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Abstract. In this article, we establish the existence of solutions for integral inclusion of fractional order. The study holds in the case when the set-valued (multi-valued) function has convex and non-convex values. Also we illustrate our results with examples to which our abstract theory applies.

1. INTRODUCTION

The class of fractional operator equations of various types plays very important role not only in mathematics but also in physics, control systems, dynamical systems and engineering. Naturally, such equations required to be solved. There are numerous books focused in this direction, that is concerning the linear and nonlinear problems involving different types of fractional derivatives as well as integral see [20, 22, 23, 27, 28]. In [13, 14, 17], El-Sayed and A. G. Ibrahim gave the concept of the definite integral of fractional order for set-valued function. Integral and integro-differential (linear and nonlinear) in abstract spaces has been studied by several authors see [3, 18, 24, 25]. Applications of this type of problem arise in the study of control systems (see [1, 2, 10]). This paper concerned with the following fractional integral inclusion.

$$u(t) - \sum_{i=1}^{n} b_i(t)u(t - \tau_i) \in I^\alpha F(t, u(t)); \ 0 < \alpha \le 1, \ t \in J := [0, T], \qquad (1.1)$$

where $\tau_i \le t \in J$, $\forall i = 1, ..., n$, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued function with nonempty values in $\mathbb{R}$, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $I^\alpha F(t, u(t))$ are the definite integral for the set-valued functions F of order α which is defined as

$$I^\alpha F(t, u(t)) = \left\{ \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau, u(\tau)) d\tau : f(t, u) \in S_F(u) \right\},$$

where
$$S_F(u) = \left\{ f \in L^1(J, \mathbb{R}) : f(t) \in F(t, u(t)) \ a.e. \ t \in J \right\}$$
denotes the set of selections of F and $b_i : J \to \mathbb{R}$ is a continuous function such that $B := \max_{1 \le i \le n} \{sup_{t \in J} \{|b_i(t)|\}\}$.

This paper will be organized as follows: In Section 2, we recall briefly some basic definitions and preliminary facts from set-valued analysis which will be used later. In Section 3, we establish the existence of the solution for the single-valued problem

$$u(t) = \sum_{i=1}^{n} b_i(t)u(t - \tau_i) + I^\alpha f(t, u(t)); \ 0 < \alpha \le 1, \ t \in J, \ f \in F(t, u(t)), \quad (1.2)$$

by using the Schauder fixed point theorem, (see [6]) and the uniqueness of the solution by using the Banach fixed point theorem (see [29]). In Section 4, we shall study the existence of solution for the set-valued problem 1.1 when F has convex values, as well when it has with non-convex values, via the single-valued problem as well as fixed point theorems of the set-valued function. In the first case (convex) a fixed point theorem due to Bohnenblust and Karlin (see [8, 31]) is used. A fixed point theorem for contraction set-valued functions due to Covitz and Nadler [11] is applied in the second one (non-convex). Also we illustrate our results with examples.

2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts from set-valued analysis which are used throughout this paper. For further background and details pertaining to this section we refer the reader to [4, 7, 15, 16, 19, 26, 30].

$\mathcal{B} := C[J, \mathbb{R}]$ is the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm
$$\|u\| = sup\{|u(t)| : t \in J\}$$
for each $u \in \mathcal{B}$. $\mathcal{L} := L^1[J, \mathbb{R}]$ denotes the Banach space of measurable functions $u : J \to \mathbb{R}$ which are Lebesgue integrable normed by

$$\|u\|_{L^1} = \int_0^T |u(t)| dt,$$

for each $u \in \mathcal{L}$. Let $(X, |.|)$ be a normed space, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, $\mathcal{P}_c(X) = \{Y \in \mathcal{P}(X) : Y \text{ is convex}\}$, $\mathcal{P}_{cl,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed and convex}\}$, $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. A set-valued function $F : X \to \mathcal{P}(X)$ is called *convex* (*closed*) valued if $F(x)$ is *convex* (*closed*) for all $x \in X$. F is called bounded valued on bounded set B if $F(B) = \bigcup_{x \in B} F(x)$ is bounded in X for all $B \in \mathcal{P}_b(X)$ i.e. $sup_{x \in B}\{sup\{|u| : u \in F(x)\}\} < \infty$. F is called upper semi-continuous (u.s.c) on X if for each $x_0 \in X$ the set $F(x_0)$ is nonempty closed subset of X and if for each open set N of X containing $F(x_0)$, there exists an open neighborhood N_0 of x_0 such that $F(N_0) \subseteq N$. F is u.s.c if

the set $F^{-1}(A) = \{x \in X : Fx \subset A\}$ is open in X for every open set A in X. F is called lower semi-continuous (l.s.c) on X if A is any open subset of X then $F^{-1}(A) = \{x \in X : Fx \bigcap A \neq \varnothing\}$ is open in X. F is called *continuous* if it is lower as well as upper semi-continuous on X. F is called compact if for every M bounded subset of X, $F(M)$ is relatively compact. Finally F is called completely continuous if it is upper semi-continuous and compact on X.

The following definitions are used in the sequel.

Definition 1. A mapping $p : J \times \mathbb{R} \to \mathbb{R}$ is said to be Carathéodory if

   (i) $t \to p(t, u)$ is measurable for each $u \in \mathbb{R}$,
   (ii) $u \to p(t, u)$ is continuous a.e. for $t \in J$.

A *Carathéodory* function $p(t, u)$ is called $L^1(J, \mathbb{R})$-*Carathéodory* if

   (iii) for each number $r > 0$ there exists a function $h_r \in L^1(J, \mathbb{R})$ such that $|p(t, u)| \leq h_r(t)$ a.e $t \in J$ for all $u \in \mathbb{R}$ with $|u| \leq r$.

A *Carathéodory* function $p(t, u)$ is called $L^1_X(J, \mathbb{R})$-*Carathéodory* if

   (iv) there exist a function $h \in L^1(J, \mathbb{R})$ such that $|p(t, u)| \leq h(t)$ a.e $t \in J$ for all $u \in \mathbb{R}$ where h is called the bounded function of p.

Definition 2. A set-valued function $F : J \to \mathcal{P}(\mathbb{R})$ is said to be measurable if for any $x \in X$, the function $t \mapsto d(x, F(t)) = inf\{|x - u| : u \in F(t)\}$ is measurable.

Definition 3. A set-valued function $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called Carathéodory if

   (i) $t \mapsto F(t, x)$ is measurable for each $x \in \mathbb{R}$, and
   (ii) $x \mapsto F(t, x)$ is u.s.c. for almost $t \in J$.

Definition 4. A set-valued function $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called L^1-Carathéodory if

   (i) F is Carathéodory and
   (ii) For each $r > 0$, there exists $h_r \in L^1(J, \mathbb{R})$ such that $\|F(t, u)\| = sup\{|f| : f \in F(t, u)\} \leq h_r(t)$ for all $|u| \leq r$ and for a.e. $t \in J$.

Definition 5 ([12])**.** A set-valued function $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is called $\mathbf{L}^1_X$-Carathéodory if there exists a function $h \in L^1(J, \mathbb{R})$ such that

$$\|F(t, u)\| = sup\{|f| : f \in F(t, u)\} \leq h(t), \ a.e. \ t \in J$$

for all $x \in \mathbb{R}$, and the function h is called a growth function of F on $J \times \mathbb{R}$.

Let $A, B \in \mathcal{P}_{cl}(X)$ and let $a \in A$. Then by

$$D(a, B) = inf\{\|a - b\| : b \in B\} \ and \ \rho(A, B) = sup\{D(a, B) : a \in A\}$$

the function $H : \mathcal{P}_{cl}(X) \times \mathcal{P}_{cl,b}(X) \to \mathbb{R}^+$ defined by

$$H(A, B) = max\{\rho(A, B), \rho(B, A)\}$$

is metric and is called Hausdorff metric on X. Moreover $(\mathcal{P}_{cl,b}(X), H)$ is a metric space and $(\mathcal{P}_{cl}(X), H)$ is a complete metric space(see [21]). It is clear that

$$H(0, C) = sup\{\|c\| : c \in C; C \in \mathcal{P}_b(X)\}.$$

Definition 6. A set-valued function $F : \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ is called

(i) γ-*Lipschitz* if and only if there exist $\gamma > 0$ such that

$$H(F(x), F(y)) \leq \gamma\|x - y\|, \, for \, each \, x, y \in X$$

the constant γ is called a Lipschitz constant.

(ii) a *contraction* if and only if it is γ-*Lipschitz* with $\gamma < 1$.

Definition 7. A set-valued function $F : J \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ is called

(i) $\gamma(t)$-Lipschitz if and only if there exists $\gamma \in L^1(J, \mathbb{R}^+)$ such that

$$H(F(t, x), F(t, y)) \leq \gamma(t)\|x - y\|, \, for \, each \, x, y \in X.$$

(ii) a contraction if and only if it is $\gamma(t)$-Lipschitz with $\|\gamma\| < 1$

The following remark and lemmas are used in the sequel.

Remark 1 ([5]). Let $M \subset X$. If $F : M \to \mathcal{P}(X)$ is closed and $F(M)$ is relatively compact then F is u.s.c. on M. And if $F : X \to \mathcal{P}(X)$ is closed and compact operator then F is u.s.c. on X.

Lemma 1 (Bohnenblust-Karlin). *Let M be a closed and convex subset of the Banach space X and $F : M \to \mathcal{P}(M)$ a set-valued function. Suppose that:*

(i) *the set $F(M)$ is relatively compact;*
(ii) *the set-value function F is u.s.c on M;*
(iii) *the set $F(x)$ is nonempty closed and convex for all $x \in M$.*

Then there is $x \in M$ such that $x \in F(x)$.

Lemma 2 (Covitz-Nadler). *Let (X, d) be a complete metric space. If $G : X \to \mathcal{P}_{cl}(X)$ is a contraction, then G has a fixed point.*

3. SINGLE-VALUED PROBLEM

In this section we prove that the integral equation 1.2 has a solution $u(t)$ on J. Let us formulate the following assumption:

(H1) The function f is $L_X^1(J, \mathbb{R})$-Carathéodory with bound function $h \in L^1(J, \mathbb{R})$ such that $|f(t, u)| \leq h(t)$ a.e. $t \in J$ for all $u \in \mathbb{R}$.

Theorem 1. *Let* (H1) *hold. If* $1 - nB > 0$ *then equation* 1.2 *has at least one solution $u(t)$ on J.*

Proof. Define an operator P by

$$(Pu)(t) := \sum_{i=1}^{n} b_i(t) u(t - \tau_i) + I^{\alpha} f(t, u(t)). \qquad (3.1)$$

Set $r := \frac{T^{\alpha} \|h\|}{\Gamma(\alpha+1)(1-nB)}$, then by the assumption of the theorem and the properties of the fractional calculus we obtain that

$$|(Pu)(t)| \leq \sum_{i=1}^{n} |b_i(t)||u(t-\tau_i)| + I^{\alpha}|f(t,u(t))|$$

$$\leq nB\|u\| + I^{\alpha}|h(t)| \leq nB\|u\| + \|h\| \frac{T^{\alpha}}{\Gamma(\alpha+1)}.$$

So we obtain that

$$\|Pu\| \leq \frac{T^{\alpha}\|h\|}{\Gamma(\alpha+1)(1-nB)}.$$

That is $P : B_r \to B_r$. Then P maps B_r into itself. In fact, P maps the convex closure of $P[B_r]$ into itself. Since f is bounded on B_r, thus $P[B_r]$ is equicontinuous and the Schauder fixed point theorem shows that P has at least one fixed point $u \in \mathcal{B} = C[J, \mathbb{R}]$ such that $Pu = u$, which is corresponding to the solution of 1.2. $\qquad \square$

For the uniqueness let us consider the following assumption

(H2) The function f satisfies that there exists a function $\ell(t) \in L^1(J, \mathbb{R}^+)$ such that for each $u, v \in C[J, \mathbb{R}]$ we have

$$|f(t,u) - f(t,v)| \leq \ell(t)\|u - v\|.$$

Theorem 2. *Let $(H2)$ hold. If $\frac{nB\Gamma(\alpha+1)+T^{\alpha}\|\ell\|}{\Gamma(\alpha+1)} < 1$, then 1.2 has a unique solution $u(t)$ on J.*

Proof. Assume the operator P defined in equation 3.1 then we have

$$|(Pu)(t)-(Pv)(t)| \leq \sum_{i=1}^{n} |b_i(t)||u(t-\tau_i)-v(t-\tau_i)| + I^{\alpha}|f(t,u(t))-f(t,v(t))|$$

$$\leq nB\|u-v\| + I^{\alpha}\ell(t)\|u-v\|$$

$$\leq nB\|u-v\| + \frac{\|\ell\|T^{\alpha}}{\Gamma(\alpha+1)}\|u-v\|$$

$$= (nB + \frac{\|\ell\|T^{\alpha}}{\Gamma(\alpha+1)})\|u-v\|.$$

$$\square$$

Hence by the assumption of the theorem we have that P is a contraction mapping. Then in view of Banach fixed point Theorem, P has a unique fixed point which is corresponding to the solution of equation 1.2.

4. SET-VALUED PROBLEM

In this section we establish the existence results for the integral inclusion 1.1 when the right hand side is convex as well as when it is non-convex valued. The study will be taken in view of the single-valued theorems as well as fixed point theorems of set-valued function. Let us illustrate the following assumption

(H3) The set-valued function $F : J \times \mathbb{R} \to \mathcal{P}_{cl,c}(\mathbb{R})$ is $L_X^1(J, \mathbb{R})$-Carathéodory with a growth function $h \in L^1(J, \mathbb{R})$ such that $\|F(t, u)\| \le h(t)$ a.e $t \in J$ for all $u \in \mathbb{R}$.

Theorem 3. *Let $(H3)$ hold. If F is lower semi-continuous (l.s.c) and $1 - nB > 0$ then the integral inclusion 1.1 has at least one solution $u(t)$ on J.*

Proof. The proof here is depending on the single-valued problem. For each $u(t)$ in $\mathbb{R}$ the set $S_F(u)$ is nonempty, since by $(H3)$, F has a non-empty measurable selection (see [9]). Thus there exists a function $f(t, u) \in F$ where f is a $L_X^1(J, \mathbb{R})$-Carathéodory function with a bounded function $h \in L^1(J, \mathbb{R})$ such that $\|f(t, u)\| \le h(t)$ a.e $t \in J$ for all $u \in \mathbb{R}$. Hence the assumptions of Theorem 2 are satisfied so the integral inclusion 1.1 has a solution. $\square$

Theorem 4. *(Convex case) Let $(H3)$ hold. If $1 - nB > 0$ then the integral inclusion 1.1 has at least one solution $u(t)$ on J.*

Proof. Now we shall show that the assumptions of Lemma 1 are satisfied. Consider the following space $\Omega := \{u : J \to \mathbb{R} | u(t) \in C[J, \mathbb{R}]\}$ which is a Banach space endow with the sup. norm. Consider the set-valued operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$$(Nu)(t) := \{u \in \Omega : u(t) = \sum_{i=1}^{n} b_i(t)u(t - \tau_i) + I^\alpha f(t, u(t)); f \in S_F(u)\}. \quad (4.1)$$

We shall show that N satisfies the assumptions of Lemma 1. The proof will be given by several steps. Define a set $K := \left\{ u \in \Omega : \|u\| \le r := \frac{T^\alpha \|h\|}{\Gamma(\alpha+1)(1-nB)} \right\}$ such that $sup\{|u(t)| : u \in \Omega\} < \infty$. It is clear that K is nonempty, bounded, closed and convex subset of Ω. Note that by (H3) $S_F(u) \neq \varnothing$. $\square$

Step 1: $N(K) \subset K$. Let $u \in K$, we must show that $N(u) \in K$. For all $h(t) \in N(u)$, we have

$$|h(t)| \le \sum_{i=1}^{n} |b_i(t)||u(t - \tau_i)| + I^\alpha |f(t, u(t))|$$

$$\leq nB\|u\| + I^{\alpha}|h(t)| \leq nB\|u\| + \frac{\|h\|_{L^1}T^{\alpha}}{\Gamma(\alpha+1)}$$

$$\leq nrB + (1-nB)r = r.$$

Step 2: $N(K)$ is relatively compact set. We must show that N is a compact operator. Since K is bounded closed and convex set and $N(K) \subset K$, it follows that $N(K)$ is bounded closed and convex set. Moreover, for $0 \leq t_1 \leq t_2 \leq T$ such that $|t_2 - t_1| < \delta$, $\delta > 0$ and $u \in K$ then for all $h(t) \in N(u)$,

$$|h(t_1) - h(t_2)|$$

$$= |\sum_{i=1}^{n} b_i(t_1)u(t_1 - \tau_i) + I^{\alpha}f(t, u(t_1)) - \sum_{i=1}^{n} b_i(t_2)u(t_2 - \tau_i) - I^{\alpha}f(t, u(t_2))|$$

$$\leq 2nB\|u\| + |I^{\alpha}f(t, u(t_1)) - I^{\alpha}f(t, u(t_2))|$$

$$\leq 2nB\|u\| + [\frac{\|f\|}{\Gamma(\alpha+1)}]|(t_1^{\alpha} - t_2^{\alpha} + 2(t_2 - t_1)^{\alpha})|$$

$$\leq 2nB\|u\| + [\frac{2\|f\|}{\Gamma(\alpha+1)}]|(t_2 - t_1)^{\alpha}|$$

$$\leq 2nB\|u\| + [\frac{2\|f\|}{\Gamma(\alpha+1)}]|(t_2 - t_1)|^{\alpha}$$

$$\leq 2[nB\|u\| + \frac{2\delta^{\alpha}\|f\|}{\Gamma(\alpha+1)},$$

which is independent of u, and that implies that $N(K)$ is equicontinuous. An application of Arzela-Ascoli Theorem yields that N maps K into a compact set in Ω, that is $N; K \to \mathcal{P}(\Omega)$ is a compact operator. Thus $N(K)$ is relatively compact set.

Step 3: N is upper semi-continuous on K. Since $N : K \to K$ is closed and compact and that $N(K)$ is relatively compact set then in view of Remark 1, we obtain that N is u.s.c on K.

Step 4: $N(u)$ is a convex. Let $h_1(t), h_2(t) \in N(u)$ then there exist two functions $f_1, f_2 \in S_F(u)$ such that

$$h_j(t) = \sum_{i=1}^{n} b_i(t)u(t - \tau_i) + I^{\alpha}f_j(t, u(t)); \ j = 1, 2.$$

Let $0 \leq \delta \leq 1$, then for each $t \in J$ we have

$$[\delta h_1 + (1-\delta)h_2](t) = \delta[\sum_{i=1}^{n} b_i(t)u(t - \tau_i) + I^{\alpha}f_1(t, u(t))]$$

$$+(1-\delta)[\sum_{i=1}^{n} b_i(t)u(t - \tau_i) + I^{\alpha}f_2(t, u(t))]$$

$$= \sum_{i=1}^{n} b_i(t)[\delta u(t-\tau_i) + (1-\delta)u(t-\tau_i)]$$
$$+[\delta I^\alpha f_1(t,u(t)) + (1-\delta)I^\alpha f_2(t,u(t))]$$

$$= \sum_{i=1}^{n} b_i(t)[\delta u(t-\tau_i) + (1-\delta)u(t-\tau_i)] + I^\alpha[\delta f_1(t,u(t)) + (1-\delta)f_2(t,u(t))]$$

$$= \sum_{i=1}^{n} b_i(t)u(t-\tau_i) + I^\alpha[\delta f_1(t,u(t)) + (1-\delta)f_2(t,u(t))]$$

by the convexity of $S_F(u)$, and $F(t,u)$ implies that $[\delta h_1 + (1-\delta)h_2](t) \in N(u)$. Hence $N(u)$ is convex. As a consequence of Lemma 1, we deduce that N has a fixed point which is a solution for the integral inclusion 1.1. Hence the proof is complete.

In order to study the existence for the problem 1.1 in non-convex case, we introduce the following assumptions.

(H4) $F : J \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$, $(t,.) \mapsto F(t,u)$ is measurable for each $u \in \mathbb{R}$.

(H5) $F : J \times \mathbb{R} \to \mathcal{P}_{cl}(\mathbb{R})$ is $\ell(t)$-*Lipschitz*

$$H(F(t,u), F(t,v)) \le \ell(t)\|u - v\|.$$

Theorem 5 (Non-convex case). *Let* $(H4)$ *and* $(H5)$ *hold. If*

$$\frac{nB\Gamma(\alpha+1) + T^\alpha\|\ell\|_{L^1}}{\Gamma(\alpha+1)} < 1,$$

then the integral inclusion 1.1 has at least one solution $u(t)$ *on* J.

Proof. For each $u(t)$ in $\mathbb{R}$ the set $S_F(u)$ is nonempty since by $(H4)$, F has a nonempty measurable selection (see [9]). Then there exists a function $f(t,u) \in F$ such that f is $\ell(t)$-*Lipschitz*. Thus by the assumption of the theorem and by Theorem 2, the inclusion 1.1 has a solution. Hence the proof is done in view of the single-valued problem. $\square$

In order to study the existence for problem 1.1 by using Lemma 2, we must show that $N : \Omega \to \mathcal{P}(\Omega)$ defined in 4.1 is a contraction set-valued mapping. The proof will be given in two steps.

Step 1: $N(u) \in \mathcal{P}_{cl}(\Omega)$ for each $u \in \Omega$. Let $\{u_m\}_{m \ge 0} \in N(u)$ such that $u_m \to \widetilde{u}$ in Ω. Then $\widetilde{u} \in \Omega$ and there exists $f_m \in S_F(u)$ such that for $t \in J$

$$u_m(t) = \sum_{i=1}^{n} b_i(t)u(t-\tau_i) + I^\alpha f_m(t,u(t)).$$

Using the fact that F has closed values, we get that f_m converges to f in $L^1(J, \mathbb{R})$ and hence $f \in S_F(u)$. Then for each $t \in J$

$$u_m(t) \to \widetilde{u}(t) = \sum_{i=1}^{n} b_i(t) u(t - \tau_i) + I^\alpha f(t, u(t)).$$

So $\widetilde{u} \in N(u)$.

Step 2: There exists $\gamma < 1$ such that

$$H(N(u), N(v)) \le \gamma \|u - v\|_\Omega, \ for \ each \ u, v \in \Omega.$$

Let $u, v \in \Omega$. Then by $(H5)$ there exists $f \in F$ that satisfies

$$|f(t, u) - f(t, v)| \le \ell(t) \|u - v\|_\Omega.$$

Then $h_1(t) \in N(u)$ where

$$h_1(t) = \sum_{i=1}^{n} b_i(t) u(t - \tau_i) + I^\alpha f(t, u(t)).$$

Then for $h_2(t) \in N(v)$, where

$$h_2(t) = \sum_{i=1}^{n} b_i(t) v(t - \tau_i) + I^\alpha f(t, v(t)),$$

we have

$$|h_1(t) - h_2(t)| \le \sum_{i=1}^{n} b_i(t) |u(t - \tau_i) - v(t - \tau_i)| + |I^\alpha f(t, u(t)) - I^\alpha f(t, v(t))|$$

$$\le \sum_{i=1}^{n} b_i(t) |u(t - \tau_i) - v(t - \tau_i)| + I^\alpha |f(t, u(t)) - f(t, v(t))|$$

$$\le nB \|u - v\|_\Omega + \frac{T^\alpha}{\Gamma(\alpha + 1)} |f(t, u(t)) - f(t, v(t))|$$

$$\le nB \|u - v\|_\Omega + \frac{T^\alpha}{\Gamma(\alpha + 1)} \ell(t) \|u - v\|_\Omega$$

$$\le nB \|u - v\|_\Omega + \frac{T^\alpha \|\ell\|_{L^1}}{\Gamma(\alpha + 1)} \|u - v\|_\Omega$$

$$= [nB + \frac{T^\alpha \|\ell\|_{L^1}}{\Gamma(\alpha + 1)}] \|u - v\|_\Omega.$$

Denote

$$\gamma := [\frac{nB\Gamma(\alpha + 1) + T^\alpha \|\ell\|_{L^1}}{\Gamma(\alpha + 1)}].$$

It follows that

$$H(N(u), N(v)) \le \gamma \|u - v\|_\Omega, \ for \ each \ u, v \in \Omega,$$

where $\gamma < 1$. It follows that N is a contraction set-valued mapping. Then, in view of Lemma 2, N has a fixed point which is corresponding to a solution of inclusion 1.1. This completes the proof.

Example 1. Let $J = [0, 1]$ denote a closed and bounded interval in $\mathbb{R}$. Consider the problem

$$u(t) \in \frac{t^3}{8}u(t - \tau_1) + \frac{t^4}{12}u(t - \tau_2) + \frac{1}{4}u(t - \tau_3) + I^{\alpha}F(t, u); \ t \in [0, 1] \quad (4.2)$$

where $n = 3, \alpha = 0.5, B = 0.25$ and

$$F(t, u) = \begin{cases} h(t), & \text{if } u < 0; \\ [h(t)\exp^{\frac{-u(t)}{2}}, h(t)], & \text{if } u \geq 0. \end{cases}$$

It is clear that F is $L_X^1(J, \mathbb{R}) - Carath\acute{e}odory$ with a growth function $h \in L^1(J, \mathbb{R}^+)$ such that $\|F(t, u)\| \leq h(t)$ a.e $t \in J$ for all $u \in \mathbb{R}$. Therefore if $\|h\| \leq \frac{\sqrt{\pi}}{8}$ then in view of Theorem 4, the problem 4.2 has a solution.

Example 2. Let $J = [0, 1]$ denote a closed and bounded interval in $\mathbb{R}$. Consider the problem

$$u(t) \in \frac{t}{5}u(t - \tau_1) + \frac{t^2}{10}u(t - \tau_2) + I^{\alpha}F(t, u); \ t \in [0, 1] \quad (4.3)$$

where $n = 2, \alpha = 0.5, B = \frac{1}{5}$ and

$$F(t, u) = \begin{cases} 0 & \text{if } u < 0; \\ [0, \frac{u(t)}{4}] & \text{if } u \geq 0. \end{cases}$$

It is clear that F is $\ell(t)$-Lipschitzian set-valued function with $\|\ell\| = \frac{1}{4}$. Therefore, for $\gamma = 0.6 < 1$, we have that F is a contraction set-valued mapping. Thus, in view of Theorem 5, the problem 4.3 has a solution.

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