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$q-$SZÁSZ SCHURER OPERATORS

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Abstract. In this paper, we introduce $q-$Szász Schurer operators and calculate their moments. The transformation properties, Korovkin type approximation theorem and rate of convergence of the operators are studied. We further obtain global estimates for $q-$Szász Schurer operators in terms of some Lipschitz classes.

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1. INTRODUCTION

Korovkin type approximation theory has been constructed on linear positive operators and has been an active research field during the last century because of its simple applicability [2]. After the works of A. Lupaş [9] and G.M. Phillips [19], where they introduced different analogues of the q-Bernstein polynomials $B_{n,q}(f;x)$, $n = 1,2,..,0 < q < \infty$, intensive research has been conducted on linear positive operators based on q-integers (see [4, 6, 8, 15–18, 20, 21]).

In [4], [3] q-Szász Mirakjan operators were defined and their approximation properties were investigated. For $0 \leq x < \dfrac{b_n}{(1-q)[n]}$, $f \in C[0,\infty)$, and $\{b_n\}$ is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$, in [4], q-Szász Mirakjan operators were defined by

$$S_{n,q}(f)(x) := E_q\left(-[n]\frac{x}{b_n}\right)\sum_{k=0}^{\infty} f\left(\frac{[k]\,b_n}{[n]}\right)\frac{[n]^k\,x^k}{[k]!b_n^k},$$

where

$$[k] = \begin{cases} (1-q^k)/(1-q) & if\ q \neq 1 \\ k & if\ q = 1 \end{cases},\ [k]! = \begin{cases} [1][2]...[k] & if\ k \geq 1 \\ 1 & if\ k = 0 \end{cases},$$

$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{[n]!} x^n = (-(1-q)x;q)_\infty ; x \in \mathbb{R}, |q| < 1, \tag{1.1}$$
$$(a;q)_\infty = \prod_{j=0}^{\infty} (1-aq^j) \qquad (a \in \mathbb{C}).$$

As it was pointed out in [11], it is better to refer to these operators as q-parametric Szász-Mirakjan-Chlodowsky operators because of their structures. In [11], Mahmudov introduced the following q-Szász-Mirakjan operators

$$S_{n,q}^*(f)(x) = \frac{1}{E_q([n]x)} \sum_{k=0}^{\infty} f\left(\frac{[k]}{q^{k-2}[n]}\right) q^{\frac{k(k-1)}{2}} \frac{[n]^k x^k}{[k]!},$$

where $x \in [0,\infty)$, $0 < q < 1$, $f \in C[0,\infty)$ and studied convergence properties of these operators. We note that these operators do not preserve linear functions. In [11], Mahmudov obtained inequalities for the weighted approximation error, and also for the rate of convergence in terms of weighted moduli of continuity, and a Voronoskaja-type formula for q-Szász-Mirakjan operators was derived, too.

Now let $E_q(x)$ be defined by (1.1). In the present paper we introduce $q-$Szász Schurer operators which is defined for fixed $p \in \mathbb{N}_0$ by

$$S_{n,q}(f;x;p) = \frac{1}{E([n+p]x)} \sum_{k=0}^{\infty} f\left(\frac{[k]}{q^{k-1}[n]}\right) q^{\frac{k(k-1)}{2}} \frac{[n+p]^k x^k}{[k]!} \tag{1.2}$$

where $x \in [0,\infty)$, $0 < q < 1$, $f \in C[0,\infty)$. It is clear from (1.1) that

$$S_{n,q}(1;x;p) = 1, \tag{1.3}$$

for each fixed $p \in \mathbb{N}_0$. Note that in the case $p = 0$, $q-$Szász Schurer operators reduces to the slightly modified form of the Mahmudov's q-Szász-Mirakjan operators. Furthermore, letting $q \to 1$ in (1.2), one can get Schurer-Szász-Mirakjan operators (see p.p. 338,[2]).

There are some recent papers related to the $q-$Szász operators. For instance, in [1] and in [10] different variants of King type $q-$Szász operators (preserving x^2 and providing better error estimation) has been considered. On the other hand, integral type $q-$Szász operators were recently investigated in [13],[14],[12] and [7]. These types of modifications of the $q-$Szász Schurer operators will be the subject of future studies.

In this paper we compute the moments of the $q-$Szász Schurer operators and investigate their transformation properties. Korovkin type approximation result and rate of convergence of the operators are obtained in Section 3. Section 4 is devoted to global estimates for $q-$Szász Schurer operators in terms of some Lipschitz classes.

2. Auxiliary lemmas

In this section we calculate the moments of the $q-$Szász Schurer operators and investigate their transformation properties. We start with the following lemma:

Lemma 1. *For $m \in \mathbb{N}_0$ and fixed $p \in \mathbb{N}_0$, we have*

$$S_{n,q}(t^{m+1};x;p) = \frac{[n+p]x}{[n]} \sum_{j=0}^{m} \binom{m}{j} \frac{1}{q^j [n]^{m-j}} S_{n,q}(t^j;x;p). \tag{2.1}$$

Using (1.3) and the above lemma, we get the following results immediately.

Corollary 1. *For fixed $p \in \mathbb{N}_0$, we have*

$$\begin{aligned} S_{n,q}(t;x;p) &= \frac{[n+p]x}{[n]}, \quad S_{n,q}(t^2;x;p) = \frac{1}{q}\left(\frac{[n+p]x}{[n]}\right)^2 + \frac{[n+p]x}{[n]^2}, \\ S_{n,q}(t^3;x;p) &= \frac{1}{q^3}\left(\frac{[n+p]x}{[n]}\right)^3 + \frac{2q+1}{q^2}\frac{[n+p]^2x^2}{[n]^3} + \frac{[n+p]x}{[n]^3}. \end{aligned}$$

Note that the operators $S_{n,q}(t;x;0)$ preserve the linear functions while $S^*_{n,q}(f)(x)$ does not preserve them. Now, using linearity of the operators, we have the following:

Corollary 2. *For the second central moment, we have*

$$S_{n,q}((t-x)^2;x;p) = \left[\left(\frac{[n+p]}{q[n]} - 2\right)\frac{[n+p]}{[n]} + 1\right]x^2 + \frac{[n+p]}{[n]^2}x.$$

Lemma 2. *For every $x \geq 0$, we have*

$$\begin{aligned} &\frac{1}{E([n+p]x)} \sum_{k=0}^{\infty} \left| \frac{[k]}{q^{k-1}[n]} - x \right| q^{\frac{k(k-1)}{2}} \frac{[n+p]^k x^k}{[k]!} \\ \leq\; & \left(\left[\left(\frac{[n+p]}{q[n]} - 2\right)\frac{[n+p]}{[n]} + 1\right]x^2 + \frac{[n+p]}{[n]^2}x \right)^{1/2}. \end{aligned}$$

From (2.1), it is not difficult to obtain the following result by induction.

Lemma 3. *For the m^{th} moment ($m = 1,2,\ldots$) we have the following:*

$$\begin{aligned} S_{n,q}(t^m;x;p) = {} & q^{\frac{-m(m-1)}{2}} \left(\frac{[n+p]x}{[n]}\right)^m \\ & + \frac{b_{m,1}(q)}{[n]}\left(\frac{[n+p]x}{[n]}\right)^{m-1} + \ldots + \frac{b_{m,m-1}(q)}{[n]^{m-1}}\frac{[n+p]x}{[n]}, \end{aligned} \tag{2.2}$$

where the coefficients $b_{m,i}(q)$ ($i = 1,2,\ldots,m-1$) are constants depending on m and q. Furthermore, $b_{m,m-1}(q) = 1$ ($m = 1,2,\ldots$).

Now let $C_B[0,\infty)$ denote the space of all real valued continuous bounded functions on $[0,\infty)$ endowed with the norm

$$\|f\| = \sup_{x\in[0,\infty)} |f(x)|, \; f \in C_B[0,\infty).$$

It is obvious by (1.3) that $S_{n,q}$ maps $C_B[0,\infty)$ into itself.

Let $r \in \mathbb{N}_0 := \{0,1,...\}$ and define the weight function μ_r as follows: $\mu_0(x) := 1$ and $\mu_r(x) := (1+x^r)^{-1}$ for $x \geq 0$ and $r \in \mathbb{N}_0$. Also for $r \in \mathbb{N}_0$, let $C_r[0,\infty)$ be the space of functions $f \in C[0,\infty)$, such that $\mu_r f$ is uniformly continuous and bounded on $[0,\infty)$, endowed with the norm

$$\|f\|_r := \sup_{x\in[0,\infty)} \mu_r(x)|f(x)| \text{ for } f \in C_r[0,\infty).$$

Using (2.2), we easily obtain the following transformation property.

Lemma 4. *For the operators $S_{n,q}$, there exists a constant $K_r(q) > 0$ depending on r and q, such that*

$$\mu_r(x)S_{n,q}\left(\frac{1}{\mu_r};x;p\right) \leq K_r(q). \tag{2.3}$$

Furthermore, for all $f \in C_r[0,\infty)$, we have

$$\left\|S_{n,q}(f)\right\|_r \leq K_r(q)\|f\|_r, \tag{2.4}$$

which guarantees that $S_{n,q}$ maps $C_r[0,\infty)$ into $C_r[0,\infty)$.

3. KOROVKIN TYPE THEOREM

In this section we consider the space

$$E := \left\{ f \in C_2[0,\infty) : \exists \lim_{x\to\infty} \frac{|f(x)|}{1+x^2} < \infty \right\},$$

endowed with the norm

$$\|f\|_2 := \sup_{x\in[0,\infty)} \mu_2(x)|f(x)| \, .$$

Now let $b > 0$. The usual modulus of continuity of f on the closed interval $[0,b]$ is defined by

$$\omega_b(f,\delta) = \sup_{\substack{|t-x|\leq\delta \\ x,t\in[0,b]}} |f(t)-f(x)|. \tag{3.1}$$

It is well known that, for a function $f \in E$, we have $\lim_{\delta\to\infty}\omega_b(f,\delta) = 0$.

We start by obtaining the rate of convergence of the operators $S_{n,q}(f;x;p)$ to $f(x)$, for all $f \in E$.

Theorem 1. *Let $f \in E$ and let $\omega_{b+1}(f,\delta)$ $(b>0)$ be its modulus of continuity on the finite interval $[0,b+1] \subset [0,\infty)$. Then for fixed $q \in (0,1)$, we have*

$$\left\| S_{n,q}(f;x;p) - f(x) \right\|_{C[0,b]} \leq 2\omega_{b+1}(f,\delta(q))$$
$$+N_f\left(1+b^2\right)\left\{\left[\left(\frac{[n+p]}{q[n]}-2\right)\frac{[n+p]}{[n]}+1\right]b^2+\frac{[n+p]}{[n]^2}b\right\}$$

where $\delta = \delta(q) = \left[\left[\left(\frac{[n+p]}{q[n]}-2\right)\frac{[n+p]}{[n]}+1\right]b^2+\frac{[n+p]}{[n]^2}b\right]^{1/2}$ *and N_f is a positive constant depending on f.*

Proof. Let $q \in (0,1)$ be fixed. For $x \in [0,b]$ and $t \leq b+1$, we can write from (3.1) that

$$|f(t)-f(x)| \leq \omega_{b+1}(f,|t-x|) \leq \left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta) \qquad (3.2)$$

where $\delta>0$. On the other hand, for $x \in [0,b]$ and $t > b+1$, using the fact that $t-x>1$, we have

$$\begin{aligned} |f(t)-f(x)| &\leq M_f(1+x^2+t^2) \leq M_f(2+3x^2+2(t-x)^2) \\ &\leq N_f\left(1+b^2\right)(t-x)^2 \qquad (3.3) \end{aligned}$$

where $N_f = 6M_f$. Combining (3.2) and (3.3), we get for all $x \in [0,b]$ and $t \geq 0$ that $|f(t)-f(x)| \leq N_f\left(1+b^2\right)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta)$ and therefore

$$\begin{aligned} \left|S_{n,q}(f;x;p)-f(x)\right| \leq\ & N_f\left(1+b^2\right)S_{n,q}((t-x)^2;x;p) \\ &+\left(1+\frac{S_{n,q}(|t-x|;x;p)}{\delta}\right)\omega_{b+1}(f,\delta). \end{aligned}$$

By Lemma 2 and Corollary 2, we obtain

$$\begin{aligned} &\left|S_{n,q}(f;x;p)-f(x)\right| \\ &\leq N_f\left(1+b^2\right)S_{n,q}((t-x)^2;x;p)+\left(1+\frac{\left[S_{n,q}((t-x)^2;x;p)\right]^{1/2}}{\delta}\right)\omega_{b+1}(f,\delta) \\ &\leq N_f\left(1+b^2\right)\left\{\left[\left(\frac{[n+p]}{q[n]}-2\right)\frac{[n+p]}{[n]}+1\right]x^2+\frac{[n+p]}{[n]^2}x\right\} \\ &+\left(1+\frac{\left[\left[\left(\frac{[n+p]}{q[n]}-2\right)\frac{[n+p]}{[n]}+1\right]x^2+\frac{[n+p]}{[n]^2}x\right]^{1/2}}{\delta}\right)\omega_{b+1}(f,\delta) \\ &\leq N_f\left(1+b^2\right)\left\{\left[\left(\frac{[n+p]}{q[n]}-2\right)\frac{[n+p]}{[n]}+1\right]b^2+\frac{[n+p]}{[n]^2}b\right\}+2\omega_{b+1}(f,\delta(q)), \end{aligned}$$

where $\delta = \delta(q) = \left[\left[\left(\frac{[n+p]}{q[n]} - 2\right)\frac{[n+p]}{[n]} + 1\right]b^2 + \frac{[n+p]}{[n]^2}b\right]^{1/2}$. Whence the result follows. □

Corollary 3. *Let $q := q_n \in (0,1)$ and fix $p \in \mathbb{N}_0$. Then for all $f \in E$, $\{S_{n,q}(f;x;p)\}$ converges uniformly to f on $[0,b]$ if and only if $\lim_{n\to\infty} q_n = 1$.*

4. Global result

In this section we consider the following Lipschitz classes:

$$\begin{aligned}
\Delta_h^2 f(x) &:= f(x+2h) - 2f(x+h) + f(x), \\
\omega_r^2(f,\delta) &:= \sup_{h\in(0,\delta]} \left\| \Delta_h^2 f \right\|_r, \\
\omega_r^1(f,\delta) &:= \sup\{\mu_r(x)\,|f(t)-f(x)| : |t-x| \le \delta \text{ and } t,x \ge 0\} \\
Lip_r^2\alpha &:= \left\{ f \in C_r[0,\infty) : \omega_r^2(f;\delta) = O(\delta^\alpha) \text{ as } \delta \to 0^+ \right\},
\end{aligned}$$

where $h > 0$ and $0 < \alpha \le 2$.

Lemma 5. *Fix $r \in \mathbb{N}$, $p \in \mathbb{N}_0$, $q \in (0,1)$. For the sequence of operators $\{S_{n,q}(f;x;p)\}$, there exists a constant $M_r(q,p)$ such that*

$$\mu_r(x) S_{n,q}\left(\frac{\psi_x^2}{\mu_r};x;p\right) \le M_r(q,p)\left\{\left[\left(\frac{[n+p]}{q^{2r+1}[n]} - \frac{2}{q^r}\right)\frac{[n+p]}{[n]} + 1\right]x^2 + \frac{[n+p]}{[n]^2}x\right\}$$

where $\psi_x^2(t) = (t-x)^2$.

Proof. Now let $r = 1$. Since, for each fixed $p \in \mathbb{N}_0, q \in (0,1)$ and for all $n \in \mathbb{N}$, we have $\frac{[n+p]}{q[n]} - 2 \le \frac{[n+p]}{q^3[n]} - \frac{2}{q}$, then we can write using Corollary 1 and Corollary 2 that

$$\begin{aligned}
&\mu_1(x) S_{n,q}\left(\frac{\psi_x^2}{\mu_1};x;p\right) \\
=\;& \mu_1(x)\left\{S_{n,q}\left(\psi_x^2;x;p\right) + S_{n,q}\left(t^3;x;p\right) - 2x S_{n,q}\left(t^2;x;p\right) + x^2 S_{n,q}\left(t;x;p\right)\right\} \\
\le\;& \mu_1(x)\left\{\left[\left(\left(\frac{[n+p]}{q^3[n]} - \frac{2}{q}\right)\frac{[n+p]}{[n]} + 1\right)x^2\right]\left(1 + \frac{[n+p]}{[n]}x\right)\right. \\
&\left. + \frac{[n+p]}{[n]^2}x\left[\frac{2q+1}{q^2}\frac{[n+p]}{[n]}x - 2x + \frac{1}{[n]} + 1\right]\right\} \\
\le\;& M_r(q,p)\left\{\left[\left(\left(\frac{[n+p]}{q^3[n]} - \frac{2}{q}\right)\frac{[n+p]}{[n]} + 1\right)x^2\right] + \frac{[n+p]}{[n]^2}x\right\}.
\end{aligned}$$

Finally, assume that $r \geq 2$. Since, for each fixed $p \in \mathbb{N}_0, q \in (0,1)$ and for all $n \in \mathbb{N}$, we have $\frac{[n+p]}{q[n]} - 2 \leq \frac{[n+p]}{q^{2r+1}[n]} - \frac{2}{q^r}$, then, we get from Lemma 3 that

$$S_{n,q}\left(\frac{\psi_x^2}{\mu_r};x;p\right) = S_{n,q}\left(\psi_x^2;x;p\right) + S_{n,q}(e_{r+2};x;p) - 2xS_{n,q}(e_{r+1};x;p)$$
$$+x^2 S_{n,q}(e_r;x;p) = S_{n,q}\left(\psi_x^2;x;p\right)$$
$$+q^{\frac{-(r-1)(r+2)}{2}}\left(\frac{[n+p]x}{[n]}\right)^{r+2} + \frac{b_{r+2,1}(q)}{[n]}\left(\frac{[n+p]x}{[n]}\right)^{r+1} + \dots + \frac{1}{[n]^{r+1}}\frac{[n+p]x}{[n]}$$
$$-2x\left\{q^{\frac{-r(r+1)}{2}}\left(\frac{[n+p]x}{[n]}\right)^{r+1} + \frac{b_{r+1,1}(q)}{[n]}\left(\frac{[n+p]x}{[n]}\right)^{r} + \dots + \frac{1}{[n]^{r}}\frac{[n+p]x}{[n]}\right\}$$
$$+x^2\left\{q^{\frac{-r(r-1)}{2}}\left(\frac{[n+p]x}{[n]}\right)^{r} + \frac{b_{r,1}(q)}{[n]}\left(\frac{[n+p]x}{[n]}\right)^{r-1} + \dots + \frac{1}{[n]^{r-1}}\frac{[n+p]x}{[n]}\right\},$$
$$= \left[\left(\frac{[n+p]}{q[n]} - 2\right)\frac{[n+p]}{[n]} + 1\right]x^2 + \frac{[n+p]}{[n]^2}x$$
$$+q^{\frac{-r(r-1)}{2}}\left[\left(\frac{[n+p]}{q^{2r+1}[n]} - \frac{2}{q^r}\right)\frac{[n+p]}{[n]} + 1\right]x^2\left(\frac{[n+p]x}{[n]}\right)^r$$
$$+\frac{[n+p]}{[n]^2}x\left[(b_{r+2}(q) - 2b_{r+1}(q) + b_r(q))\left(\frac{[n+p]x}{[n]}\right)^r + \dots + \frac{1}{[n]^r}\right]$$

which implies that

$$\mu_r(x)S_{n,q}\left(\frac{\psi_x^2}{\mu_r};x;p\right)$$
$$\leq M_r(q,p)\left\{\left[\left(\frac{[n+p]}{q^{2r+1}[n]} - \frac{2}{q^r}\right)\frac{[n+p]}{[n]} + 1\right]x^2 + \frac{[n+p]}{[n]^2}x\right\},$$

whence the result holds. □

Now, for fixed $p \in \mathbb{N}_0$, consider the space

$$C_p^2[0,\infty) := \left\{f \in C_p[0,\infty) : f'' \in C_p[0,\infty)\right\}.$$

Then we have the following result.

Lemma 6. *Let $g \in C_p^2[0,\infty)$. For the operators $S_{n,q}^*(f;x;p) = S_{n,q}(f;x;p) - f(\frac{[n+p]x}{[n]}) + f(x)$, there exists a positive constant $M_r(q,p)$ such that, for all $x \in [0,\infty)$ and $n \in \mathbb{N}$, we have*

$$\mu_r(x)\left|S_{n,q}^*(g;x;p) - g(x)\right| \leq M_r(q,p)\left\|g''\right\|_r$$
$$\times\left\{\left[\left(1 + \frac{1}{q^{2r+1}}\right)\frac{[n+p]^2}{[n]^2} - 2\left(1 + \frac{1}{q^r}\right)\frac{[n+p]}{[n]} + 2\right]x^2 + \frac{[n+p]x}{[n]^2}\right\}.$$

Proof. Let $\psi_x(y) = y - x$. Using Taylor formula, we may write that

$$g(y) - g(x) = \psi_x(y) g'(x) + \int_x^y \psi_t(y) g''(t)\,dt, \quad y \in [0, \infty).$$

Since $S^*_{n,q}(\psi_x(y); x; p) = 0$, we get

$$\begin{aligned}\left|S^*_{n,q}(g;x;p) - g(x)\right| &= \left|S^*_{n,q}(g(y) - g(x); x; p)\right| \\ &= \left|S^*_{n,q}\left(\int_x^y \psi_t(y) g''(t)\,dt; x; p\right)\right|\end{aligned}$$

$$= \left|S_{n,q}\left(\int_x^y \psi_t(y) g''(t)\,dt; x; p\right) - \int_x^{\frac{[n+p]x}{[n]}} \psi_t\left(\frac{[n+p]x}{[n]}\right) g''(t)\,dt\right|.$$

Using the fact that

$$\left|\int_x^y \psi_t(y) g''(t)\,dt\right| \leq \frac{\|g''\|_r \, \psi_x^2(y)}{2}\left(\frac{1}{\mu_r(x)} + \frac{1}{\mu_r(y)}\right)$$

and

$$\left|\int_x^{\frac{[n+p]x}{[n]}} \psi_t\left(\frac{[n+p]x}{[n]}\right) g''(t)\,dt\right| \leq \frac{\|g''\|_r}{2\mu_r(x)}\left(\frac{[n+p]x}{[n]} - x\right)^2,$$

it follows from Lemma 5 that

$$\begin{aligned}\mu_r(x)\left|S^*_{n,q}(g;x;p) - g(x)\right| &\leq \frac{\|g''\|_r}{2}\left\{S_{n,q}(\psi_x^2; x) + \mu_r(x) S_{n,q}\left(\frac{\psi_x^2}{\mu_r}; x\right)\right\} \\ &+ \frac{\|g''\|_r}{2}\left(\frac{[n+p]x}{[n]} - x\right)^2 \\ \leq\ & M_r(q,p)\left\|g''\right\|_r \\ &\times\left\{\left[\left(1 + \frac{1}{q^{2r+1}}\right)\frac{[n+p]^2}{[n]^2} - 2\left(1 + \frac{1}{q^r}\right)\frac{[n+p]}{[n]} + 2\right]x^2 + \frac{[n+p]x}{[n]^2}\right\}.\end{aligned}$$

The Lemma is proved. □

The main result of the section is the following.

Theorem 2. *For each fixed $r \in \mathbb{N}, p \in \mathbb{N}_0$ and for all $n \in \mathbb{N},\ f \in C_r[0,\infty)$ and $x \in [0,\infty)$ and fixed $q \in (0,1)$, there exists an absolute constant $M_r(q,p) > 0$ such that*

$$\mu_r(x)\left|S_{n,q}(f;x;p) - f(x)\right| \le \omega_r^1(f;\frac{[n+p]x}{[n]} - x) + M_r(q,p)$$

$$\times\omega_r^2\left(f,\sqrt{\left[\left(1+\frac{1}{q^{2r+1}}\right)\frac{[n+p]^2}{[n]^2} - 2\left(1+\frac{1}{q^r}\right)\frac{[n+p]}{[n]} + 2\right]x^2 + \frac{[n+p]x}{[n]^2}}\right),$$

where μ_r is the same as in Section 2. Particularly, if $f \in Lip_r^2\alpha$ for some $\alpha \in (0,2]$, then

$$\mu_r(x)\left|S_{n,q}(f;x;p) - f(x)\right| \le \omega_r^1(f;\frac{[n+p]x}{[n]} - x) + M_r(q,p)$$

$$\times\left(\left[\left(1+\frac{1}{q^{2r+1}}\right)\frac{[n+p]^2}{[n]^2} - 2\left(1+\frac{1}{q^r}\right)\frac{[n+p]}{[n]} + 2\right]x^2 + \frac{[n+p]x}{[n]^2}\right)^{\frac{\alpha}{2}}$$

holds.

We first consider the modified Steklov means (see [5]) of a function $f \in C_r[0,\infty)$ as follows:

$$f_h(y) := \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\{2f(y+s+t) - f(y+2(s+t))\}\,dsdt,$$

where $h > 0$ and $y \ge 0$. In this case, it is clear that

$$f(y) - f_h(y) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\Delta_{s+t}^2 f(y)\,dsdt,$$

which guarantees that

$$\|f - f_h\|_r \le \omega_r^2(f;h). \tag{4.1}$$

Furthermore, we have $f_h''(y) = \frac{1}{h^2}\left(8\Delta_{h/2}^2 f(y) - \Delta_h^2 f(y)\right)$, which implies

$$\|f_h''\|_r \le \frac{9}{h^2}\omega_r^2(f;h). \tag{4.2}$$

Then, combining (4.1) with (4.2) we conclude that the Steklov means f_h corresponding to $f \in C_r[0,\infty)$ belongs to $C_r^2[0,\infty)$.

Proof. Let $r \in \mathbb{N},\ f \in C_r[0,\infty)$ and $x \in [0,\infty)$ be fixed. Assume that, for $h > 0$, f_h denotes the Steklov means of f. For any $n \in \mathbb{N}$, the following inequality holds:

$$\left|S_{n,q}(f;x;p) - f(x)\right| \le S_{n,q}^*(|f(y) - f_h(y)|;x;p) + |f(x) - f_h(x)|$$

$$+\left|S^*_{n,q}(f_h;x;p)-f_h(x)\right|+\left|f\left(\frac{[n+p]x}{[n]}\right)-f(x)\right|.$$

Since $f_h \in C_r^2[0,\infty)$, it follows from Lemma 6 that

$$\begin{aligned}
&\mu_r(x)\left|S_{n,q}(f;x;p)-f(x)\right| \le \|f-f_h\|_r\left\{\mu_r(x)S^*_{n,q}\left(\frac{1}{\mu_r};x;p\right)+1\right\}\\
&+M_r(q,p)\left\|f_h''\right\|_p\left\{\left[\left(1+\frac{1}{q^{2r+1}}\right)\frac{[n+p]^2}{[n]^2}-2\left(1+\frac{1}{q^r}\right)\frac{[n+p]}{[n]}+2\right]x^2\right.\\
&\left.+\frac{[n+p]x}{[n]^2}\right\}+\mu_r(x)\left|f\left(\frac{[n+p]x}{[n]}\right)-f(x)\right|\\
\le\;& \|f-f_h\|_p\left\{\mu_r(x)S_{n,q}\left(\frac{1}{\mu_r};x\right)+3\right\}\\
&+M_r(q,p)\left\|f_h''\right\|_r\left\{\left[\left(1+\frac{1}{q^{2r+1}}\right)\frac{[n+p]^2}{[n]^2}-2\left(1+\frac{1}{q^r}\right)\frac{[n+p]}{[n]}+2\right]x^2\right.\\
&\left.+\frac{[n+p]x}{[n]^2}\right\}+\mu_r(x)\left|f\left(\frac{[n+p]x}{[n]}\right)-f(x)\right|.
\end{aligned}$$

By (4.1) and (4.2), the last inequality yields that

$$\begin{aligned}
&\mu_r(x)\left|S_{n,q}(f;x;p)-f(x)\right| \le M_r(q)\omega_r^2(f;h)\\
&\times\left\{1+\frac{1}{h^2}\left\{\left[\left(1+\frac{1}{q^{2r+1}}\right)\frac{[n+p]^2}{[n]^2}-2\left(1+\frac{1}{q^r}\right)\frac{[n+p]}{[n]}+2\right]x^2\right.\right.\\
&\left.\left.+\frac{[n+p]x}{[n]^2}\right\}\right\}+\omega_r^1(f;f\left(\frac{[n+p]x}{[n]}\right)-f(x)).
\end{aligned}$$

Thus, choosing $h=\sqrt{\left[\left(1+\frac{1}{q^{2r+1}}\right)\frac{[n+p]^2}{[n]^2}-2\left(1+\frac{1}{q^r}\right)\frac{[n+p]}{[n]}+2\right]x^2+\frac{[n+p]x}{[n]^2}}$, the first part of the proof is completed. The remaining part of the proof can be easily obtained from the definition of the space $Lip_p^2\alpha$. □

REFERENCES

[1] O. Agratini and O. Doğru, "Weighted approximation by q-Szász-King type operators," *Taiwanese J. Math.*, vol. 14, no. 4, pp. 1283–1296, 2010.

[2] F. Altomare and M. Campiti, *Korovkin-type approximation theory and its applications*, ser. de Gruyter Stud. Math. Berlin: Walter de Gruyter & Co., 1994, vol. 17.

[3] A. Aral, "A generalization of Szász-Mirakyan operators based on q-integers," *Math. Comput. Modelling*, vol. 47, no. 9-10, pp. 1052–1062, 2008.

[4] A. Aral and V. Gupta, "The q-derivative and applications to q-Szász Mirakyan operators," *Calcolo*, vol. 43, no. 3, pp. 151–170, 2006.

[5] M. Becker, "Global approximation theorems for Szász-Mirakjan and Baskakov operators in polynomial weight spaces," *Indiana Univ. Math. J.*, vol. 27, no. 1, pp. 127–142, 1978.

[6] V. Gupta, "Some approximation properties of q-Durrmeyer operators," *Appl. Math. Comput.*, vol. 197, no. 1, pp. 172–178, 2008.

[7] V. Gupta and A. Aral, "Convergence of the q analogue of Szász-beta operators," *Appl. Math. Comput.*, vol. 216, no. 2, pp. 374–380, 2010.

[8] A. Il'inskii and S. Ostrovska, "Convergence of generalized Bernstein polynomials," *J. Approx. Theory*, vol. 116, no. 1, pp. 100–112, 2002.

[9] A. Lupaş, "A q-analogue of the Bernstein operator," in *Seminar on Numerical and Statistical Calculus (Cluj-Napoca, 1987)*, ser. Preprint, vol. 87-9. Cluj-Napoca: Univ. "Babeş-Bolyai", 1987, pp. 85–92.

[10] N. I. Mahmudov, "q-Szász-Mirakjan operators which preserve x^2," *J. Comput. Appl. Math.*, vol. 235, no. 16, pp. 4621–4628, 2011.

[11] N. I. Mahmudov, "On q-parametric Szász-Mirakjan operators," *Mediterr. J. Math.*, vol. 7, no. 3, pp. 297–311, 2010.

[12] N. I. Mahmudov and V. Gupta, "On certain q-analogue of Szász Kantorovich operators," *J. Appl. Math. Comput.*, vol. 37, no. 1-2, pp. 407–419, 2011.

[13] N. I. Mahmudov and H. Kaffaoğlu, "On q-Szász-Durrmeyer operators," *Cent. Eur. J. Math.*, vol. 8, no. 2, pp. 399–409, 2010.

[14] M. Örkcü and O. Doğru, "Weighted statistical approximation by Kantorovich type q-Szász-Mirakjan operators," *Appl. Math. Comput.*, vol. 217, no. 20, pp. 7913–7919, 2011.

[15] S. Ostrovska, "q-Bernstein polynomials and their iterates," *J. Approx. Theory*, vol. 123, no. 2, pp. 232–255, 2003.

[16] S. Ostrovska, "On the limit q-Bernstein operator," *Math. Balkanica (N.S.)*, vol. 18, no. 1-2, pp. 165–172, 2004.

[17] M. A. Óżarslan and O. Duman, "Approximation theorems by Meyer-König and Zeller type operators," *Chaos Solitons Fractals*, vol. 41, no. 1, pp. 451–456, 2009.

[18] M. Özarslan, "q-Laguerre type linear positive operators," *Studia Sci. Math. Hungar.*, vol. 44, no. 1, pp. 65–80, 2007.

[19] G. M. Phillips, "On generalized Bernstein polynomials," in *Numerical analysis*. River Edge, NJ: World Sci. Publ., 1996, pp. 263–269.

[20] G. M. Phillips, "Bernstein polynomials based on the q-integers," *Ann. Numer. Math.*, vol. 4, no. 1-4, pp. 511–518, 1997.

[21] T. Trif, "Meyer-König and Zeller operators based on the q-integers," *Rev. Anal. Numér. Théor. Approx.*, vol. 29, no. 2, pp. 221–229, 2000.

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