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STATISTICAL APPROXIMATION BY SOME POSITIVE LINEAR OPERATORS OF DISCRETE TYPE

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Abstract. The aim of this paper is to present a class of linear positive operators of discrete type and its statistical approximation properties obtained by using a Bohman–Korovkin type theorem.

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1. Introduction

The study of the statistical convergence for sequences of positive linear operators was attempted in the year 2002 by A. G. Gadjiev and C. Orhan [8]. The research orientation was proved to be extremely fertile, many researchers approaching this subject recently [2–4]. Motivated by this research direction, we construct a general class of positive linear operators of discrete type and study its statistical approximation properties.

In order to construct the operators, we need some notation on A-statistical convergence. Let $A := (a_{kn})_{k,n\in\mathbb{N}}$ be a non-negative regular summability matrix. For a given sequence of real numbers, $x := (x_n)_{n\in\mathbb{N}}$, the sequence $Ax := ((Ax)_k)$ defined by the formula

$$(Ax)_k := \sum_{n=1}^{\infty} a_{kn} x_n$$

is called the A-transform of x whenever the series converges for each $k \in \mathbb{N}$. A sequence x is said to be A-statistically convergent to a real number L if for every $\varepsilon > 0$, one has

$$\lim_k \sum_{n:|x_n - L| \geq \varepsilon} a_{kn} = 0.$$

We denote this limit by $\mathrm{st}_A - \lim x = L$ (see [6]).

2. Definition of operators

We set $\mathbb{R}_+ := [0,\infty)$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Let D be a given interval of the real line. We denote by $C(D)$ the space of all real-valued continuous functions on D. For each $n \in \mathbb{N}$ we consider a set of indices I_n and a net on D called $(x_{n,j})_{j\in I_n}$. We set $e_i(x) = x^i$, $i \geq 0$, $x \in D$. Following [1], let $(l_n)_{n\geq 1}$ be a sequence of positive linear operators of discrete type, defined by the equality

$$(l_n f)(x) = \sum_{j\in I_n} u_{n,j}(x) f(x_{n,j}), \quad x \in D,\ f \in C(D), \tag{2.1}$$

where $(u_{n,j})_{j\in I_n}$ is a family of continuous functions on D satisfying the following conditions

$$u_{n,j}(x) \geq 0, \quad x \in D, \tag{2.2}$$

$$\sum_{j\in I_n} u_{n,j}(x) = e_0(x), \quad x \in D, \tag{2.3}$$

$$\sum_{j\in I_n} u_{n,j}(x) x_{n,j} = e_1(x), \quad x \in D, \tag{2.4}$$

$$\sum_{j\in I_n} u_{n,j}(x) x_{n,j}^2 = e_2(x) + \psi_n(x), \quad x \in D, \tag{2.5}$$

where $\psi_n \in C(D)$.

Under this assumptions the sequence $(l_n)_{n\geq 1}$ can be indicated by the following system

$$l_n : \langle D, I_n, x_{n,j}, u_{n,j}(x); \psi_n \rangle, \quad (n,j) \in \mathbb{N} \times I_n,\ x \in D. \tag{2.6}$$

We denote by $C_B(D)$ the space of all continuous functions on D and bounded on the entire line, i. e.,

$$|f(x)| \leq M_f \quad \text{for all } x \in \mathbb{R},$$

where M_f is a constant depending on f. $C_B(D)$ is a Banach space with respect to the supremal norm $\|\cdot\|$.

In [1], compounding two sequences of operators given by (2.6), the author constructed a sequence of positive linear operators $(L_{n,\lambda})_{n\geq 1}$ acting on $C(\mathbb{R}_+)$.

In what follows, we will replace the conditions (2.3)–(2.5) imposed on the sequence $(u_{n,j})_{j\in I_n}$ by the following ones:

$$\mathrm{st}_A - \lim_n \left\| \sum_{j\in I_n} u_{n,j} - e_0 \right\| = 0, \tag{2.7}$$

$$\mathrm{st}_A - \lim_n \left\| \sum_{j\in I_n} u_{n,j} x_{n,j} - e_1 \right\| = 0, \tag{2.8}$$

$$\mathrm{st}_A - \lim_n \left\| \sum_{j\in I_n} u_{n,j} x_{n,j}^2 - e_2 \right\| = 0. \tag{2.9}$$

A sequence of positive linear operators of the form (2.1) which satisfies the conditions (2.2) and (2.7)–(2.9) will be denoted by

$$\widetilde{l}_n : \left\langle D, I_n, x_{n,j}, u_{n,j}(x) \right\rangle, \quad (n,j) \in \mathbb{N} \times I_n,\ x \in D. \tag{2.10}$$

Further on we will consider two sequences of operators of the type (2.10) and (2.6), respectively,

$$\widetilde{l}_n : \left\langle [0,1], I_n, x_{n,j}, u_{n,j}(x) \right\rangle, \quad (n,j) \in \mathbb{N} \times I_n,\ x \in [0,1],$$

and

$$l_n : \left\langle [0,b], J_n, y_{n,j}, v_{n,j}(x); \psi_n \right\rangle, \quad (n,j) \in \mathbb{N} \times J_n,\ x \in [0,b],$$

such that $1 \le b$ and, for any $n \in \mathbb{N}$, there is a function $\widetilde{\psi}_n \in C([0,b])$ with the property

$$x_{n,j} \psi_j(x) = \widetilde{\psi}_n(x), \quad j \in I_n,\ x \in [0,b]. \tag{2.11}$$

Let us consider a continuous function $\lambda : [0,b] \to [0,1]$. Now we are ready to introduce the operator $\widetilde{L}_{n,\lambda}$ by putting

$$\left(\widetilde{L}_{n,\lambda} f\right)(x) = \sum_{j\in I_n} \sum_{s\in J_j} u_{n,j}\left(\lambda(x)\right) v_{j,s}(x)\, f\left(x_{n,j} y_{j,s} + (1 - x_{n,j})x\right) \tag{2.12}$$

for all $x \in [0,b]$, $f \in C_B([0,b])$, and $n \in \mathbb{N}$. We observe that these operators are positive and linear.

3. A Bohman–Korovkin type theorem

In [8], Gadjiev and Orhan proved the following Bohman–Korovkin type statistical approximation theorem.

Theorem A. *If a sequence of positive linear operators $A_n : C_B([a,b]) \to B([a,b])$ satisfies the conditions*

$$\mathrm{st} - \lim_n \|A_n e_i - e_i\| = 0 \quad \textit{for} i = 0,1,2,$$

then, for any function $f \in C_B([a,b])$, we have

$$\mathrm{st} - \lim_n \|A_n f - f\| = 0,$$

where $B([a,b])$ is the space of all real-valued functions bounded on $[a,b]$.

We note that the above theorem is given for statistical convergence, but it also stands for A-statistical convergence. To obtain our main result we need the next lemma.

Lemma 1. *Let $A := (a_{kn})_{k,n\in\mathbb{N}}$ be a non-negative regular summability matrix and let the operators $\widetilde{L}_{n,\lambda}$ be defined by (2.12) such that the following conditions hold*

(1) *the sequence $\left(\|\widetilde{\psi}_n\|\right)_{n\in\mathbb{N}}$ is bounded,*
(2) $\mathrm{st}_A - \lim_n \|\widetilde{\psi}_n\| = 0$.

Then the following identities hold:

$$\mathrm{st}_A - \lim_n \left\| \widetilde{L}_{n,\lambda} e_i - e_i \right\| = 0 \quad \textit{for } i = 0,1,2. \tag{3.1}$$

Proof. From (2.12) and (2.3) it follows that

$$\left(\widetilde{L}_{n,\lambda} e_0\right)(x) = \sum_{j\in I_n} u_{n,j}\left(\lambda(x)\right), \quad x \in [0,b].$$

Since $e_0\left(\lambda(x)\right) = e_0(x) = 1$ for all $x \in [0,b]$, we obtain

$$\left|\left(\widetilde{L}_{n,\lambda} e_0\right)(x) - e_0(x)\right| = \left|\sum_{j\in I_n} u_{n,j}\left(\lambda(x)\right) - e_0(x)\right| \le \left\|\sum_{j\in I_n} u_{n,j} - e_0\right\|.$$

Now using the above relation and (2.7), we get (3.1) for $i = 0$.

By the definition (2.12) of the operator $\widetilde{L}_{n,\lambda}$ we have

$$\begin{aligned}\left(\widetilde{L}_{n,\lambda} e_1\right)(x) = &\sum_{j\in I_n} u_{n,j}\left(\lambda(x)\right) x_{n,j} \sum_{s\in J_j} v_{j,s}(x) y_{j,s} \\ &+ x \sum_{j\in I_n} u_{n,j}\left(\lambda(x)\right) \sum_{s\in J_j} v_{j,s}(x) \\ &- x \sum_{j\in I_n} u_{n,j}\left(\lambda(x)\right) x_{n,j} \sum_{s\in J_j} v_{j,s}(x).\end{aligned}$$

Using (2.3) and (2.4) we obtain

$$\left(\widetilde{L}_{n,\lambda} e_1\right)(x) = x \sum_{j\in I_n} u_{n,j}\left(\lambda(x)\right), \quad x \in [0,b].$$

Hence, we get

$$\begin{aligned}\left|\left(\widetilde{L}_{n,\lambda} e_1\right)(x) - e_1(x)\right| &= \left| x \sum_{j\in I_n} u_{n,j}\left(\lambda(x)\right) - e_1(x)\right| \\ &= |x| \left|\sum_{j\in I_n} u_{n,j}\left(\lambda(x)\right) - e_0(x)\right| \le b \left\|\sum_{j\in I_n} u_{n,j} - e_0\right\|.\end{aligned}$$

Since for a given $\varepsilon > 0$ we have

$$T_1 := \left\{ n \in \mathbb{N} : \left\| \widetilde{L}_{n,\lambda} e_1 - e_1 \right\| \ge \varepsilon \right\} \subseteq \left\{ n \in \mathbb{N} : \left\| \sum_{j\in I_n} u_{n,j} - e_0 \right\| \ge \frac{\varepsilon}{b} \right\} := T_2,$$

we get

$$\sum_{T_1} a_{kn} \le \sum_{T_2} a_{kn}.$$

Taking $k \to \infty$, we obtain (3.1) for $i = 1$.

By (2.3)–(2.5), (2.11), and an elementary calculus it follows that

$$\begin{aligned}\big(\widetilde{L}_{n,\lambda}e_2\big)(x) &= \sum_{j\in I_n} u_{n,j}\big(\lambda(x)\big)x_{n,j}^2\psi_j(x) + x^2\sum_{j\in I_n} u_{n,j}\big(\lambda(x)\big)\\ &= \sum_{j\in I_n} u_{n,j}\big(\lambda(x)\big)x_{n,j}\widetilde{\psi}_n(x) + x^2\sum_{j\in I_n} u_{n,j}\big(\lambda(x)\big).\end{aligned}$$

Let $M := \sup_{n\in\mathbb{N}}\{\|\widetilde{\psi}_n\|\}$. Then

$$\begin{aligned}\Big|\big(\widetilde{L}_{n,\lambda}e_2\big)(x) - e_2(x)\Big| &\le \Big|\widetilde{\psi}_n(x)\Big(\sum_{j\in I_n} u_{n,j}\big(\lambda(x)\big)x_{n,j} - e_1\big(\lambda(x)\big)\Big)\Big|\\ &\quad + \big|\lambda(x)\widetilde{\psi}_n(x)\big| + \Big|x^2\Big(\sum_{j\in I_n} u_{n,j}\big(\lambda(x)\big) - e_0(x)\Big)\Big|\\ &\le M\Big\|\sum_{j\in I_n} u_{n,j}x_{n,j} - e_1\Big\| + b\big\|\widetilde{\psi}_n\big\| + b^2\Big\|\sum_{j\in I_n} u_{n,j} - e_0\Big\|.\end{aligned}$$

Let us set $K := \max\{M, b^2\}$. Then

$$\big\|\widetilde{L}_{n,\lambda}e_2 - e_2\big\| \le K\Big(\Big\|\sum_{j\in I_n} u_{n,j}x_{n,j} - e_1\Big\| + \big\|\widetilde{\psi}_n\big\| + \Big\|\sum_{j\in I_n} u_{n,j} - e_0\Big\|\Big). \tag{3.2}$$

For a given $\varepsilon > 0$, we put

$$U := \Big\{n\in\mathbb{N} : \Big\|\sum_{j\in I_n} u_{n,j}x_{n,j} - e_1\Big\| + \big\|\widetilde{\psi}_n\big\| + \Big\|\sum_{j\in I_n} u_{n,j} - e_0\Big\| \ge \frac{\varepsilon}{K}\Big\},$$

$$U_1 := \Big\{n\in\mathbb{N} : \Big\|\sum_{j\in I_n} u_{n,j}x_{n,j} - e_1\Big\| \ge \frac{\varepsilon}{3K}\Big\},$$

$$U_2 := \Big\{n\in\mathbb{N} : \big\|\widetilde{\psi}_n\big\| \ge \frac{\varepsilon}{3K}\Big\},$$

and

$$U_3 := \Big\{n\in\mathbb{N} : \Big\|\sum_{j\in I_n} u_{n,j} - e_0\Big\| \ge \frac{\varepsilon}{3K}\Big\}.$$

It is obvious that $U \subset U_1 \cup U_2 \cup U_3$. Inequality (3.2) implies that

$$\sum_{n:\|\widetilde{L}_{n,\lambda}e_2 - e_2\|\ge\varepsilon} a_{kn} \le \sum_{n\in U} a_{kn} \le \sum_{n\in U_1} a_{kn} + \sum_{n\in U_2} a_{kn} + \sum_{n\in U_3} a_{kn}.$$

Passing to the limit as $k \to \infty$, we complete the proof. $\square$

Remark 1. If the non-negative regular summability matrix A is the Cesáro matrix of order one, then the A-statistical convergence reduces to the statistical convergence [5, 7]. Consequently, Lemma 1 also stands for statistical convergence.

Using the above lemma and Theorem A, we obtain the following result.

Theorem 1. *Let $A := (a_{kn})_{k,n\in\mathbb{N}}$ be a non-negative regular summability matrix and let the operators $\widetilde{L}_{n,\lambda}$ be defined by (2.12). If the conditions of Lemma 1 hold, then for any function $f \in C_B([0,b])$ we have*

$$\mathrm{st}_A - \lim_n \left\| \widetilde{L}_{n,\lambda} f - f \right\| = 0.$$

4. Example

In what follows we present a particular sequence of positive linear operators of the form (2.12) which is statistically convergent with respect to the sup-norm to the approximated function but it is not uniformly convergent.

Let

$$\widetilde{l}_n : \left\langle [0,1], \{0,\dots,n\}, \frac{j}{n}, u_{n,j}(x) \right\rangle, \quad (n,j) \in \mathbb{N} \times \{0,\dots,n\},\ x \in [0,1]$$

be such that

$$u_{n,j}(x) = \mathrm{C}_n^j \left(\phi_n(x)\right)^j \left(1 - \phi_n(x)\right)^{n-j}, \quad x \in [0,1],$$

and

$$\phi_n = \begin{cases} e_1 & \text{if } \lg n \notin \mathbb{N}_0, \\ n e_1 & \text{if } \lg n \in \mathbb{N}_0 \end{cases}$$

for all $(n,j) \in \mathbb{N} \times \{0,\dots,n\}$. It is clear that taking $\phi_n = e_1$ for all $n \in \mathbb{N}$, $\widetilde{l}_n$ becomes the nth Bernstein polynomial.

We must should now check that $(u_{n,j})$ satisfies conditions (2.2), (2.7)–(2.9). It is obvious that (2.2) and (2.7) are fulfilled. By using an elementary calculus, we obtain

$$\left| \sum_{j=0}^{n} u_{n,j}(x) \frac{j}{n} - e_1(x) \right| = \left| \phi_n(x) - e_1(x) \right| = 0$$

for all n with the property $\lg n \notin \mathbb{N}_0$. Letting $\varepsilon > 0$ we obtain

$$\left\{ n \in \mathbb{N} : \left\| \sum_{j=0}^{n} u_{n,j} \frac{j}{n} - e_1 \right\| \geq \varepsilon \right\} = \left\{ n \in \mathbb{N} : \lg n \in \mathbb{N}_0 \right\},$$

and (2.8) is thus satisfied.

It remains only to verify (2.9). Indeed,

$$\left|\sum_{j=0}^{n} u_{n,j}(x)\left(\frac{j}{n}\right)^2 - e_2(x)\right| = \left|\frac{\phi_n(x)}{n} + \frac{n-1}{n}(\phi_n(x))^2 - e_2(x)\right|$$
$$= \left|\frac{x}{n} + \frac{n-1}{n}x^2 - x^2\right| \le \frac{x}{n} \le \frac{1}{n}$$

for all n with the property $\lg n \notin \mathbb{N}_0$. Letting $\varepsilon > 0$ we get

$$\left\{n \in \mathbb{N} : \left\|\sum_{j=0}^{n} u_{n,j}\left(\frac{j}{n}\right)^2 - e_2\right\| \ge \varepsilon\right\} \subseteq \left\{n \in \mathbb{N} : \lg n \in \mathbb{N}_0 \text{ and } \frac{1}{n} \ge \varepsilon\right\}$$

and, therefore, (2.9) is also satisfied.

Choosing

$$l_n = s_n : \left\langle [0,\infty), \mathbb{N}_0, \frac{j}{n}, e^{-nx}\frac{(nx)^j}{j!}; \frac{e_1}{n}\right\rangle, \quad (n,j) \in \mathbb{N}\times\mathbb{N}_0,\ x \in [0,\infty)$$

(the Favard–Szász–Mirakjan operators), we are able to define the operators $\widetilde{L}_{n,\lambda}$ by putting

$$(\widetilde{L}_{n,\lambda} f)(x) = \sum_{j=0}^{n}\sum_{s=0}^{\infty} C_n^j (\phi_n(\lambda(x)))^j (1-\phi_n(\lambda(x)))^{n-j}$$
$$\times e^{-jx}\frac{(jx)^s}{s} f\left(\frac{s}{n} + \left(1-\frac{j}{n}\right)x\right) \quad (4.1)$$

for all $x \ge 0$, $f \in C_B(\mathbb{R}_+)$, and $n \in \mathbb{N}$, where $\lambda : [0,\infty) \to [0,1]$ is a continuous function.

On the basis of Lemma 1 and Theorem 1 we deduce that the sequence of operators $(\widetilde{L}_{n,\lambda} f)$ defined by (4.1) is A-statistically convergent to f for any function $f \in C_B(D)$, where D is a compact interval on the positive semiaxis.

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