (2.10)$$

Combining (2.6) and (2.10), we get the final result:

$$\sum_{k=1}^n (-1)^{kr} L_{kt} = \frac{(-1)^{nr} F_{t(n+1)} + (-1)^{nr-r} F_{tn} - F_t}{F_t},$$

as claimed.

Finally by taking  $a = 1, p = t + 1, q = 0$  in Lemma 2, the second claim is obtained similary to the first claim.  $\square$

**Theorem 1.** *i) For positive odd  $m$ ,*

$$\begin{aligned} \sum_{k=1}^n (-1)^k F_k^{2m} &= \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{i(n+1)+n} \binom{2m}{i} \frac{F_{(m-i)(2n+1)}}{F_{m-i}} \\ &\quad - \frac{1}{5^m} \binom{2m-1}{m} - \frac{1}{5^m} \binom{2m}{m} n, \end{aligned}$$

*and, for positive even  $m$ ,*

$$\begin{aligned} &\sum_{k=1}^n (-1)^k F_k^{2m} \\ &= \begin{cases} \frac{1}{5^m} \left( \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} + \binom{2m-1}{m} \right) & \text{if } n \text{ is even,} \\ -\frac{1}{5^m} \left( \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} + \binom{2m-1}{m-1} \right) & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

and

$$\begin{aligned}
 ii) \sum_{k=1}^n (-1)^k F_k^{2m+1} &= \frac{1}{5^m} \sum_{i=0}^m \binom{2m+1}{i} \frac{(-1)^{i(n+1)+n}}{L_{2m-2i+1}} \\
 &\quad \times \sum_{j=0}^{2(m-i)} (-1)^{j(i-2)} F_{n(2m-2i+1)+j+\frac{1-3(-1)^i}{2}} \\
 &\quad - \frac{1}{5^m} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \frac{F_{2m-2i+1}}{L_{2m-2i+1}}.
 \end{aligned}$$

*Proof.* i) For odd  $m$ , consider

$$\begin{aligned}
 &\sum_{k=1}^n (-1)^k F_k^{2m} \\
 &= \sum_{k=1}^n (-1)^k \left( \frac{\alpha^k - \beta^k}{\alpha - \beta} \right)^{2m} \\
 &= \sum_{k=1}^n (-1)^k \frac{1}{(\alpha - \beta)^{2m}} \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \alpha^{ki} \beta^{k(2m-i)} \\
 &= \sum_{k=1}^n \frac{(-1)^k}{5^m} \left( \sum_{i=0}^m (-1)^i \binom{2m}{i} (\alpha^{k(2m-i)} \beta^{ki} + \alpha^{ki} \beta^{k(2m-i)}) \right. \\
 &\quad \left. - (-1)^m \binom{2m}{m} (\alpha\beta)^{km} \right) \\
 &= \frac{1}{5^m} \left( \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{k=1}^n (-1)^{k(i+1)} L_{2k(m-i)} \right. \\
 &\quad \left. + \binom{2m}{m} \sum_{k=1}^n (-1)^{m(k+1)+k} \right).
 \end{aligned}$$

By taking  $i+1 = r$  and  $2(m-i) = t$  in Corollary 1, we write

$$\begin{aligned}
 &\sum_{k=1}^n (-1)^k F_k^{2m} \\
 &= \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \\
 &\quad \times \left( \frac{(-1)^{n(i+1)} F_{2(m-i)(n+1)} + (-1)^{(n+1)(i+1)} F_{2(m-i)n} - F_{2(m-i)}}{F_{2(m-i)}} \right)
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{5^m} \binom{2m}{m} \sum_{k=1}^n (-1)^{m(k+1)+k} \\
= & \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{i(n+1)+n} \binom{2m}{i} \left( \frac{F_{2(m-i)(n+1)} - (-1)^i F_{2(m-i)n}}{F_{2(m-i)}} \right) \\
& - \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} + \frac{1}{5^m} \binom{2m}{m} \sum_{k=1}^n (-1)^{m(k+1)+k}.
\end{aligned}$$

Using (ii) in Lemma 1, we have the claimed result. For even  $m$ , the desired result is also obtained.

ii) Consider

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k F_k^{2m+1} \\
= & \sum_{k=1}^n (-1)^k \left( \frac{\alpha^k - \beta^k}{\alpha - \beta} \right)^{2m+1} \\
= & \sum_{k=1}^n \frac{(-1)^k}{(\alpha - \beta)^{2m+1}} \sum_{i=0}^{2m+1} \binom{2m+1}{i} (-1)^{i+1} \alpha^{ki} \beta^{k(2m+1-i)} \\
= & \sum_{k=1}^n \frac{(-1)^k}{(\alpha - \beta)^{2m+1}} \sum_{i=0}^m \binom{2m+1}{i} (-1)^i \left( \alpha^{k(2m+1-i)} \beta^{ki} - \alpha^{ki} \beta^{k(2m+1-i)} \right) \\
= & \frac{1}{5^m} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \sum_{k=1}^n (-1)^{k(i+1)} F_{k(2m-2i+1)}.
\end{aligned}$$

By taking  $r = i + 1$  and  $t = 2(m - i)$  in Corollary 1, we write

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k F_k^{2m+1} \\
= & \frac{1}{5^m} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \times \\
& \frac{(-1)^{(n+1)(i+1)} F_{(2m-2i+1)n} + (-1)^{n(i+1)} F_{(2m-2i+1)(n+1)} - F_{2m-2i+1}}{L_{2m-2i+1}} \\
= & \frac{1}{5^m} \sum_{i=0}^m (-1)^{i+n(i+1)} \binom{2m+1}{i} \frac{(-1)^{i+1} F_{(2m-2i+1)n} + F_{(2m-2i+1)(n+1)}}{L_{2m-2i+1}}
\end{aligned}$$



$$-\frac{1}{5^m} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \frac{F_{2m-2i+1}}{L_{2m-2i+1}}.$$

By Lemma 1, we obtain the claimed result.  $\square$

For further use, we state a consequence of Lemma 2:

**Corollary 2.** *For even positive integer  $t$  and  $n > 0$ ,*

$$\sum_{k=1}^n (-1)^k L_{2tk} = \frac{(-1)^n L_{t(2n+1)}}{L_t} - 1.$$

*Proof.* Substituting  $a = 1$ ,  $p = 2t$  and  $q = 0$  in Lemma 2, we get

$$\begin{aligned} \sum_{k=1}^n (-1)^k L_{2tk} &= \frac{(-1)^n L_{2t(n+1)} + (-1)^n L_{2tn} - L_{2t} - 2}{2 + L_{2t}} \\ &= \frac{(-1)^n (L_{2t(n+1)} + L_{2tn}) - (L_{2t} + 2)}{2 + L_{2t}}. \end{aligned}$$

For even  $t$ , from [1, 5], we have that

$$L_{c+t} + L_{c-t} = L_c L_t \quad (2.11)$$

and for any  $c$ ,

$$L_{2c} + (-1)^c 2 = L_c^2. \quad (2.12)$$

Thus we obtain

$$\sum_{k=1}^n (-1)^k L_{2tk} = (-1)^n \frac{L_{(2n+1)t}}{L_t} - 1,$$

as claimed.  $\square$

**Theorem 2.** *For  $m > 0$ ,*

$$\begin{aligned} i) \quad & \sum_{k=1}^n (-1)^k F_{2k}^{2m} \\ = \quad & \begin{cases} \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} + \frac{(-1)^m}{5^m} \binom{2m-1}{m-1} & \text{if } n \text{ is even,} \\ \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{i+1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - \frac{(-1)^m}{5^m} \binom{2m-1}{m-1} & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

and

$$ii) \quad \sum_{k=1}^n (-1)^k F_{2k}^{2m+1}$$

$$= \frac{1}{5^{m+1}} \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \frac{(-1)^n L_{(2m-2i+1)(2n+1)} - L_{2m-2i+1}}{F_{2m-2i+1}}.$$

*Proof.* i) We write

$$\begin{aligned} \sum_{k=1}^n (-1)^k F_{2k}^{2m} &= \sum_{k=1}^n (-1)^k \left( \frac{\alpha^{2k} - \beta^{2k}}{\alpha - \beta} \right)^{2m} \\ &= \sum_{k=1}^n (-1)^k \frac{1}{(\alpha - \beta)^{2m}} \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \alpha^{2ki} \beta^{2k(2m-i)} \\ &= \sum_{k=1}^n \frac{(-1)^k}{5^m} \left( \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} (\alpha^{2k(2m-i)} \beta^{2ki} + \alpha^{2ki} \beta^{2k(2m-i)}) \right. \\ &\quad \left. + (-1)^m \binom{2m}{m} (\alpha\beta)^{2km} \right) \\ &= \sum_{k=1}^n \frac{(-1)^k}{5^m} \left( \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} L_{4k(m-i)} + (-1)^m \binom{2m}{m} \right) \\ &= \frac{1}{5^m} \left( \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{k=1}^n (-1)^k L_{4k(m-i)} + (-1)^m \binom{2m}{m} \sum_{k=1}^n (-1)^k \right). \end{aligned}$$

If we take  $t = 2(m-i)$  in Corollary 2, we write

$$\begin{aligned} &\sum_{k=1}^n (-1)^k F_{2k}^{2m} \\ &= \frac{1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} (-1)^i \left( \frac{(-1)^n L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 1 \right) \\ &\quad + \frac{1}{5^m} \binom{2m}{m} (-1)^m \sum_{k=1}^n (-1)^k \\ &= \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{n+i} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \\ &\quad + \frac{1}{5^m} \binom{2m}{m} (-1)^m \sum_{k=1}^n (-1)^k \\ &= \frac{1}{5^m} \sum_{i=0}^{m-1} (-1)^{n+i} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} \end{aligned}$$

$$+ \frac{1}{5^m} (-1)^m \left( \binom{2m-1}{m-1} + \binom{2m}{m} \left( \frac{(-1)^n - 1}{2} \right) \right).$$

According to the choice of  $n$  as an odd or as an even number, we have the conclusion.

*ii*) The proof is obtained similiary to the proof of (*i*).  $\square$

For further use, we state the following result:

**Corollary 3.** For positive even integer  $t$  and  $n > 0$ ,

$$\sum_{k=1}^n (-1)^k L_{(2k+1)t} = \frac{(-1)^n L_{2t(n+1)} - L_{2t}}{L_t}.$$

*Proof.* Substituting  $a = 1$ ,  $p = 2t$  and  $q = t$  in Lemma 2, we get

$$\begin{aligned} \sum_{k=1}^n (-1)^k L_{(2k+1)t} &= \frac{(-1)^n L_{(2n+3)t} + (-1)^n L_{(2n+1)t} - L_{3t} - L_t}{2 + L_{2t}} \\ &= \frac{(-1)^n (L_{(2n+3)t} + L_{(2n+1)t}) - (L_{3t} + L_t)}{2 + L_{2t}}. \end{aligned}$$

By (2.11) and (2.12), we rewrite the last equation as for even  $t$ ,

$$\begin{aligned} \sum_{k=1}^n (-1)^k L_{(2k+1)t} &= (-1)^n \frac{L_t L_{2t(n+1)}}{L_t^2} - \frac{L_{3t} + L_t}{L_t^2} \\ &= (-1)^n \frac{L_{2t(n+1)}}{L_t} - \frac{L_{2t}}{L_t}. \end{aligned}$$

Thus we have the conclusion.  $\square$

**Theorem 3.** For  $m > 0$ ,

$$\begin{aligned} &i) \sum_{k=1}^n (-1)^k F_{2k+1}^{2m} \\ &= \begin{cases} \frac{-1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{4(m-i)(n+1)} + L_{4(m-i)}}{L_{2(m-i)}} - \frac{1}{5^m} \binom{2m}{m} & \text{if } n \text{ is odd} \\ \frac{1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} & \text{if } n \text{ is even,} \end{cases} \end{aligned}$$

and

$$ii) \sum_{k=1}^n (-1)^k F_{2k+1}^{2m+1} = \frac{1}{5^{m+1}} \sum_{i=0}^m \binom{2m+1}{i} \frac{(-1)^n L_{(4m-4i+2)(n+1)} - L_{4m-4i+2}}{F_{2m-2i+1}}.$$

*Proof.* *i)* Consider

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k F_{2k+1}^{2m} \\
&= \sum_{k=1}^n (-1)^k \left( \frac{\alpha^{2k+1} - \beta^{2k+1}}{\alpha - \beta} \right)^{2m} \\
&= \sum_{k=1}^n \frac{(-1)^k}{(\alpha - \beta)^{2m}} \sum_{i=0}^{2m} (-1)^i \binom{2m}{i} \alpha^{(2k+1)i} \beta^{(2k+1)(2m-i)} \\
&= \sum_{k=1}^n \frac{(-1)^k}{5^m} \left( \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \right. \\
&\quad \left. \left( \alpha^{(2k+1)(2m-i)} \beta^{(2k+1)i} + \alpha^{(2k+1)i} \beta^{(2k+1)(2m-i)} \right) \right. \\
&\quad \left. + \binom{2m}{m} (\alpha\beta)^{2km} \right) \\
&= \sum_{k=1}^n \frac{(-1)^k}{5^m} \left( \sum_{i=0}^{m-1} \binom{2m}{i} L_{2(2k+1)(m-i)} + \binom{2m}{m} \right) \\
&= \frac{1}{5^m} \left( \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{k=1}^n (-1)^k L_{2(2k+1)(m-i)} + \binom{2m}{m} \sum_{k=1}^n (-1)^k \right)
\end{aligned}$$

If we take  $t = 2(m - i)$  in Corollary 3, we write

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k F_{2k+1}^{2m} \\
&= \frac{1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{(-1)^n L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} + \frac{1}{5^m} \binom{2m}{m} \sum_{k=1}^n (-1)^k \\
&= \frac{(-1)^n}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{4(m-i)(n+1)}}{L_{2(m-i)}} \\
&\quad - \frac{1}{5^m} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{4(m-i)}}{L_{2(m-i)}} + \frac{1}{5^m} \binom{2m}{m} \left( \frac{(-1)^n - 1}{2} \right).
\end{aligned}$$

According to the choice of  $n$  as an odd or as an even number, we have the conclusion.

*ii)* The proof of (*ii*) is obtained similarly to the proof of (*i*).  $\square$

## 3. ALTERNATING MELHAM'S SUM FOR LUCAS NUMBERS

**Theorem 4.** *i) For positive odd  $m$ ,*

$$\sum_{k=1}^n (-1)^k L_k^{2m} = \sum_{i=0}^{m-1} (-1)^{(i+1)n} \binom{2m}{i} \frac{F_{(m-i)(2n+1)}}{F_{m-i}} - 2^{2m-1} + \binom{2m-1}{m} + \binom{2m}{m} n,$$

*and, for even  $m$ ,*

$$\sum_{k=1}^n (-1)^k L_k^{2m} = \begin{cases} \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} - 2^{2m-1} + \binom{2m-1}{m} & \text{if } n \text{ is even,} \\ \sum_{i=0}^{m-1} (-1)^{i+1} \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} - 2^{2m-1} + \binom{2m-1}{m} - \binom{2m}{m} & \text{if } n \text{ is odd,} \end{cases}$$

and

$$\begin{aligned} ii) \sum_{k=1}^n (-1)^k L_k^{2m+1} &= \sum_{i=0}^m \binom{2m+1}{i} \frac{(-1)^{(i+1)n+1}}{L_{2m-2i+1}} \\ &\quad \times \sum_{j=0}^{2(m-i)} (-1)^{j+1} L_{n(2m-2i+1)+j+\frac{1-3(-1)^i}{2}} \\ &\quad + 2 \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \frac{1}{L_{2m-2i+1}} - 2^{2m}. \end{aligned}$$

*Proof.* *i)* By the Binet formula of  $\{L_n\}$ , we have

$$\begin{aligned} \sum_{k=1}^n (-1)^k L_k^{2m} &= \sum_{k=1}^n (-1)^k (\alpha^k + \beta^k)^{2m} \\ &= \sum_{k=1}^n (-1)^k \left( \sum_{i=0}^{m-1} \binom{2m}{i} (\alpha^{k(2m-i)} \beta^{ki} + \alpha^{ki} \beta^{k(2m-i)}) + \binom{2m}{m} (\alpha\beta)^{km} \right). \end{aligned}$$

Since  $\alpha\beta = -1$ , we get

$$\sum_{k=1}^n (-1)^k L_{2k}^{2m} = \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{k=1}^n (-1)^{k(i+1)} L_{2k(m-i)} + \binom{2m}{m} \sum_{k=1}^n (-1)^{k(m+1)}.$$

From Corollary 1, we write

$$\begin{aligned} & \sum_{k=1}^n (-1)^k L_{2k}^{2m} \\ = & \sum_{i=0}^{m-1} \binom{2m}{i} \\ & \left( \frac{(-1)^{(i+1)n} F_{2(m-i)(n+1)} + (-1)^{(n+1)(i+1)} F_{2(m-i)n} - F_{2(m-i)}}{F_{2(m-i)}} \right) \\ & + \binom{2m}{m} \sum_{k=1}^n (-1)^{k(m+1)} \\ = & \sum_{i=0}^{m-1} (-1)^{(i+1)n} \binom{2m}{i} \frac{F_{2(m-i)(n+1)} + (-1)^{i+1} F_{2(m-i)n}}{F_{2(m-i)}} \\ & - \sum_{i=0}^{m-1} \binom{2m}{i} + \binom{2m}{m} \sum_{k=1}^n (-1)^{k(m+1)} \\ = & \left\{ \begin{array}{l} \sum_{i=0}^{m-1} (-1)^{(i+1)n} \binom{2m}{i} \frac{F_{(m-i)(2n+1)}}{F_{m-i}} - 2^{2m-1} \\ \quad + \binom{2m-1}{m} + \binom{2m}{m} n \end{array} \right. \quad \text{if } m \text{ is odd,} \\ & \left\{ \begin{array}{l} \sum_{i=0}^{m-1} (-1)^{(i+1)n} \binom{2m}{i} \frac{L_{(m-i)(2n+1)}}{L_{m-i}} - 2^{2m-1} \\ \quad + \binom{2m-1}{m} + \binom{2m}{m} \left( \frac{(-1)^n - 1}{2} \right) \end{array} \right. \quad \text{if } m \text{ is even.} \end{aligned}$$

According to the choice of  $n$ , the claim is obtained.

*i)* The proof is similar to the proof of *i)*. □

**Theorem 5.** *i)* For  $m > 0$  and even  $n > 0$ ,

$$\sum_{k=1}^n (-1)^k L_{2k}^{2m} = \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 2^{2m-1} + \binom{2m-1}{m},$$

and, for odd  $n > 0$ ,

$$\sum_{k=1}^n (-1)^k L_{2k}^{2m} = - \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 2^{2m-1} - \binom{2m-1}{m}.$$

*i i)* For positive integers  $m$  and  $n$ ,

$$\sum_{k=1}^n (-1)^k L_{2k}^{2m+1} = (-1)^n \sum_{i=0}^m \binom{2m+1}{i} \frac{F_{(2m-2i+1)(2n+1)}}{F_{2m-2i+1}} - 2^{2m}.$$

*Proof.* *i)* We write

$$\begin{aligned} & \sum_{k=1}^n (-1)^k L_{2k}^{2m} \\ &= \sum_{k=1}^n (-1)^k (\alpha^{2k} + \beta^{2k})^{2m} \\ &= \sum_{k=1}^n (-1)^k \\ & \quad \left( \sum_{i=0}^{m-1} \binom{2m}{i} (\alpha^{2k(2m-i)} \beta^{2ki} + \alpha^{2ki} \beta^{2k(2m-i)}) + \binom{2m}{m} (\alpha\beta)^{2km} \right) \\ &= \sum_{i=0}^{m-1} \binom{2m}{i} \sum_{k=1}^n (-1)^k L_{4k(m-i)} + \binom{2m}{m} \sum_{k=1}^n (-1)^k. \end{aligned}$$

If we take  $t = 2(m-i)$  in Corollary 2, we can write

$$\begin{aligned} & \sum_{k=1}^n (-1)^k L_{2k}^{2m} \\ &= \sum_{i=0}^{m-1} \binom{2m}{i} \left( \frac{(-1)^n L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 1 \right) + \binom{2m}{m} \sum_{k=1}^n (-1)^k \\ &= (-1)^n \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - \sum_{i=0}^{m-1} \binom{2m}{i} + \binom{2m}{m} \sum_{k=1}^n (-1)^k \\ &= (-1)^n \sum_{i=0}^{m-1} \binom{2m}{i} \frac{L_{2(m-i)(2n+1)}}{L_{2(m-i)}} - 2^{2m-1} + \binom{2m-1}{m-1} + \binom{2m}{m} \left( \frac{(-1)^n - 1}{2} \right). \end{aligned}$$

According to the choice of  $n$ , the proof is complete by the fact that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

*ii*) The proof of (*ii*) is similar to the proof of (*i*).  $\square$

**Theorem 6.** For  $m > 0$ ,

$$\begin{aligned} & i) \sum_{k=1}^n (-1)^k L_{2k+1}^{2m} \\ &= \begin{cases} \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} & \text{if } n \text{ is even,} \\ (-1)^{m+1} \binom{2m}{m} - \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)(n+1)} + L_{4(m-i)}}{L_{2(m-i)}} & \text{if } n \text{ is odd,} \end{cases} \end{aligned}$$

and

$$\begin{aligned} & ii) \sum_{k=1}^n (-1)^k L_{2k+1}^{2m+1} = \\ & \sum_{i=0}^m (-1)^i \binom{2m+1}{i} \frac{(-1)^n F_{2(2m-2i+1)(n+1)} - F_{2(2m-2i+1)}}{F_{2m-2i+1}}. \end{aligned}$$

*Proof.* *i*) Using the Binet formula of  $\{L_n\}$ , we have

$$\begin{aligned} & \sum_{k=1}^n (-1)^k L_{2k+1}^{2m} \\ &= \sum_{k=1}^n (-1)^k (\alpha^{2k+1} + \beta^{2k+1})^{2m} \\ &= \sum_{k=1}^n (-1)^k \left( \sum_{i=0}^{m-1} \binom{2m}{i} (\alpha^{(2k+1)(2m-i)} \beta^{(2k+1)i} + \alpha^{(2k+1)i} \beta^{(2k+1)(2m-i)}) \right) \\ & \quad + \binom{2m}{m} (\alpha\beta)^{(2k+1)m} \\ &= \sum_{k=1}^n (-1)^k \left( \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} (\alpha^{(2k+1)(2m-2i)} + \beta^{(2k+1)(2m-2i)}) \right) \end{aligned}$$



$$\begin{aligned}
& + \binom{2m}{m} (-1)^m \\
& = \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \sum_{k=1}^n (-1)^k L_{2(2k+1)(m-i)} + \binom{2m}{m} (-1)^m \sum_{k=1}^n (-1)^k.
\end{aligned}$$

If we take  $t = 2(m-i)$  in Corollary 3, we get

$$\begin{aligned}
& \sum_{k=1}^n (-1)^k L_{2k+1} \\
& = \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \left( \frac{(-1)^n L_{4(m-i)(n+1)} - L_{4(m-i)}}{L_{2(m-i)}} \right) \\
& \quad + \binom{2m}{m} (-1)^m \sum_{k=1}^n (-1)^k \\
& = \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{(-1)^n L_{4(m-i)(n+1)}}{L_{2(m-i)}} - \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)}}{L_{2(m-i)}} \\
& \quad + \binom{2m}{m} (-1)^m \sum_{k=1}^n (-1)^k \\
& = (-1)^n \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)(n+1)}}{L_{2(m-i)}} - \sum_{i=0}^{m-1} (-1)^i \binom{2m}{i} \frac{L_{4(m-i)}}{L_{2(m-i)}} \\
& \quad + \binom{2m}{m} (-1)^m \left( \frac{(-1)^n - 1}{2} \right).
\end{aligned}$$

According to the choice of  $n$ , the claimed result is clear.

*ii)* The proof is similar to the proof of *i)*. □

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