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LOCAL APPROXIMATION BEHAVIOR OF MODIFIED SMK OPERATORS

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Abstract. In this paper, for a general modification of the classical Szász–Mirakjan–Kantorovich operators, we obtain many local approximation results including the classical cases. In particular, we obtain a Korovkin theorem, a Voronovskaya theorem, and some local estimates for these operators.

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1. INTRODUCTION

As usual, let $C[0,\infty)$ denote the space of all continuous functions on $[0,\infty)$. The classical Szász–Mirakjan–Kantorovich (SMK) operators [4] are given by the relation

$$K_n(f;x) := ne^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_{I_{n,k}} f(t)dt, \tag{1.1}$$

where $I_{n,k} = \left[\frac{k}{n}, \frac{k+1}{n}\right]$ and f belongs to an appropriate subspace of $C[0,\infty)$ for which the above series is convergent. Among of these subspaces, we can take the space $C_B[0,\infty)$ of all bounded and continuous functions on $[0,\infty)$, or, the weighted space $C_\gamma[0,+\infty)$, $\gamma > 0$, defined by the equality

$$C_\gamma[0,+\infty) := \left\{f \in C[0,+\infty) : |f(t)| \le M(1+t)^\gamma \text{ for some } M > 0\right\}.$$

Assume now that (u_n) is a sequence of functions on $[0,\infty)$ such that, for a fixed $a \ge 0$,

$$0 \le u_n(x) \le x \quad \text{for every } x \in [a,\infty) \text{ and } n \in \mathbb{N}. \tag{1.2}$$

Then, we consider the following modification of SMK operators:

$$L_n(f;x) := \sum_{k=0}^{\infty} p_{k,n}(x)\int_{I_{n,k}} f(t)dt, \quad n \in \mathbb{N},\ x \in [a,\infty), \tag{1.3}$$

where

$$p_{k,n}(x) := ne^{-nu_n(x)}\frac{(nu_n(x))^k}{k!}, \quad k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \tag{1.4}$$

Throughout the paper we use the following test functions

$$e_i(y) = y^i, \quad i = 0,1,2,3,4,$$

and the moment function

$$\psi_x(y) = y - x.$$

So, using the fundamental properties of the classical SMK operators, one can get the following lemmas.

Lemma 1. *For the operators L_n, we have*

(i) $L_n(e_0;x) = 1$,

(ii) $L_n(e_1;x) = u_n(x) + \frac{1}{2n}$,

(iii) $L_n(e_2;x) = u_n^2(x) + \frac{2u_n(x)}{n} + \frac{1}{3n^2}$,

(iv) $L_n(e_3;x) = u_n^3(x) + \frac{9u_n^2(x)}{2n} + \frac{7u_n(x)}{2n^2} + \frac{1}{4n^3}$,

(v) $L_n(e_4;x) = u_n^4(x) + \frac{8u_n^3(x)}{n} + \frac{15u_n^2(x)}{n^2} + \frac{6u_n(x)}{n^3} + \frac{1}{5n^4}$.

Lemma 2. *For the operators L_n, we have*

(i) $L_n(\psi_x;x) = u_n(x) - x + \frac{1}{2n}$,

(ii) $L_n(\psi_x^2;x) = (u_n(x)-x)^2 + \frac{2u_n(x)-x}{n} + \frac{1}{3n^2}$,

(iii) $L_n(\psi_x^3;x) = (u_n(x)-x)^3 + \frac{3(3u_n(x)-x)(u_n(x)-x)}{2n} + \frac{7u_n(x)-2x}{2n^2} + \frac{1}{4n^3}$,

(iv) $L_n(\psi_x^4;x) = (u_n(x)-x)^4 + \frac{2(4u_n(x)-x)(x-u_n(x))^2}{n}$
$+ \frac{15u_n^2(x)-14xu_n(x)+2x^2}{n^2} + \frac{6u_n(x)-x^3}{n^3} + \frac{1}{5n^4}$.

Then, we see from Lemma 1 that, with some suitable choices of u_n, our operators L_n may preserve the linear functions or the test function e_2. For example, taking $a = \frac{1}{2}$, if we choose $u_n(x) = x - \frac{1}{2n}$ for $x \in \left[\frac{1}{2},\infty\right)$ and $n \in \mathbb{N}$, then the corresponding operators L_n preserve the linear functions, i.e., they preserve the test functions e_0 and e_1 (see [8]). In this case, we know from [8] that the operators L_n have a better error estimation than the classical SMK operators on $\left[\frac{1}{2},\infty\right)$. Also, taking $a = \frac{1}{3}$ and

$$u_n(x) := \frac{\sqrt{3n^2x^2+2}-\sqrt{3}}{n\sqrt{3}} \quad \text{for } x \in \left[\frac{1}{\sqrt{3}},\infty\right) \text{ and } n \in \mathbb{N},$$

we see that the corresponding operators L_n preserve the test functions e_0 and e_2. Finally, for $a = 0$ and

$$u_n(x) := \frac{-1+\sqrt{4n^2x^2+1}}{2n}, \quad x \geq 0 \text{ and } n \in \mathbb{N},$$

the corresponding operators L_n becomes the Kantorovich variant of the modified Szász–Mirakjan operators (see [6, 11]).

The first study regarding the preservation of e_0 and e_2 for the linear positive operators in order to get better error estimation, was first presented by King. In [9], King introduced a modification of the classical Bernstein polynomials and had a better error estimation than the classical ones on the interval $\left[0, \frac{1}{3}\right]$. Later, similar problems were accomplished for Szász–Mirakjan operators [6], Szász–Mirakjan–Beta operators [7], Meyer–König and Zeller operators [11], Bernstein–Chlodovsky operators [1], q-Bernstein operators [10], Baskakov operators and Stancu operators [12], and some other kinds of summation-type positive linear operators [2].

However, in the present paper, for a general sequence (u_n) satisfying (1.2), we study the local approximation behavior of the operators L_n defined by (1.3) and (1.4). First of all, we get the following Korovkin-type approximation theorem for these operators.

Theorem 1. *Let (u_n) be a sequence of functions on $[0,\infty)$ satisfying (1.2) for a fixed $a \geq 0$. If*

$$\lim_{n\to\infty} u_n(x) = x \tag{1.5}$$

uniformly with respect to $x \in [a,b]$ with $b > a$, then, for all $f \in C_\gamma[0,+\infty)$ with $\gamma \geq 2$, we have

$$\lim_{n\to\infty} L_n(f;x) = f(x)$$

uniformly with respect to $x \in [a,b]$.

Proof. For a fixed $b > 0$, consider the lattice homomorphism $H_b : C[0,+\infty) \to C[a,b]$ defined by $H_b(f) := f|_{[a,b]}$ for every $f \in C[0,+\infty)$. In this case, from (1.5), we see that, for each $i = 0,1,2$,

$$\lim_{n\to\infty} H_b(T_n(e_i)) = H_b(e_i)$$

uniformly on $[a,b]$. Hence, using the universal Korovkin-type property with respect to monotone operators (see Theorem 4.1.4(vi) of [3, p. 199]), we obtain that, for all $f \in C_\gamma[0,+\infty)$, $\gamma \geq 2$,

$$\lim_{n\to\infty} L_n(f;x) = f(x)$$

uniformly with respect to $x \in [a,b]$. □

Finally, using Proposition 4.2.5(2) of [3], we can state the following approximation result in the space L_p:

Corollary 1. *Let $1 \leq p < \infty$. Then for all $f \in L_p$, we have*

$$\lim_{n\to\infty} L_n(f;x) = f(x)$$

uniformly with respect to $x \in [a,\infty)$ with $a \geq 0$.

2. A VORONOVSKAYA-TYPE THEOREM

In order to get a Voronovskaya-type theorem for the operators L_n given by (1.3) and (1.4), we need the following lemma.

Lemma 3. *Let (u_n) be a sequence of functions on $[0,\infty)$ satisfying (1.2) for a fixed $a \geq 0$. If*

$$\lim_{n\to\infty} \sqrt{n}\,(x - u_n(x)) = 0 \tag{2.1}$$

uniformly with respect to $x \in [a,b]$, $b > a$, then we have

$$\lim_{n\to\infty} n^2 L_n(\psi_x^4;x) = 3x^2 \tag{2.2}$$

uniformly with respect to $x \in [a,b]$.

Proof. Let $x \in [a,b]$, $b > a$, be fixed. Then, by (1.2), since

$$0 \leq x - u_n(x) \leq \sqrt{n}(x - u_n(x)) \quad \text{for every } n \in \mathbb{N},$$

it follows from (2.1) that

$$\lim_{n\to\infty} u_n(x) = x \tag{2.3}$$

uniformly with respect to $x \in [a,b]$. Also, since

$$0 \leq \frac{u_n(x)}{n} = \frac{u_n(x) - x}{n} + \frac{x}{n} \leq x - u_n(x) + \frac{x}{n},$$

we obtain from (2.1) that

$$\lim_{n\to\infty} \frac{u_n(x)}{n} = 0 \tag{2.4}$$

uniformly with respect to $x \in [a,b]$. Observe now that, by Lemma 2(iv),

$$\begin{aligned} n^2 L_n(\psi_x^4;x) = &\left\{\sqrt{n}(x - u_n(x))\right\}^4 \\ &+ 2\left\{\sqrt{n}(x - u_n(x))\right\}^2 (4u_n(x) - x) \\ &+ \left\{15u_n^2(x) - 14xu_n(x) + 2x^2\right\} + \frac{6}{n}u_n(x) - \frac{x}{n} + \frac{1}{5n^2}. \end{aligned}$$

Taking limit as $n \to \infty$ in the both sides of the last equality and also using (2.1), (2.3), (2.4), we immediately see that

$$\lim_{n\to\infty} n^2 L_n(\psi_x^4;x) = 3x^2$$

uniformly with respect to $x \in [a,b]$. The proof is complete. □

We now get the following result.

Theorem 2. *Let (u_n) be a sequence of functions on $[0,\infty)$ satisfying (1.2) and (2.1) for a fixed $a \geq 0$. Assume further that there exists a function ξ defined on $[a,\infty)$ such that*

$$\lim_{n\to\infty} n\left(x-u_n(x)\right)=\xi(x) \tag{2.5}$$

uniformly with respect to $x\in[a,b]$, $b>a$. Then, for every $f\in C_\gamma[0,+\infty)$, $\gamma\geq 4$, such that $f', f''\in C_\gamma[0,+\infty)$, we have

$$\lim_{n\to\infty} n\{L_n(f;x)-f(x)\}=\frac{1}{2}xf''(x)+\left(\frac{1}{2}-\xi(x)\right)f'(x)$$

uniformly with respect to $x\in[a,b]$.

Proof. Let $f, f', f''\in C_\gamma[0,+\infty)$ with $\gamma\geq 4$. Define

$$\Psi(y,x)=\begin{cases}\frac{f(y)-f(x)-(y-x)f'(x)-\frac{1}{2}(y-x)^2f''(x)}{(y-x)^2} & \text{for } y\neq x,\\ 0 & \text{for } y=x.\end{cases}$$

Then, it is clear that $\Psi(x,x)=0$ and that the function $\Psi(\cdot,x)$ belongs to $C_\gamma[0,+\infty)$. Hence, it follows from the Taylor theorem that

$$f(y)-f(x)=\psi_x(y)f'(x)+\frac{1}{2}\psi_x^2(y)f''(x)+\psi_x^2(y)\Psi(y,x).$$

Now, by Lemma 2(ii) and (iii), we get

$$\begin{aligned}n\{L_n(f;x)-f(x)\}&=nf'(x)L_n\left(\psi_x;x\right)+\frac{n}{2}f''(x)L_n\left(\psi_x^2;x\right)\\&\quad+nL_n(\psi_x^2(y)\Psi(y,x);x),\end{aligned}$$

which gives

$$\begin{aligned}n\{L_n(f;x)-f(x)\}&=f'(x)\left\{n\left(u_n(x)-x\right)+\frac{1}{2}\right\}\\&\quad+\frac{f''(x)}{2}\left\{\left(\sqrt{n}\left(u_n(x)-x\right)\right)^2+2u_n(x)-x\right\}\\&\quad+nL_n(\psi_x^2(y)\Psi(y,x);x).\end{aligned} \tag{2.6}$$

If we apply the Cauchy–Schwarz inequality for the last term on the right-hand side of (2.6), then we conclude that

$$n\left|L_n\left(\psi_x^2(y)\Psi(y,x);x\right)\right|\leq\left(n^2L_n(\psi_x^4(y);x)\right)^{1/2}\left(L_n(\Psi^2(y,x);x)\right)^{1/2}. \tag{2.7}$$

Let $\eta(y,x):=\Psi^2(y,x)$. In this case, observe that $\eta(x,x)=0$ and $\eta(\cdot,x)\in C_\gamma[0,+\infty)$. Then it follows from Theorem 1 that

$$\lim_{n\to\infty} L_n\left(\Psi^2(y,x);x\right)=\lim_{n\to\infty} L_n\left(\eta(y,x);x\right)=\eta(x,x)=0 \tag{2.8}$$

uniformly with respect to $x \in [a,b]$, $b > a$. So, considering (2.5), (2.7) and (2.8), and also using Lemma 3, we immediately see that

$$\lim_{n\to\infty} nL_n\left(\psi_x^2(y)\Psi(y,x);x\right) = 0 \tag{2.9}$$

uniformly with respect to $x \in [a,b]$. Taking limit as $n \to \infty$ in (2.6) and also using (2.1), (2.3), (2.5), (2.9) we have

$$\lim_{n\to\infty} n\{L_n(f;x) - f(x)\} = \frac{1}{2}xf''(x) + \left(\frac{1}{2} - \xi(x)\right)f'(x)$$

uniformly with respect to $x \in [a,b]$. The proof is complete. □

We should note that one can find a sequence (u_n) satisfying all assumptions (1.2), (2.1) and (2.5) in Theorem 2. For example, if we take $a = 0$ and $u_n(x) = x$, then our operators in (1.3) turn out to be the classical SMK operators K_n defined by (1.1). In this case, we have $\xi(x) = 0$. Hence, we obtain the following result.

Corollary 2. *For the operators (1.1), if $f \in C_\gamma[0,+\infty)$, $\gamma \geq 4$, such that $f', f'' \in C_\gamma[0,+\infty)$, then we have*

$$\lim_{n\to\infty} n\{K_n(f;x) - f(x)\} = \frac{1}{2}xf''(x) + \frac{1}{2}f'(x)$$

uniformly with respect to $x \in [0,b]$, $b > 0$.

Now, if take $a = 0$ and

$$u_n(x) := u_n^{[1]}(x) = \frac{-1+\sqrt{4n^2x^2+1}}{2n}, \quad x \in [0,\infty),\ n \in \mathbb{N}, \tag{2.10}$$

then our operators L_n in (1.3) turn out to be

$$L_n^{[1]}(f;x) := ne^{-\left(-1+\sqrt{4n^2x^2+1}\right)/2}\sum_{k=0}^{\infty}\frac{\left(-1+\sqrt{4n^2x^2+1}\right)^k}{2^k k!}\int_{I_{n,k}} f(t)dt. \tag{2.11}$$

In this case, observe that

$$\xi(x) = \lim_{n\to\infty} n\left(x - u_n^{[1]}(x)\right) = \begin{cases} 0 & \text{if } x = 0, \\ \frac{1}{2} & \text{if } x > 0. \end{cases}$$

So, the next result immediately follows from Theorem 2.

Corollary 3. *For the operators (2.11), if $f \in C_\gamma[0,+\infty)$, $\gamma \geq 4$, such that $f', f'' \in C_\gamma[0,+\infty)$, then we have*

$$\lim_{n\to\infty} n\{L_n^{[1]}(f;x) - f(x)\} = \begin{cases} f'(0)/2 & \text{if } x = 0, \\ xf''(x)/2 & \text{if } x > 0. \end{cases}$$

Furthermore, if we choose $a = \frac{1}{2}$ and

$$u_n(x) := u_n^{[2]}(x) = x - \frac{1}{2n}, \quad x \in \left[\frac{1}{2}, \infty\right), \ n \in \mathbb{N},$$

then the operators in (1.3) reduce to the following operators (see [8]):

$$L_n^{[2]}(f;x) := ne^{\frac{1-2nx}{2}} \sum_{k=0}^{\infty} \frac{(2nx-1)^k}{2^k k!} \int_{I_{n,k}} f(t)dt. \tag{2.12}$$

Then, we observe that

$$\xi(x) = \lim_{n\to\infty} n\left(x - u_n^{[2]}(x)\right) = \frac{1}{2}.$$

Therefore, we get the next result at once.

Corollary 4 ([8])**.** *For the operators* (2.12)*, if* $f \in C_\gamma[0,+\infty)$*,* $\gamma \geq 4$*, such that* $f', f'' \in C_\gamma[0,+\infty)$*, then we have*

$$\lim_{n\to\infty} n\{L_n^{[2]}(f;x) - f(x)\} = \frac{1}{2} x f''(x)$$

uniformly with respect to $x \in [1/2, b]$*,* $b > 1/2$*.*

Finally, taking $a = \frac{1}{\sqrt{3}}$ and

$$u_n(x) := u_n^{[3]}(x) = \frac{\sqrt{3n^2x^2+2} - \sqrt{3}}{n\sqrt{3}}, \quad x \in \left[\frac{1}{\sqrt{3}}, \infty\right), \ n \in \mathbb{N}, \tag{2.13}$$

we get the following positive linear operators:

$$L_n^{[3]}(f;x) := ne^{\frac{\sqrt{3}-\sqrt{3n^2x^2+2}}{\sqrt{3}}} \sum_{k=0}^{\infty} \frac{\left(\sqrt{3n^2x^2+2} - \sqrt{3}\right)^k}{3^{k/2} k!} \int_{I_{n,k}} f(t)dt. \tag{2.14}$$

In this case, we find that

$$\xi(x) = \lim_{n\to\infty} n\left(x - u_n^{[3]}(x)\right) = 1.$$

Then, for the corresponding operators, we have the following

Corollary 5. *For the operators* (2.14)*, if* $f \in C_\gamma[0,+\infty)$*,* $\gamma \geq 4$*, such that* $f', f'' \in C_\gamma[0,+\infty)$*, then we have*

$$\lim_{n\to\infty} n\{L_n^{[3]}(f;x) - f(x)\} = \frac{1}{2} x f''(x) - \frac{1}{2} f'(x)$$

uniformly with respect to $x \in \left[\frac{1}{\sqrt{3}}, b\right]$*,* $b > \frac{1}{\sqrt{3}}$*.*

3. Local approximation results for the operators L_n

In order to study various local approximation properties of the operators L_n we mainly use the (usual) modulus of continuity, the second modulus of smoothness, and Peetre's K-functional.

By $C_B^2[0,\infty)$ we denote the space of all functions $f \in C_B[0,\infty)$ such that $f', f'' \in C_B[0,\infty)$. Let $\|f\|$ denote the usual supremum norm of a bounded function f. Then, the classical Peetre's K-functional and the second modulus of smoothness of a function $f \in C_B[0,\infty)$ are defined respectively by

$$K(f,\delta) := \inf_{g \in C_B^2[0,\infty)} \left\{ \|f-g\| + \delta \|g''\| \right\}$$

and

$$\omega_2(f,\delta) := \sup_{0<h\leq\delta,\ x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|,$$

where $\delta > 0$. Then, by Theorem 2.4 of [5, p. 177], there exists a constant $C > 0$ such that

$$K(f,\delta) \leq C\omega_2\big(f,\sqrt{\delta}\big). \tag{3.1}$$

Also, as usual, by $\omega(f,\delta)$, $\delta > 0$, we denote the usual modulus of continuity of f.

Then, we first get the following local approximation result.

Theorem 3. *Let (u_n) be a sequence of functions on $[0,\infty)$ satisfying (1.2) for a fixed $a \geq 0$. For any $f \in C_B[0,\infty)$ and for every $x \in [a,\infty)$, $n \in \mathbb{N}$, we have*

$$|L_n(f;x) - f(x)| \leq C\omega_2\left(f, \sqrt{\delta_n(x)}\right) + \omega\left(f, \left|u_n(x) - x + \frac{1}{2n}\right|\right)$$

for some constant $C > 0$, where

$$\delta_n(x) := (u_n(x) - x)^2 + \frac{2u_n(x) - x}{n} + \frac{1}{3n^2}. \tag{3.2}$$

Proof. Define an operator $\Omega_n : C_B[0,\infty) \to C_B[0,\infty)$ by

$$\Omega_n(f;x) := L_n(f;x) - f\left(u_n(x) + \frac{1}{2n}\right) + f(x). \tag{3.3}$$

So, by Lemma 2(ii), we get

$$\Omega_n(\psi_x;x) = L_n(\psi_x;x) - u_n(x) - \frac{1}{2n} + x = 0. \tag{3.4}$$

Let $g \in C_B^2[0,\infty)$, the space of all functions having the second continuous derivative on $[0,\infty)$, and let $x \in [0,\infty)$. Then, it follows from the well-known Taylor formula that

$$g(y) - g(x) = \psi_x(y)g'(x) + \int_x^y \psi_t(y)g''(t)\,dt, \quad y \in [0,\infty).$$

By (3.4), we get

$$\begin{aligned}|\Omega_n(g;x)-g(x)| &= |\Omega_n(g(y)-g(x);x)|\\ &= \left|\Omega_n\left(\int_x^y \psi_t(y)g''(t)dt;x\right)\right|\\ &= \Bigg|L_n\left(\int_x^y \psi_t(y)g''(t)dt;x\right)\\ &\quad -\int_x^{u_n(x)+\frac{1}{2n}} \psi_t\left(u_n(x)+\frac{1}{2n}\right)g''(t)dt\Bigg|.\end{aligned}$$

Using (3.3) and Lemma 2(ii), we obtain that

$$\begin{aligned}|\Omega_n(g;x)-g(x)| &\leq \frac{\|g''\|}{2}L_n\left(\psi_x^2;x\right)+\frac{\|g''\|}{2}\psi_x^2\left(u_n(x)+\frac{1}{2n}\right)\\ &= \frac{\|g''\|}{2}\left\{\left((u_n(x)-x)^2+\frac{2u_n(x)-x}{n}+\frac{1}{3n^2}\right)\right.\\ &\quad \left.+\left(u_n(x)-x+\frac{1}{2n}\right)^2\right\},\end{aligned}$$

which implies that

$$|\Omega_n(g;x)-g(x)| \leq \|g''\|\delta_n(x), \tag{3.5}$$

where $\delta_n(x)$ is given by (3.2). Then, for any $f\in C_B[0,\infty)$, it follows from (3.5) that

$$\begin{aligned}|L_n(f;x)-f(x)| &\leq |\Omega_n(f-g;x)-(f-g)(x)|\\ &\quad +|\Omega_n(g;x)-g(x)|+\left|f\left(u_n(x)+\frac{1}{2n}\right)-f(x)\right|\\ &\leq 2\|f-g\|+\delta_n(x)\|g''\|+\left|f\left(u_n(x)+\frac{1}{2n}\right)-f(x)\right|\\ &\leq 2\{\|f-g\|+\delta_n(x)\|g''\|\}+\left|f\left(u_n(x)+\frac{1}{2n}\right)-f(x)\right|.\end{aligned}$$

Hence, by (3.1), we deduce that

$$\begin{aligned}|L_n(f;x)-f(x)| &\leq 2\{\|f-g\|+\delta_n(x)\|g''\|\}+\omega\left(f,\left|u_n(x)-x+\frac{1}{2n}\right|\right)\\ &\leq 2K\left(f,\delta_n(x)\right)+\omega\left(f,\left|u_n(x)-x+\frac{1}{2n}\right|\right)\\ &\leq C\omega_2\left(f,\sqrt{\delta_n(x)}\right)+\omega\left(f,\left|u_n(x)-x+\frac{1}{2n}\right|\right)\end{aligned}$$

which completes the proof. □

Using Theorem 3, one can get the following special cases.

Corollary 6. *For the classical SMK operators (1.1), we have, for any $x \geq 0$, $n \in \mathbb{N}$ and $f \in C_B[0,\infty)$,*

$$|K_n(f;x) - f(x)| \leq C\omega_2\left(f, \sqrt{\frac{x}{n} + \frac{1}{3n^2}}\right) + \omega\left(f, \frac{1}{2n}\right).$$

Corollary 7. *For the operators (2.11), we have, for any $f \in C_B[0,\infty)$, $x \geq 0$ and $n \in \mathbb{N}$,*

$$\left|L_n^{[1]}(f;x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{\delta_n^{[1]}(x)}\right) + \omega\left(f, \frac{\sqrt{4n^2x^2+1} - 2nx}{2n}\right),$$

where

$$\delta_n^{[1]}(x) := 2x^2 - \frac{1}{6n^2} + \frac{(1-2nx)\sqrt{4n^2x^2+1}}{2n^2}.$$

Corollary 8. *For the operators (2.12), we have, for any $f \in C_B[0,\infty)$, $x \geq \frac{1}{2}$ and $n \in \mathbb{N}$,*

$$\left|L_n^{[2]}(f;x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{\delta_n^{[2]}(x)}\right),$$

where

$$\delta_n^{[2]}(x) := \frac{x}{n} - \frac{5}{12n^2}.$$

Corollary 9. *For the operators (2.14) we have, for any $f \in C_B[0,\infty)$, $x \geq \frac{1}{\sqrt{3}}$ and $n \in \mathbb{N}$,*

$$\left|L_n^{[3]}(f;x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{\delta_n^{[3]}(x)}\right) + \omega\left(f, \frac{2\sqrt{3}\sqrt{3n^2x^2+2} - 6(nx+1) + 3}{6n}\right),$$

where

$$\delta_n^{[3]}(x) := 2x^2 + \frac{x\left(3 - 2\sqrt{3}\sqrt{3n^2x^2+2}\right)}{3n}.$$

4. Estimates for Lipschitz-type functions

In this section, for a fixed $a \geq 0$, we consider the following Lipschitz-type space

$$\text{Lip}_M^*(r) := \left\{ f \in C_B[0,\infty) : |f(t) - f(x)| \leq M\frac{|t-x|^r}{(t+x)^{r/2}};\ x,t \in (a,\infty)\right\},$$

where M is any positive constant and $0 < r \leq 1$.

In order to give an estimation in approximating the functions in $\text{Lip}_M^*(r)$ we need the next lemma.

Lemma 4. *Let (u_n) be a sequence of functions on $[0,\infty)$ satisfying (1.2) for a fixed $a \geq 0$. For every $x > a$ and $n \in \mathbb{N}$, we have*

$$\sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x|\, dt \leq \sqrt{\frac{\delta_n(x)}{n}}, \tag{4.1}$$

where $p_{n,k}(x)$ and $\delta_n(x)$ are given by (1.4) and (3.2), respectively.

Proof. Applying the Cauchy–Schwarz inequality to the series in the left hand side of (4.1), we get

$$\sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x|\, dt \leq \left\{ \sum_{k=0}^{\infty} p_{n,k}(x) \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x|\, dt \right)^2 \right\}^{1/2}.$$

If we again apply the Cauchy–Schwarz inequality to the integral in the right-hand side of the last inequality and also use Lemma 2(ii), then we see that

$$\begin{aligned} \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x|\, dt &\leq \frac{1}{\sqrt{n}} \left\{ \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} (t-x)^2\, dt \right\}^{1/2} \\ &= \frac{1}{\sqrt{n}} \sqrt{L_n\left(\psi_x^2; x\right)} \\ &= \sqrt{\frac{\delta_n(x)}{n}}, \end{aligned}$$

whence the result follows. □

Now we are in position to give our result.

Theorem 4. *Let (u_n) be a sequence of functions on $[0,\infty)$ satisfying (1.2) for a fixed $a \geq 0$. Then, for any $f \in \mathrm{Lip}_M^*(r)$, $r \in (0,1]$, and for every $n \in \mathbb{N}$ and $x \in (a,\infty)$, we have*

$$|L_n(f;x) - f(x)| \leq \frac{M \delta_n^{r/2}(x)}{n^{1-r+r/2} x^{r/2}}, \tag{4.2}$$

where $\delta_n(x)$ is given by (3.2).

Proof. We first assume that $r = 1$. So, let $f \in \mathrm{Lip}_M^*(1)$ and $x \in (a,\infty)$. Then, we get

$$\begin{aligned} |L_n(f;x) - f(x)| &\leq \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t) - f(x)|\, dt \\ &\leq M \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \frac{|t-x|}{\sqrt{t+x}}\, dt. \end{aligned}$$

Since $\frac{1}{\sqrt{t+x}} \leq \frac{1}{\sqrt{x}}$, we may write that

$$|L_n(f;x) - f(x)| \leq \frac{M}{\sqrt{x}} \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x|\, dt.$$

Now, by Lemma 4, we conclude that

$$|L_n(f;x) - f(x)| \leq M \sqrt{\frac{\delta_n(x)}{nx}}, \tag{4.3}$$

which gives the desired result for $r = 1$. Assume now that $r \in (0,1)$. Then, taking $p = \frac{1}{r}$ and $q = \frac{1}{1-r}$, for any $f \in \mathrm{Lip}_M^a(r)$, if we apply the Hölder inequality two times, then we obtain that

$$\begin{aligned} |L_n(f;x) - f(x)| &\leq \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t) - f(x)|\, dt \\ &\leq \left\{ \sum_{k=0}^{\infty} p_{n,k}(x) \left(\int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t) - f(x)|\, dt \right)^{1/r} \right\}^r \\ &\leq \frac{1}{n^{1-r}} \left\{ \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t) - f(x)|^{1/r}\, dt \right\}^r. \end{aligned}$$

Using the definition of the space $\mathrm{Lip}_M^*(r)$ and also considering Lemma 4, we get

$$\begin{aligned} |L_n(f;x) - f(x)| &\leq \frac{M}{n^{1-r}} \left\{ \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \frac{|t-x|}{\sqrt{t+x}} dt \right\}^r \\ &\leq \frac{M}{n^{1-r} x^{r/2}} \left\{ \sum_{k=0}^{\infty} p_{n,k}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x|\, dt \right\}^r \\ &\leq \frac{M \delta_n^{r/2}(x)}{n^{1-r+r/2} x^{r/2}}. \end{aligned}$$

Thus, the proof is complete. □

Finally, it should be noted that our Theorem 4 includes many special cases as in the previous sections. However, we omit the details.

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