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LAWS AND IDENTITIES FOR SOME UPPER TRIANGULAR MATRICES

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Abstract. J. C. Robson has investigated the ideal I_n of all polynomials in the free associative algebra $R\langle x\rangle$ over a non-commutative ring R generated by x and the n^2 entries of an $n \times n$ matrix $\alpha = (a_{ij})$, which are satisfied by α. He found four cubics generating the ideal for $n = 2$ and proved its finite generation for any n. Ts. Rashkova has considered the ideal I_2 for matrix algebras with involution over a noncommutative ring and over a field of characteristic zero. In the paper the ideal I_3 is described for some special upper triangular matrices over a field of characteristic 0. The T-ideal $T(U_2(G))$ is investigated as well for G denoting the infinite dimensional Grassmann algebra.

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1. INTRODUCTION

The Cayley–Hamilton theorem and the corresponding trace identity play a fundamental role in proving classical results about the polynomial and trace identities of the $n \times n$ matrix algebra $M_n(K)$ over a field K of characteristic zero.

The structure theory of semisimple rings and quantum matrices for example show the importance of matrices over non-commutative rings in the theory of PI-algebras and other branches of algebra as well.

J. C. Robson investigated in [12–14] the ideal I_n of all polynomials (including nonmonics) in the free associative algebra $R\langle x\rangle$ over a non-commutative ring R generated by x and the n^2 entries of an $n \times n$ matrix $\alpha = (a_{ij})$, which are satisfied by α. Those polynomials we call the laws over R of a non-commutative $n \times n$ matrix α. These are not polynomial identities since the entries of α are allowed as coefficients in the laws and they vary with the choice of α.

Robson showed that I_n is an insertive ideal (meaning that its homogeneous elements are closed under inner multiplication by a constant a_{ij} in some fixed position) and as such is finitely generated (see [13] and [12, Theorem 2.3]).

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The minimal degree of polynomials in I_n remains unknown. However, for the case $n=2$, Robson [12, Proposition 3.2] found four polynomials of degree 3 (least possible) in I_2 and Pearson showed in [10, Corollary] that these four Robson cubics do indeed generate I_2 as an insertive ideal.

2. Results

The first study of the non-commutative case for $n=3$ was done in [9]. The results there provide further evidence of the tantalizing complexity of even these small matrices. Each of the four found laws of degree 7 has 1156 terms. Thus, special cases of 3×3 matrices even over a field are of interest.

In [11] we study the ideal I_3 for some special 3×3 upper triangular matrices considered in [7]. These algebras could be endowed with involution $*$ and their $*$-codimensions have important properties.

Here we give a complete answer for the existing laws in these algebras and investigate the T-ideal $T(U_2(G))$ of the 2×2 upper triangular matrices over G, where G stands for the infinite dimensional Grassmann algebra.

2.1. *Laws for upper triangular matrices over a field*

2.1.1. *Special upper triangular matrices*

In [7] D. La Mattina and P. Misso study some associative algebras with involution generating $*$-varieties of algebras with linear or linearly bounded sequences of $*$-codimensions. Questions concerning laws for them were discussed in [11]. Here we give the complete answers.

Let K be a field of characteristic zero and

$$M_2(K)=\left\{x=\begin{pmatrix} a & b & c\\ 0 & a & b\\ 0 & 0 & a\end{pmatrix} : a,b,c\in K\right\}.$$

All coefficients in the polynomials below mean the corresponding scalar matrix, i. e., $a=aE$ for example.

Theorem 1. *The only Robson cubic for a matrix from $M_2(K)$ in the general case is $r(x)=(x-a)^3$. For $b=0$, $c\neq 0$ a law of minimal degree is $(x-a)^2$ and for $b=c=0$ it is $x-a$.*

Proof. For a matrix x from the considered algebra we prove directly that $(x-a)^3=0$. Due to [8, Lemma 2, p.432] considering the subring T of the diagonal elements of a generic upper triangular matrix A and $w(x)\in T\langle x\rangle$, then $w(A)=0$ iff $w(x)\in\langle x-a_{11}\rangle\langle x-a_{22}\rangle\cdots\langle x-a_{nn}\rangle$, where $\langle f\rangle$ is the ideal generated by f. In the cited Lemma T is a subring of a non-commutative ring. In Theorem 1 the elements of the matrix are elements of a field and then $T\langle x\rangle=K\langle x\rangle$ and the possible laws of degree less or equal to 3 could be given with the linear combination

$$A = \alpha_1 + \alpha_2 x + \alpha_3(x-a) + \alpha_4 x^2 + \alpha_5(x-a)^2 + \alpha_6 x(x-a).$$

Equating to zero the corresponding entries we get the system

$$\begin{aligned} \alpha_1 + a\alpha_2 + a^2\alpha_4 &= 0, \\ b\alpha_2 + b\alpha_3 + 2ab\alpha_4 + ab\alpha_6 &= 0, \\ c\alpha_2 + c\alpha_3 + (2ac+b^2)\alpha_4 + b^2\alpha_5 + (ac+b^2)\alpha_6 &= 0. \end{aligned}$$

For $b \neq 0$ the system has a solution

$$\begin{aligned} \alpha_5 &= -\alpha_4 - \alpha_6, \\ \alpha_1 &= a\alpha_3 + a^2\alpha_4 + a^2\alpha_6, \\ \alpha_2 &= -\alpha_3 - 2a\alpha_4 - a\alpha_6. \end{aligned}$$

In this case A, is identically equal to zero. Considering $b = 0$, we get the rest of the statement. □

Let us put

$$M_3(K) = \left\{ x = \begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & a \end{pmatrix} : a, b, c, d \in K \right\}.$$

Theorem 2. *The only Robson cubic for a matrix from $M_3(K)$ in the general case is $r(x) = (x-a)^2 x$. For the three cases $b \neq 0$, $c = d = 0$; $d \neq 0$, $b = c = 0$, and $b = c = d = 0$ a law of minimal degree is $(x-a)x$.*

Proof. Directly we calculate that for a matrix x from the considered algebra we have $(x-a)^2 x = 0$. Then, due to Lemma 2 from [8, p. 432], considerations analogous to the proof of Theorem 1 lead us to the linear combination

$$A = \alpha_1 + \alpha_2 x + \alpha_3(x-a) + \alpha_4 x^2 + \alpha_5(x-a)^2 + \alpha_6 x(x-a)$$

of all possible laws of degree ≤ 3. Equating to zero the corresponding entries, we obtain the system

$$\begin{aligned} \alpha_1 + a\alpha_2 + a^2\alpha_4 &= 0, \\ b\alpha_2 + b\alpha_3 + ab\alpha_4 - ab\alpha_5 &= 0, \\ c\alpha_2 + c\alpha_3 + (2ac+bd)\alpha_4 + bd\alpha_5 + (ac+bd)\alpha_6 &= 0, \\ \alpha_1 - a\alpha_3 + a^2\alpha_5 &= 0, \\ d\alpha_2 + d\alpha_3 + ad\alpha_4 - ad\alpha_5 &= 0. \end{aligned}$$

For $b \neq 0$, $d \neq 0$ the system has a solution

$$\alpha_6 = -\alpha_4 - \alpha_5,$$
$$\alpha_1 = a\alpha_3 - a^2\alpha_5,$$
$$\alpha_2 = -\alpha_3 - a\alpha_4 + a\alpha_5.$$

In this case A is identically equal to zero.

The three cases $b \neq 0$, $c = d = 0$; $d \neq 0$, $b = c = 0$, and $b = c = d = 0$ give the law $(x-a)x$. □

Let us put

$$M_4(K) = \left\{ x = \begin{pmatrix} 0 & b & c \\ 0 & a & d \\ 0 & 0 & 0 \end{pmatrix} : a,b,c,d \in K \right\}.$$

Theorem 3. *The only Robson cubic for a matrix from $M4(K)$ in the general case is $r(x) = (x-a)x^2$. For $c = d = 0$ or $b = c = 0$ a law of minimal degree is $(x-a)$.*

Proof. It follows the above pattern of proof. Directly we get $(x-a)x^2 = 0$. The corresponding system is

$$\alpha_1 + a\alpha_2 + a^2\alpha_4 = 0,$$
$$b\alpha_2 + b\alpha_3 + ab\alpha_4 - ab\alpha_5 = 0,$$
$$c\alpha_2 + c\alpha_3 + bd\alpha_4 + (bd - 2ac)\alpha_5 + (bd - ac)\alpha_6 = 0,$$
$$\alpha_1 - a\alpha_3 + a^2\alpha_5 = 0,$$
$$d\alpha_2 + d\alpha_3 + ad\alpha_4 - ad\alpha_5 = 0.$$

For $b \neq 0$, $d \neq 0$ the system has the same solution as in the previous theorem and A is identically equal to zero.

The two cases $c = d = 0$ and $b = c = 0$ give the law $(x-a)x$. □

2.1.2. *The general upper triangular case*

Now we consider the general case, namely, the algebra $U_3(K,*)$ of the upper triangular matrices of order 3 with the involution $*$ reflection along the second diagonal. We could find the analogues of the Robson cubics for the $*$-symmetric matrices and for the $*$-skew-symmetric matrices as well.

Theorem 4. *The only law of minimal degree for a symmetric matrix $x \in U_3^+(K,*)$, where*

$$U_3^+(K,*) = \left\{ x = \begin{pmatrix} a & b & d \\ 0 & c & b \\ 0 & 0 & a \end{pmatrix} : a,b,c \in K \right\},$$

is $(x-a)^2(x-c)$. For $b = d = 0$ it is $(x-a)(x-c)$.

Proof. Directly we calculate that $(x-a)^2(x-c)=0$. Then as explained in the above proofs we have to form the linear combination

$$A=\alpha_1+\alpha_2(x-a)+\alpha_3(x-a)^2+\alpha_4(x-c)+\alpha_5(x-c)^2+\alpha_6(x-a)(x-c)$$

of all possible laws of degree ≤ 3. Equating to zero the corresponding entries we get the system

$$\begin{aligned}
\alpha_1+(a-c)\alpha_4+(a-c)^2\alpha_5&=0,\\
b\alpha_2+b(c-a)\alpha_3+b\alpha_4+b(a-c)\alpha_5&=0,\\
d\alpha_2+b^2\alpha_3+d\alpha_4+(2d(a-c)+b^2)\alpha_5+(d(a-c)+b^2)\alpha_6&=0,\\
\alpha_1+(c-a)\alpha_2+(c-a)^2\alpha_3&=0,\\
b\alpha_2+b(c-a)\alpha_3+b\alpha_4+b(a-c)\alpha_5&=0,\\
\alpha_1+(a-c)\alpha_4+(a-c)^2\alpha_5&=0.
\end{aligned}$$

For $b\neq 0$, $d\neq 0$ the system has a solution

$$\begin{aligned}
\alpha_6&=-\alpha_3-\alpha_5,\\
\alpha_1&=-(a-c)\alpha_4-(a-c)^2\alpha_5,\\
\alpha_2&=-(-a+c)\alpha_3-\alpha_4-(a-c)\alpha_5.
\end{aligned}$$

In this case A, is identically equal to zero. For $b=d=0$, we obtain the quadratic law $(x-a)(x-c)$. □

We point that in the general case $(x-a)^2(x-c)=0$ is in fact the Cayley–Hamilton theorem in a factor form, i. e., $x^3-(2a+c)x^3+a(a+2c)x-a^2c=0$.

Theorem 5. *The only law of minimal degree for a skew-symmetric matrix $x\in U_3^-(K,*)$, where*

$$U_3^-(K,*)=\left\{x=\begin{pmatrix}a&b&0\\0&c&-b\\0&0&-a\end{pmatrix}:a,b,c\in K\right\},$$

is $(x-a)(x-c)(x+a)$. If $a=b=0$, then a law is $x(x-c)$.

Proof. The law $(x-a)(x-c)(x+a)=0$ is checked directly. Then we form the linear combination

$$\begin{aligned}
A=\alpha_1&+\alpha_2(x-a)+\alpha_3(x-a)^2+\alpha_4(x-c)\\
&+\alpha_5(x-c)^2+\alpha_6(x+a)+\alpha_7(x+a)^2\\
&+\alpha_8(x-a)(x-c)+\alpha_9(x-a)(x+a)+\alpha_{10}(x-c)(x+a)
\end{aligned}$$

of all possible laws of degree ≤ 3. Equating to zero the corresponding entries, we obtain the system

$$\begin{aligned}
\alpha_1+(a-c)\alpha_4+(a-c)^2\alpha_5+2a\alpha_6+4a^2\alpha_7+2a(a-c)\alpha_{10}&=0,\\
b\alpha_2+b(c-a)\alpha_3+b\alpha_4+b(a-c)\alpha_5+b\alpha_6&\\
+b(c+3a)\alpha_7+b(c+a)\alpha_9+2ab\alpha_{10}&=0,\\
-b^2\alpha_3-b^2\alpha_5-b^2\alpha_7-b^2\alpha_8-b^2\alpha_9-b^2\alpha_{10}&=0,\\
\alpha_1+(c-a)\alpha_2+(c-a)^2\alpha_3+(c+a)\alpha_6+(c+a)^2\alpha_7+(c-a)(c+a)\alpha_9&=0,\\
-b\alpha_2-b(c-3a)\alpha_3-b\alpha_4+b(a+c)\alpha_5-b\alpha_6&\\
-b(c+a)\alpha_7+2ab\alpha_8-b(c-a)\alpha_9&=0,\\
\alpha_1-2a\alpha_2+4a^2\alpha_3-(a+c)\alpha_4+(a+c)^2\alpha_5+2a(a+c)\alpha_8&=0.
\end{aligned}$$

Its solution is

$$\begin{aligned}
\alpha_1&=-2a(a-c)\alpha_{10}-(a-c)\alpha_4-(a-c)^2\alpha_5-2a\alpha_6-4a^2\alpha_7,\\
\alpha_8&=-\alpha_{10}-\alpha_3-\alpha_5-\alpha_7-\alpha_9,\\
\alpha_2&=-2a\alpha_{10}-(-a+c)\alpha_3-\alpha_4-(a-c)\alpha_5-\alpha_6-(3a+c)\alpha_7-(a+c)\alpha_9.
\end{aligned}$$

In this case A, is identically equal to zero. The case $a=b=0$ leads one directly to the validity of the law $x(x-c)$. $\square$

The law in the general case illustrates the Cayley–Hamilton theorem in a factor form, namely $x^3-cx^2-a^2x+a^2b=0$. All the computations are made using the computer algebra system *Mathematica*.

2.2. *Laws and identities for upper triangular matrices over the Grassmann algebra*

2.2.1. *Preliminaries*

We consider the matrix algebra of the 2×2 upper triangular matrices $U_2(G)$ over the Grassmann algebra G.

We recall the definition of the infinite dimensional Grassmann algebra G, namely,

$$G=G(V)=K\langle v_1,v_2,\ldots \mid v_iv_j+v_jv_i=0,\ i,j=1,2,\ldots\rangle.$$

The algebra G' (without 1) has a basis $v_{i_1}v_{i_2}\ldots v_{i_k}$, where $1\leq i_1<i_2<\ldots<i_k$. The elements v_i are called generators of G' while the elements $v_{i_1}v_{i_2}\ldots v_{i_k}$ for $1\leq i_1<i_2<\ldots<i_k$ are called basic monomials of G'. For $G=G'\cup 1$, a generator is 1 as well. The algebras G and G' are PI-equivalent (they satisfy one and the same identities). It is easy to be seen that $G'=J(G)$ for $J(G)$ being the Jacobson radical of the algebra.

The algebra G is in the mainstream of recent research in PI theory. Its importance is connected mainly with the structure theory for the T-ideals of identities of associative algebras developed by Kemer in [5]. Kemer proved [5, Theorem 1.2] that any T-prime T-ideal can be obtained as the T-ideal of identities of one of the following algebras: $M_n(K)$, $M_n(G)$ and $M_{n,u}(G)$, the latter being the algebra of $n\times n$ supermatrices over $G = G_0 \oplus G_1$ with G_0 blocks (with entries of even degree) of sizes $u\times u$ and $(n-u)\times(n-u)$ and with G_1 blocks (with entries of odd degree) of sizes $u\times(n-u)$ and $(n-u)\times u$.

Well known facts concerning the algebra G are the following:

Proposition 1 ([6, Corollary, p. 437])**.** *The T-ideal $T(G)$ is generated by the identity* $[x_1,x_2,x_3]=0$.

Proposition 2 ([4, Exercise 5.3])**.** *For $G_k = G(V_k)$ over k-dimensional vector space V_k all identities follow from the identity $[x_1,x_2,x_3]=0$ and the standard identity $S_{2p}(x_1,\dots,x_{2p}) = \sum_{\sigma\in\mathrm{Sym}(2p)}(-1)^{\sigma}x_{\sigma(1)}\dots x_{\sigma(2p)}=0$, where p is the minimal integer with $2p>k$.*

Remark 1. In the monograph [4, Exercise 5.3] one could see that the identity $[x_1,x_2]\dots[x_{2p+1},x_{2p+2}]=0$ on G_{2p+1} is equivalent to the standard identity of degree $2p+2$.

Remark 2. It could be seen [4, Exercise 5.8] that the T-ideal of the algebra $M_2(K)$ from Theorem 1 is generated by the identities $[x_1,x_2,x_3]=0$ and $S_4(x_1,\dots,x_4)=0$. Nevertheless, the algebra $M_2(K)$ is not isomorphic to the Grassmann algebra G_2 of the two-dimensional vector space.

For the rest of the paper we will use capital letters for the matrices with entries from the Grassmann algebra.

2.2.2. *The T-ideal $T(U_2(G))$*

Theorem 6. *The identity $[X_1,X_2,X_3][X_4,X_5,X_6]=0$ holds on the algebra $U_2(G)$.*

Proof. As the considered polynomial is multilinear we could rely on [16, Remark 3.1] stating that it is an identity on $M_2(G)$ if and only if for every choice of the matrix units e_{a_i,b_i} and either $v_i^*=v_i$ or $v_i^*=1$, the substitution $x_i\to e_{a_i,b_i}v_i^*$ in the polynomial gives zero.

We take the matrices $X_i = \begin{pmatrix} a_{1i} & b_{1i}\\ 0 & c_{1i}\end{pmatrix}$ for $i=1,2,3$ belonging to $U_2(G)$ with entries being generators of G. It is easy to see that

$$[X_1,X_2,X_3] = \begin{pmatrix} [a_{11},a_{12},a_{13}] & *\\ 0 & [c_{11},c_{12},c_{13}]\end{pmatrix} = \begin{pmatrix} 0 & *\\ 0 & 0\end{pmatrix}.$$

The same form has the matrix $[X_4,X_5,X_6]$. As the only possibly non-zero entry of the considered matrices is the (1,2)-entry, the multiplication gives 0. □

Remark 3. Theorem 6 could be reformulated as follows. Let $f(x,y,z)=[x,y,z]$. The 2×2 upper triangular matrices over the Grassmann algebra satisfy the identity $f^2=0$ while their entries satisfy the identity $f=0$.

In [3, Theorem 3.1], Domokos gave a compact form of a theorem of Szigeti from [15], namely,

Proposition 3. *For any* 2×2 *matrix* X *over a* K*-algebra* S *satisfying the identity* $[x_1,x_2,x_3]=0$ *we have that*

$$X^4-2X^3(\operatorname{tr}X)+X^2(2\operatorname{tr}^2X-\operatorname{tr}X^2)+X\Big(\frac{1}{2}\operatorname{tr}X\circ\operatorname{tr}X^2-\operatorname{tr}^3X\Big)$$
$$+\frac{1}{4}\Big(\operatorname{tr}^4X+\operatorname{tr}^2X^2+\frac{1}{2}\operatorname{tr}^2X\operatorname{tr}X^2-\frac{5}{2}\operatorname{tr}X^2\operatorname{tr}^2X+2[\operatorname{tr}X^3,\operatorname{tr}X]\Big)E$$

and

$$X^4-2(\operatorname{tr}X)X^3+(2\operatorname{tr}^2X-\operatorname{tr}X^2)X^2+\Big(\frac{1}{2}\operatorname{tr}X\circ\operatorname{tr}X^2-\operatorname{tr}^3X\Big)X$$
$$+\frac{1}{4}\Big(\operatorname{tr}^4X+\operatorname{tr}^2X^2-\frac{5}{2}\operatorname{tr}^2X\operatorname{tr}X^2+\frac{1}{2}\operatorname{tr}X^2\operatorname{tr}^2X-2[\operatorname{tr}X^3,\operatorname{tr}X]\Big)E$$

are equal to zero in $S^{2\times 2}$.

In [15], Szigeti developed a new theory of determinants of $n\times n$ matrices over rings satisfying the polynomial identity of m-Lie nilpotency

$$[[[\cdots[x_1,x_2],x_3],\cdots],x_m],x_{m+1}]=0.$$

As the Grassmann algebra is 2-Lie nilpotent the defined in [15] right m-adjoint of a matrix, the right m-determinant of a matrix rd_m and the right m-characteristic polynomial $p(x)$ of a matrix and their properties could be interpreted for the matrix algebra $U_2(G)$.

Proposition 4 ([15, Theorem 4.2])**.** *If* $p(x)=\lambda_0+\lambda_1x+\cdots+\lambda_dx^d$ *is the right* m*-characteristic polynomial of a* $n\times n$ *matrix* $A\in M_n(R)$ *over a* m*-Lie nilpotent ring* R *then the left substitution of* A *into* $p(x)$ *is zero:* $(A)p=E\lambda_0+A\lambda_1+\cdots+A^d\lambda_d=0$.

Again in [15], Szigeti pointed out the identity of "algebraicity" for matrices over the Grassmann algebra.

Proposition 5 ([15, Theorem 5.1])**.** *The polynomial identity*

$$S_{2n^2}\big([Y^{2n^2},Z],[Y^{2n^2-1},Z],\ldots,[Y^2,Z],[Y,Z]\big)=0$$

holds on $M_n(G)$ *for any two matrices* Y *and* Z.

Now we give some laws and identities for the upper triangular matrices over the Grassmann algebra G.

Theorem 7. *Let the matrix* $X = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ *belong to* $U_2(J(G))$ *for* a, b, c *being basic monomials of* G'.

(I) *The following laws are valid for* X:

$$(X-a)(X-c) = 0,$$
$$X^3(\operatorname{tr} X) = 0,$$
$$(\operatorname{tr} X)X^3 = 0.$$

(II) *Two identities hold for any matrices* X *and* Y *of the considered type, namely* $X^2Y^2 = 0$ *and* $(X^2Y)^2 = 0$.

Thus any matrix X *is nilpotent of index* 4. *A matrix* X *with* $\operatorname{tr} X = 0$ *is nilpotent of index* 3.

Proof. (I) Direct calculations give the validity of the three stated laws for a matrix X.

(II) Again applying direct calculations we get that the only non-zero entries in X^2, Y^2 and X^2Y are the $(1,2)$ entries and the corresponding multiplication gives zero. □

We see that both Proposition 3 and Proposition 4 for $n = 2$ are compatible with Theorem 7 as for such a matrix $A \in U_2(G)$ we have $\operatorname{rdet}_2 A = 0$ and $p(x) = \operatorname{rdet}_2(A - Ex) = x^4 - 2 \operatorname{rdet} A x^3$. Thus $A^4 - 2(\operatorname{tr} A)A^3 = 0$.

Corollary 1. *An identity of degree* 9 *holds for any two matrices* Y *and* Z *from* $U_2(G)$ *with entries being basic monomials, namely,* $S_3([Y^3, Z], [Y^2, Z], [Y, Z]) = 0$.

Proof. Applying Proposition 5 and the index of nilpotency of the matrix Y. □

Remark 4. If we consider the Grassmann algebra over a finite dimensional space, the corresponding identities have much smaller degrees.

Theorem 8. *The following two assertions hold:*

(I) *On the algebra* $U_2(J(G_2))$ *we get the identity* $XYZ = 0$ *(respectively,* $X^3 = 0$*).*

(II) *Any matrix* $X = \left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right)$ *from* $U_2(J(G_2))$ *satisfies the law* $(\operatorname{tr} X)X^2 = 0$*, respectively,* $X^2(\operatorname{tr} X) = 0$.

Proof. In the considered algebra the square of every element is zero. The identity and the law are proved directly. □

Multiplying two linear combinations of the basic elements e_1, e_2 and e_1e_2 we get only αe_1e_2. Its product with any other linear combination of e_1, e_2 and e_1e_2 gives zero.

Analogous considerations are valid for the linear combinations of the basic elements of any finite dimensional Grassmann algebra (over a n-dimensional vector space). The multiplication of n linear combinations will result in $\alpha e_1e_2\dots e_n$ and the result of the next multiplication will be zero. Thus, we come to the following

Corollary 2. *The matrix algebra $U_k(J(G_n))$ (respectively, $M_k(J(G_n))$) is nilpotent of class $\le n+1$.*

Corollary 3. *The polynomial identity $S_{n-1}([Y^{n-1},Z],[Y^{n-2},Z],\dots,[Y,Z])=0$ holds on $M_k(J(G_n))$ for $n\ge 3$ and $k\ge 2$.*

As the algebra $U_2(G)$ is a subalgebra of $M_2(G)$ we turn to the Hall identity, the four degree standard identity for $M_2(K)$ and the product commutator identity for $U_2(K)$ considering the Grassmann algebra G instead of the field K. We get

Theorem 9. *The polynomials $[[x_1,x_2]^2,x_1]$, $S_4(x_1,x_2,x_3,x_4)$, and $[x_1,x_2][x_3,x_4]$ are not identities for the algebra $U_2(G)$.*

Proof. A counter example for the validity of the first and the third identity gives the matrices

$$X_1=\begin{pmatrix} e_1 & e_2\\ 0 & e_3\end{pmatrix}$$

and

$$X_2=\begin{pmatrix} e_4e_5+e_6 & 1\\ 0 & e_1+e_3\end{pmatrix}.$$

We get that the $(1,2)$-entry of $[x_1,x_2]^2$ is nonzero, namely,

$$2e_1e_3e_6+2e_1e_2e_4e_5e_6-2e_1e_2e_3e_4e_5+4e_1e_2e_3e_6.$$

The $(1,2)$-entry of $[[x_1,x_2]^2,x_1]$ is $-2e_1e_2e_3e_4e_5e_6$. For the second statement, we rely on the general case considered later. □

In [1, 2], a connection is given between the identities on $M_n(K)$ and those on $M_n(G)$.

Proposition 6 ([2, Proposition 2.1])**.** *Let $f_1,\dots,f_d\in K\langle x_1,\dots,x_m\rangle$ be elements of the T-ideal of identities of $M_n(K)$. If $d>\frac{1}{2}n^2m$, then $f_1f_2\cdots f_d=0$ is an identity on $M_n(G)$.*

Remark 5. Applying the result to $M_2(G)$ and the standard polynomial S_4 we get that $S_4^9=0$ is an identity on $M_2(G)$. However, this is not the best possible result. Really on $M_2(G)$ we get the identities $S_4^5=0$ and $[[x,y]^2,x]^5=0$. Thus we get two identities of degree 20 and 25, respectively, for $U_2(G)$.

The above Proposition 6 has an analogue for the upper triangular matrices U_n.

Theorem 10. *Let $f_1,\dots,f_d\in K\langle x_1,\dots,x_m\rangle$ be elements of the T-ideal of identities of $U_n(K)$. If $d>\frac{1}{4}n(n+1)m$, then $f_1f_2\cdots f_d=0$ is an identity on $U_n(G)$.*

Proof. It follows the proof of Proposition 6 taking into account the dimension of $U_n(K)$ and the fact that the relatively free algebra $F(U_n(K))$ has as a basis the monomials

$$x_1^{a_1}\cdots x_m^{a_m}[x_{i_{11}},x_{i_{21}},\dots,x_{i_{p_1 1}}]\cdots[x_{i_{1r}},x_{i_{2r}},\dots,x_{i_{p_r r}}],$$

where the number r of participating commutators is $\le n-1$ and the indices in each commutator $[x_{i_{1s}},x_{i_{2s}},\dots,x_{i_{p_s s}}]$ satisfy the relations $i_{1s}>i_{2s}\le\cdots<i_{p_s s}$. □

Proposition 7 ([1, Lemma, p. 1509])**.** *The algebra $M_n(G)$ satisfies the identity S_{2n}^k for some $k>1$ but satisfies neither S_{2n} nor identities of the form S_m^k for any k when $m<2n$.*

Theorem 11. *The algebra $U_n(G)$ does not satisfy the identity $S_{2n}=0$.*

Proof. We have

$$S_{2n}(e_{11},e_{12},e_{22},e_{23},\dots,e_{n-1,n-1},e_{n-1,n},ee_{nn},fe_{nn})=2efe_{1n}\neq 0$$

for $e,f\in G$ such that $ef=-fe\neq 0$. □

In [16, Proposition 4.1 and Corollary 6.1], Vishne described an efficient way to use the S_n-module structure in the computation of the multilinear identities of degree n of a given algebra. He used the method to show that the minimal degree of an identity for $M_2(G)$ is 8 and gave explicit identities of degree 8. He described a class of identities for $M_2(G)$, namely,

Proposition 8 ([16, Corollary 4.3])**.** *Let f be a multilinear polynomial of degree 8. If $\operatorname{tr} f(x_{\sigma(1)},\dots,x_{\sigma(8)})=0$ for every $x_1,\dots,x_8\in M_2(G)$, then f is an identity of $M_2(G)$.*

We use the notation $A_n=\sum_{\sigma\in\mathrm{Sym}(n)}x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$. By G_0' we denote the even part of G' and by G_1' its odd part.

Proposition 9. *The following identities hold:*

(1) *On $U_2(G_1')$ we have $A_k^2=0$ for every integer k.*
(2) *On $U_2(G_0')$ we have $S_k^2=0$ for every integer k.*

Proof. For $x_1,x_2\in U_2(G_1')$ and $A_2=(a_{ij})$ we get $a_{11}=a_{21}=a_{22}=0$. Thus $A_2^2=0$. Then we use induction. Let $A_{k-1}(x_1,\dots,x_{k-1})$ have only one nonzero entry, namely the $(1,2)$-entry. We have

$$\begin{aligned}A_k(x_1,\dots,x_k)&=A_{k-1}(x_1,\dots,x_{k-1})x_k+A_{k-1}(x_1,\dots,x_{k-2},x_k)x_{k-1}\\&\quad+A_{k-1}(x_1,\dots,x_{k-3},x_{k-1},x_k)x_{k-2}+\cdots\\&\quad+A_{k-1}(x_1,x_3,\dots,x_k)x_2+A_{k-1}(x_2,\dots,x_k)x_1.\end{aligned}$$

The multiplication by x_i keeps the three zero entries in every summand. So for $A_k=(b_{ij})$ we have $b_{11}=b_{21}=b_{22}=0$ and thus $A_k^2=0$.

The arguments for $S_2^2 = 0$ are similar as for $x_1, x_2 \in U_2(G_0')$ and for $S_2 = (c_{ij})$ we have $c_{11} = c_{21} = c_{22} = 0$. The recursive formulas

$$S_k(x_1,\dots,x_k) = \sum_{i=1}^{k} x_i(-1)^{k-1} S_{k-1}(x_{i+1},\dots,x_k,x_1,\dots,x_{i-1})$$

for k even and

$$S_k(x_1,\dots,x_k) = \sum_{i=1}^{k} x_i S_{k-1}(x_{i+1},\dots,x_k,x_1,\dots,x_{i-1})$$

for k odd show that for $S_k = (d_{ij})$ we have $d_{11} = d_{21} = d_{22} = 0$ and, therefore, $S_k^2 = 0$. $\square$

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