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MAXIMUM PRINCIPLE AND EXISTENCE OF SOLUTIONS FOR NON-NECESSARILY COOPERATIVE SYSTEMS INVOLVING SCHRÖDINGER OPERATORS

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Abstract. In this paper, we obtain the necessary and sufficient conditions for having the maximum principle and existence of positive solutions for some cooperative systems involving Schrödinger operators defined on unbounded domains. Then, we deduce the existence of solutions for semi-linear systems. Finally we discuss the generalized maximum principle (ϕ_q- positivity) for non-cooperative systems.

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1. Introduction

We consider the following semilinear system

$$\begin{cases} LY + QY = Ag(x)Y + F(x,Y) \quad \text{in} \quad \Omega, \\ Y \to 0 \quad \text{as} \quad |x| \to \infty, \\ Y = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{S}$$

where Ω is an unbounded domain of R^N, L is an $n \times n$ diagonal matrix of Laplace operators, Q is an $n \times n$ diagonal matrix of potential functions q_i $(1 \le i \le n)$, $g(x)$ is a weight function tending to zero at infinity, F is a given n-vector function and $A = (a_{ij})$ is a constant $n \times n$ cooperative matrix such that

$$a_{ij} \ge 0 \quad \text{for all} \quad i \ne j. \tag{1}$$

It is well known that the maximum principle plays an important role in the theory of partial differential equations. An analogous theory has been appeared for semilinear systems in [1,5–9,11,14,22]. In [7,9], the authors studied system (S) with $q_i = 0$ and $g(x) = 1$, defined on bounded domains with Dirichlet conditions. The problem with $q_i = 0$ defined on the whole space R^N has been established in [12, 13]. The system with equal potentials defined on R^N has been considered in [3,4].

Some applications concerning the optimal control of systems like (S) have been introduced in [15–21].

Here, we extend these results to system (S). In section two, we obtain necessary or sufficient conditions for having the maximum principle and existence of positive solutions for cooperative linear systems. Then, we study semilinear systems in section three; we adapt the method of sub and super solutions for proving the existence of nonnegative solutions. Finally, in section four, we study the generalized maximum principle (φ_q-positivity) for non-cooperative systems.

To prove our theorems, we make use of earlier results by Djellit and Yechoui [10] who proved that, for $N > 2$ and $q > 0$, if there exist $\alpha > 0$, $\beta \geq 1$, $\alpha > \beta$, and $k, c > 0$ such that

$$0 < g(x) \leq \frac{k}{(1+|x|^2)^\alpha}, \qquad 0 < q(x) \leq \frac{c}{(1+|x|^2)^\beta}, \tag{2}$$

then the eigenvalue problem

$$\begin{cases} (-\Delta + q)y = \lambda g(x)y & \text{in} \quad \Omega, \\ y \to 0 \quad \text{as} \quad |x| \to \infty, \\ y = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{E}$$

has simple principal eigenvalue (λ_q^+) which is associated with positive eigenfunction φ_q on V.

Moreover (λ_q^+) is characterized by

$$\lambda_q^+ \int_\Omega g(x)|y|^2 dx \leq |\nabla y|^2 + q|y|^2, \tag{3}$$

where

$$V(\Omega) = \left\{ y \in D'(\Omega) \mid |\left(1+|x|^2\right)^{-\frac{1}{2}} y \in L^2(\Omega),\ \nabla y \in L^2(\Omega) \right\}$$

is a Hilbert space with an inner product $(y, \psi)_V = \int_\Omega \left(\nabla y . \nabla \psi + \frac{1}{1+|x|^2} y \psi\right) dx$ and a norm

$$\|y\|_V = \left(\int_\Omega \left(|\nabla y|^2 + \frac{1}{1+|x|^2} |y|^2 \right) dx \right)^{\frac{1}{2}}$$

which is equivalent to

$$\|y\|_q = \left(\int_\Omega \left(|\nabla y|^2 + q|y|^2 \right) dx \right)^{\frac{1}{2}}.$$

We also introduce the Hilbert space

$$\mathcal{H} = \left\{ y : \Omega \to R \mid \int_\Omega g y^2 dx \leq \infty \right\} = L_g^2(\Omega)$$

with an inner product

$$(y,\psi)_g = \int_\Omega g y \psi dx.$$

2. Cooperative linear systems

In this section, we study the maximum principle and existence of positive solutions for system (S) when the right-hand side is linear.

Definition 1. We say that the maximum principle holds for system (S) if $F \geq 0$ and Y is a solution of (S), then $Y \geq 0$.

Definition 2. A non-singular square matrix $B = (b_{ij})$ is said to be an M-matrix if $b_{ij} \leq 0$ for $i \neq j$, $b_{ii} > 0$ for $i = 1, \dots, n$, and if all the principal minors extracted from B are positive.

The ith equation of system (S) can be written as

$$\begin{cases} (-\Delta + q_i) y_i = g(x) \sum_{j=1}^{n} a_{ij} y_j + f_i & \text{in} \quad \Omega, \\ y_i \to 0 \quad \text{as} \quad |x| \to \infty, \\ y_i = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{S_i}$$

Theorem 1. *Assume that* (1) *and* (2) *with* $q = q_i$ *hold, and* $f_i \geq 0$. *System* (S) *satisfies the maximum principle if*

the matrix $(\Lambda_Q^+ - A)$ *is a non-singular* M*-matrix, where* (4)

$$\Lambda_Q^+ = \begin{pmatrix} \lambda_{q_1}^+ & 0 & \dots & 0 \\ 0 & \lambda_{q_2}^+ & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_{q_n}^+ \end{pmatrix}. \tag{5}$$

Moreover, if the maximum principle holds for system (S), *then*

the matrix $(\Lambda_q^+ - A)$ *is a non-singular* M*-matrix, where* (6)

$$q = \max\{q_i : 1 \leq i \leq n\}, \qquad \Lambda_q^+ = \begin{pmatrix} \lambda_q^+ & 0 & \dots & 0 \\ 0 & \lambda_q^+ & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_q^+ \end{pmatrix}.$$

Proof. First assume that $f_i \geq 0$ and $(y_i)_{i=1}^n \in \prod_{i=1}^n V_{q_i}$ is a solution of (S). Multiplying (S_i) by $y_i^- = \max\{-y_i, 0\}$ and integrating over Ω, we obtain by Green's formula that

$$\int_\Omega \nabla y_i . \nabla y_i^- dx + \int_\Omega q_i y_i y_i^- dx = \sum_{j=1}^n a_{ij} \int_\Omega g(x) y_j y_i^- dx + \int_\Omega f_i y_i^- dx,$$

i. e.,

$$\int_\Omega |\nabla y_i^-|^2 dx + \int_\Omega q_i |y_i^-|^2 dx$$
$$= a_{ii} \int_\Omega g(x)|y_i^-|^2 dx - \sum_{j\neq i}^n a_{ij} \int_\Omega g(x) y_j y_i^- dx - \int_\Omega f_i y_i^- dx,$$

and thus, by (3), we get

$$\lambda_{q_i}^+ \int_\Omega g(x)|y_i^-|^2 dx$$
$$\leq a_{ii} \int_\Omega g(x)|y_i^-|^2 dx - \sum_{j\neq i}^n a_{ij} \int_\Omega g(x) y_j y_i^- dx - \int_\Omega f_i y_i^- dx.$$

Therefore

$$(\lambda_{q_i}^+ - a_{ii}) \int_\Omega |\sqrt{g} y_i^-|^2 dx \leq \sum_{j\neq i}^n a_{ij} \int_\Omega g(x) y_j^- y_i^- dx$$

and, by the Cauchy-Schwartz inequality, we have

$$(\lambda_{q_i}^+ - a_{ii}) \left(\int_\Omega |\sqrt{g} y_i^-|^2 dx \right)^{\frac{1}{2}} - \sum_{j\neq i}^n a_{ij} \left(\int_\Omega |\sqrt{g} y_j^-|^2 dx \right)^{\frac{1}{2}} \leq 0,$$

which can be rewritten to the form

$$\begin{pmatrix} \lambda_{q_1}^+ - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda_{q_2}^+ - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda_{q_n}^+ - a_{nn} \end{pmatrix} \begin{pmatrix} \left(\int_\Omega |\sqrt{g} y_1^-|^2 dx\right)^{1/2} \\ \left(\int_\Omega |\sqrt{g} y_2^-|^2 dx\right)^{1/2} \\ \vdots \\ \left(\int_\Omega |\sqrt{g} y_n^-|^2 dx\right)^{1/2} \end{pmatrix} \leq 0.$$

Now, (4), $y_1^- = y_2^- = \cdots = y_n^- = 0$ and hence $y_1, y_2, \ldots, y_n \geq 0$.

Assume now that $0 \leq f_i \in L^2_{1/g}(\Omega)$ and that the maximum principle holds for system (S). We rewrite (S_i) as follows:

$$\begin{cases} (-\Delta + q)y_i = g(x)\sum_{j=1}^{n} a_{ij} y_j + H_i \quad \text{in} \quad \Omega \\ y_i \to 0 \quad \text{as} \quad |x| \to \infty \\ y_i = 0 \quad \text{on} \quad \partial\Omega, \end{cases}$$

where $0 \leq H_i = (q - q_i)y_i + f_i \in L^2_{1/g}(\Omega)$.

Multiplying by φ_q (the eigenfunction corresponding λ_q^+) and integrating over Ω, we get

$$\int_\Omega (-\Delta + q)y_i \varphi_q dx = \sum_{j=1}^{n} a_{ij} \int_\Omega g(x) y_j \varphi_q dx + \int_\Omega H_i \varphi_q dx$$

and, by using Green's formula and (3), we obtain

$$(\lambda_q^+ - a_{ii}) \int_\Omega g(x) y_i \varphi_q dx - \sum_{j \neq i}^{n} a_{ij} \int_\Omega g(x) y_j \varphi_q dx = \int_\Omega H_i \varphi_q dx \qquad (7)$$

which is a Cramer system in $X_i = \int_\Omega g(x) y_i \varphi_q dx$, $1 \leq i \leq n$. Since the right-hand side is non-negative as well as X_i, we obtain

$$\begin{vmatrix} \lambda_q^+ - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda_q^+ - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda_q^+ - a_{nn} \end{vmatrix} = \left| \Lambda_q^+ - A \right| > 0.$$

The inequality $\lambda_q^+ - a_{ii} > 0$ is satisfied from the scalar case (see [13]), because the functions g, y_i, H_i and the coefficients a_{ij} for $i \neq j$ are non-negative. □

Remark 1. If $q_i = q$ for $1 \leq i \leq n$, then condition (6) is the necessary and sufficient condition for having the maximum principle for system (S).

Theorem 2. *Let* (1) *and* (2) *with* $q = q_i$ *hold. Then, for* $F \geq 0$, *system* (S) *has a unique positive solution if condition* (4) *is satisfied.*

Proof. We consider the bilinear form $a: \prod_{i=1}^{n} V_{q_i} \times \prod_{i=1}^{n} V_{q_i} \to R$ defined by

$$\begin{aligned} a(Y, \Psi) &= a((y_1, y_2, \ldots, y_n), (\psi_1, \psi_2, \ldots, \psi_n)) \\ &= \sum_{i=1}^{n} \int_\Omega \nabla y_i . \nabla \psi_i dx + \sum_{i=1}^{n} \int_\Omega q_i y_i \psi_i dx \end{aligned}$$

$$-\sum_{j\neq i}^{n}\int_{\Omega} g(xa_{ij}y_j\psi_i\,dx-\sum_{i=1}^{n}\int_{\Omega} g(x)a_{ii}y_i\psi_i\,dx$$

We choose $m\geq 0$ such that $m+a_{ii}>0$. Then, we have

$$\begin{aligned}a(Y,Y)=&\sum_{i=1}^{n}\int_{\Omega}\left[|\nabla y_i|^2+q_i y_i^2\right]dx\\&-\sum_{j\neq i}^{n}\int_{\Omega} g(x)a_{ij}y_iy_j\,dx-\sum_{i=1}^{n}\int_{\Omega} g(x)a_{ii}y_i^2\,dx\\=&\sum_{i=1}^{n}\int_{\Omega}\left[|\nabla y_i|^2+(q_i+mg)y_i^2\right]dx\\&-\sum_{j\neq i}^{n}\int_{\Omega} g(x)a_{ij}y_iy_j\,dx-\sum_{i=1}^{n}(m+a_{ii})\int_{\Omega} g(x)y_i^2\,dx\end{aligned}$$

and, by the Cauchy-Schwartz inequality and (3), we get

$$\begin{aligned}a(Y,Y)\geq&\sum_{i=1}^{n}\int_{\Omega}\left[|\nabla y_i|^2+(q_i+mg)y_i^2\right]dx\\&-\sum_{j\neq i}^{n}a_{ij}\left(\int_{\Omega}(\sqrt{g}y_i)^2dx\right)^{\frac{1}{2}}\left(\int_{\Omega}(\sqrt{g}y_j)^2dx\right)^{\frac{1}{2}}\\&-\sum_{i=1}^{n}(m+a_{ii})\int_{\Omega} g(x)y_i^2\,dx\\\geq&\sum_{i=1}^{n}\left(1-\frac{m+a_{ii}}{m+\lambda_{q_i}^+}\right)\int_{\Omega}\left[|\nabla y_i|^2+(q_i+mg)y_i^2\right]dx\\&-\sum_{j\neq i}^{n}\frac{a_{ij}}{\sqrt{(m+\lambda_{q_i}^+)(m+\lambda_{q_j}^+)}}\left(\int_{\Omega}\left[|\nabla y_i|^2+(q_i+mg)y_i^2\right]dx\right)^{\frac{1}{2}}\\&\times\left(\int_{\Omega}\left[|\nabla y_j|^2+(q_j+mg)y_j^2\right]dx\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore, using (4), it follows that

$$a(Y,Y)\geq C\sum_{i=1}^{n}\int_{\Omega}\left[|\nabla y_i|^2+(q_i+mg)(y_i)^2\right]dx$$

$$\geq C \sum_{i=1}^{n} \int_{\Omega} \left[|\nabla y_i|^2 + q_i (y_i)^2 \right] dx = C \sum_{i=1}^{n} \|y_i\|_{q_i}^2,$$

where $C > 0$. Then, by the Lax Milgram lemma, since a is a continuous coercive bilinear form, there exists a unique solution $Y = (y_i)_{i=1}^n \in \prod_{i=1}^n V_{q_i}$. This solution is non-negative by the maximum principle. □

3. SEMILINEAR SYSTEMS

In this section, we adapt the method of sub and super solutions, to prove the existence of solutions for semilinear cooperative system (S). The proof here is analogous to that of [3, 13].

We assume that $f_i(x, Y) = f_i(x, y_1, y_2, \ldots, y_n)$ is a Carathéodory function such that

$$0 \leq F(x, Y) = (f_1(x, Y), f_2(x, Y), \ldots, f_n(x, Y)) \leq N g(x) Y + h' \tag{8}$$

for all $Y \geq 0$ and $x \in \Omega$, where N is a positive constant and $0 < h' = (h, h, \ldots, h)$ is a bounded vector-function in $\left(L^2_{1/g}(\Omega)\right)^n$.

Theorem 3. *Let* (1), (2) *with* $q = q_i$, *and* (8) *be satisfied. Then there exists a positive solution of system* (S) *if*

$$\text{the matrix } \left(\Lambda_Q^+ - (A + NI)\right) \text{ is a non-singular } M\text{-matrix,} \tag{9}$$

where Λ_Q^+ *is defined by the relation* (5) *and* I *denotes the identity matrix.*

Proof. We divide the proof into several steps.
Step (i): *Construction of sub and super solutions.* It is clear that

$$Y^0 = (y_1^0, y_2^0, \ldots, y_n^0) = (0, 0, \ldots, 0)$$

is a sub solution of (S), because

$$LY^0 + QY^0 - g(x)AY^0 - F(x, Y^0) \leq 0.$$

Consider now the system

$$\begin{cases} LY + QY = g(x)(A + NI)Y + h' & \text{in} \quad \Omega. \\ Y \to 0 & \text{as} \quad |x| \to \infty, \\ Y = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{10}$$

It follows from Theorem 2 that, under condition (9), system (10) has a unique positive solution

$$Y^* = (y_1^*, y_2^*, \ldots, y_n^*).$$

By (8), we have

$$0 \leq (L + Q)Y^* - g(x)AY^* - F(x, Y^*), \tag{11}$$

i. e., $Y^* = (y_1^*, y_2^*, \ldots, y_n^*)$ is a super solution of (S).

Step (ii): *Definition of a compact operator.* We introduce

$$T : \mathcal{H}^n \ni Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \geq 0 \mapsto \Psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} = TY \in \prod_{i=1}^{n} V_{q_i},$$

where Ψ is a unique solution of the following system:

$$\begin{cases} (L + Q + mg(x)I)\Psi = (mI + A)g(x)Y + F(x, Y) & \text{in} \quad \Omega, \\ \Psi \to 0 \quad \text{as} \quad |x| \to \infty, \\ \Psi = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{12}$$

where $m \geq 0$ is such that $m + a_{ii} > 0$. Since the equation in (12) can be rewritten as

$$(L + Q)\Psi = -mg(x)\Psi + \bar{F}, \quad \bar{F} = (mI + A)g(x)Y + F(x, Y) \geq 0,$$

system (12) has a unique solution $\Psi \in \prod_{i=1}^{n} V_{q_i}$ and therefore T is well-defined.

Step (iii): *$K = [Y^0, Y^*]$ is invariant by T, i. e., $T(K) \subseteq K$.* For every $Y \in \prod_{i=1}^{n} V_{q_i}$, $Y \geq 0$, we have $T(Y) = \Psi \geq 0$. We show now that if $Y \leq Y^*$, then $\Psi \leq Y^*$. From (10) and (12), we obtain

$$(L + Q + mg(x)I)(Y^* - \Psi) = \tau,$$

where $\tau := (mI + A)g(x)(Y^* - Y) + Ng(x)Y^* - F(x, Y) \geq 0$. Therefore

$$(L + Q)(Y^* - \Psi) = -mg(x)(Y^* - \Psi) + \tau$$

and thus $Y^* - \Psi$ is non-negative, i. e., $\Psi \leq Y^*$.

Step (iv): *T is a continuous operator.* Let $Y_k \to Y$ in $\mathcal{H}^n$. Then we get

$$F(x, Y_k) \to F(x, Y) \quad \text{in} \quad (\mathcal{H}')^n.$$

If we denote $\Psi_k = T(Y_k)$, by (12) it follows that

$$(L + Q + mg(x)I)(\Psi - \Psi_k) = (mI + A)g(x)(Y - Y_k) + F(x, Y) - F(x, Y_k).$$

Multiplying by $(\Psi - \Psi_k)$ and integrating over Ω, we get

$$\begin{aligned} \int_\Omega (L + Q + mg(x)I)(\Psi - \Psi_k).(\Psi - \Psi_k)dx \\ = (mI + A) \int_\Omega g(x)(Y - Y_k).(\Psi - \Psi_k)dx \\ + \int_\Omega \big(F(x, Y) - F(x, Y_k)\big).(\Psi - \Psi_k)dx, \end{aligned}$$

which, by virtue of the Green formula, yields that

$$\int_\Omega \left|\nabla(\Psi - \Psi_k)\right|^2 dx + \int_\Omega Q|\Psi - \Psi_k|^2 dx + m \int_\Omega g(x)|\Psi - \Psi_k|^2 dx$$

$$= (mI + A) \int_{\Omega} g(x)(Y - Y_k).(\Psi - \Psi_k)dx$$
$$+ \int_{\Omega} \big(F(x,Y) - F(x,Y_k)|\big).(\Psi - \Psi_k)dx.$$

Now, using the Cauchy-Schwarz inequality, we get

$$\int_{\Omega} \big|\nabla(\Psi - \Psi_k)\big|^2 dx + \int_{\Omega} Q|\Psi - \Psi_k|^2 dx + m\int_{\Omega} g(x)|\Psi - \Psi_k|^2 dx$$
$$\leq (mI + A)\|Y - Y_k\|_{\mathcal{H}^n}\|\Psi - \Psi_k\|_{\mathcal{H}^n}$$
$$+ \|F(x,Y) - F(x,Y_k)\|_{(\mathcal{H}')^n}\|\Psi - \Psi_k\|_{\mathcal{H}^n}$$

and thus

$$\|\Psi - \Psi_k\|_Q^2 \leq (mI + A)\|Y - Y_k\|_{\mathcal{H}^n}\|\Psi - \Psi_k\|_{\mathcal{H}^n}$$
$$+ \|F(x,Y) - F(x,Y_k)\|_{(\mathcal{H}')^n}\|\Psi - \Psi_k\|_{\mathcal{H}^n},$$

where $\|\cdot\|_Q$ denotes the norm in $\prod_{i=1}^n V_{q_i}$. Since $\|Y - Y_k\|_{\mathcal{H}} \to 0$, we have $\|\Psi - \Psi_k\|_Q \to 0$.

Step (v): *T is a compact operator.* First note the following: Let $Y \geq 0$ and $\Psi = T(Y)$. Multiplying (12) by Ψ and integrating over Ω, we get

$$\int_{\Omega} \big|\nabla\Psi\big|^2 dx + \int_{\Omega} Q|\Psi|^2 dx + m\int_{\Omega} g(x)|\Psi|^2 dx$$
$$= (mI + A)\int_{\Omega} g(x)Y\Psi dx + \int_{\Omega} F(x,Y)\Psi dx,$$

which, in view of (8), yields that

$$\int_{\Omega} \big|\nabla\Psi\big|^2 dx + \int_{\Omega} Q|\Psi|^2 dx + m\int_{\Omega} g(x)|\Psi|^2 dx$$
$$\leq (mI + A)\int_{\Omega} g(x)Y\Psi dx + N\int_{\Omega} g(x)Y\Psi dx + \int_{\Omega} \sqrt{g}\Psi\frac{h}{\sqrt{g}}dx$$
$$= (mI + A + NI)\int_{\Omega} g(x)Y\Psi dx + \int_{\Omega} \sqrt{g}\Psi\frac{h}{\sqrt{g}}dx.$$

By the Cauchy-Schwarz inequality, we obtain

$$\|\psi\|_Q \leq C\left(\|Y\|_{(\mathcal{H})^n} + \|h\|_{\mathcal{H}'}\right) \leq C\left(\|Y\|_Q + \|h\|_{\mathcal{H}'}\right)$$

Therefore if $\{Y_k\}_{k\in N}$ is a bounded sequence in $\prod_{i=1}^n V_{q_i}$, the associated sequence $\{\Psi_k\}_{k\in N}$ is bounded in $\prod_{i=1}^n V_{q_i}$. We show now that $\{\Psi_k\}_{k\in N}$ is a Cauchy sequence in $\prod_{i=1}^n V_{q_i}$.

Suppose that $\|Y_k\|_Q^2 \le E$ for $k \in N$, E is a constant. In view of (2), we can choose R large enough such that

$$(1+|x|^2)g(x) < \frac{\epsilon}{8\gamma E} \quad \text{for} \quad |x| \ge R, \tag{13}$$

where $\epsilon > 0$ is fixed and γ is given by

$$\int_\Omega (1+|x|^2)^{-1} y^2 dx \le \gamma \int_\Omega |\nabla y|^2 dx. \tag{14}$$

Let $B = \{x \in \Omega \mid |x| < R\}$ and $B' = \{x \in \Omega \mid |x| \ge R\}$. Since $\{Y_k\}_{k\in N}$ is bounded in $\prod_{i=1}^n V_{q_i}$, Y_k is bounded in $(\mathcal{H}(B))^n$. However B is bounded and therefore the embedding $(\mathcal{H}(B))^n$ into $(L^2(B))^n$ is compact. Hence there exists a convergent subsequence still denoted by $\{Y_k\}_{k\in N}$, which is a Cauchy sequence and thus, for every j and k large enough, we have

$$\int_B g(x)|Y_k - Y_j|^2 dx \le \int_B |Y_k - Y_j|^2 dx < \frac{\epsilon}{2}.$$

Moreover using (13) and (14), we obtain

$$\begin{aligned}\int_{B'} g(x)|Y_k - Y_j|^2 dx &= \int_{B'} (1+|x|^2)g(x)\frac{1}{1+|x|^2}|Y_k - Y_j|^2 dx \\ &\le \frac{\varepsilon}{8\gamma E}\int_{B'} \frac{1}{1+|x|^2}|Y_k - Y_j|^2 dx \\ &\le \frac{\varepsilon}{8\gamma E}\gamma \|Y_k - Y_j\|_Q^2\end{aligned}$$

so $\int_{B'} g(x)|Y_k - Y_j|^2 dx < \frac{\epsilon}{2}$.

Then $\{\Psi_k\}_{k\in N}$ is a Cauchy sequence in $\prod_{i=1}^n V_{q_i}$. Hence it converges to Ψ and therefore T is compact in $\prod_{i=1}^n V_{q_i}$. By Schauder fixed point theorem, there exists at least one positive solution $Y = (y_i)_{i=1}^n \in \prod_{i=1}^n V_{q_i}$ of system (S) satisfying $Y^0 \le Y \le Y^*$. □

4. Non-cooperative systems

In this section, we study the generalized maximum principle (φ_q-positivity) for $n \times n$ non-cooperative systems.

Definition 3. System (S) (with $q_i = q$) satisfies the generalized maximum principle if $F \ge 0$ and $Y = (y_1, y_2, \ldots, y_n)$ is a solution of (S), then there exists $C > 0$ such that

$$y_i \ge C\varphi_q \quad \text{in} \quad \Omega \quad \text{for all} \quad i = 1, 2, \ldots, n.$$

We start with 2×2 non cooperative systems:

$$\begin{cases} L_q y_1 = a_{11}g(x)y_1 + a_{12}g(x)y_2 + f_1 & \text{in} \quad \Omega, \\ L_q y_2 = a_{21}g(x)y_1 + a_{22}g(x)y_2 + f_2 & \text{in} \quad \Omega, \\ y_1, y_2 \to 0 \quad \text{as} \quad |x| \to \infty, \\ y_1 = y_2 = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{S_2}$$

As in [2], we can prove

Theorem 4. *Assume that* $a_{12} < 0$, $a_{21} > 0$, $a_{11} > a_{22}$, $f_1 - \xi_2 f_2 \geq 0$, *and* $(a_{11} - a_{22})^2 + 4a_{12}a_{21} \geq 0$ *are satisfied. The generalized maximum principle holds for system* (S_2) *if*

$$a_{11} - \lambda_q < r < \lambda_q - a_{22}.$$

Now, let us consider the following 3×3 non cooperative system

$$\begin{cases} L_q y_1 = a_{11}g(x)y_1 + a_{12}g(x)y_2 + a_{13}g(x)y_3 + f_1 & \text{in} \quad \Omega \\ L_q y_2 = a_{21}g(x)y_1 + a_{22}g(x)y_2 + a_{23}g(x)y_3 + f_2 & \text{in} \quad \Omega \\ L_q y_3 = a_{31}g(x)y_1 + a_{32}g(x)y_2 + a_{33}g(x)y_3 + f_3 & \text{in} \quad \Omega \\ y_1, y_2, y_3 \to 0 \quad \text{as} \quad |x| \to \infty \\ y_1 = y_2 = y_3 = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{S_3}$$

Assume that

$$a_{12}, a_{13} < 0, \quad a_{21}, a_{23}, a_{31}, a_{32} \geq 0 \tag{15}$$

We insert our 3×3 non-cooperative system into a 4×4 cooperative one. We introduce the following fourth equation of a new variable $y_4 = y_1 - \xi_2 y_2 - \xi_3 y_3$:

$$\begin{aligned} L_q y_4 = {} & (a_{11} - \xi_2 a_{21} - \xi_3 a_{31} - s)g(x)y_1 \\ & + (a_{12} - \xi_2 a_{22} - \xi_3 a_{32} + s\xi_2)g(x)y_2 \\ & + (a_{13} - \xi_2 a_{23} - \xi_3 a_{33} + s\xi_3)g(x)y_3 + sg(x)y_4 + f_4, \end{aligned}$$

where ξ_2 and ξ_3 are positive numbers and $f_4 = f_1 - \xi_2 f_2 - \xi_3 f_3$. Then system ($S_3$) can be changed into system of the form

$$\begin{cases} L_q Y = \mathcal{N} g(x)Y + F & \text{in} \quad \Omega \\ Y \to 0 \quad \text{as} \quad |x| \to \infty \\ Y = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{S_3'}$$

where

$$L_q = \begin{pmatrix} -\Delta + q & 0 & 0 & 0 \\ 0 & -\Delta + q & 0 & 0 \\ 0 & 0 & -\Delta + q & 0 \\ 0 & 0 & 0 & -\Delta + q \end{pmatrix},$$

$Y = (y_1, y_2, y_2, y_4)$, $F = (f_1, f_2, f_3, f_4)$, and

$$\mathcal{N} = \begin{pmatrix} a_{11}-r & a_{12}+r\xi_2 & a_{13}+r\xi_3 & r \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ b_1 & b_2 & b_3 & s \end{pmatrix}$$

with

$$\begin{gathered} b_1 = a_{11}-\xi_2 a_{21}-\xi_3 a_{31}-s, \quad b_2 = a_{12}-\xi_2 a_{22}-\xi_3 a_{32}+s\xi_2, \\ b_3 = a_{13}-\xi_2 a_{23}-\xi_3 a_{33}+s\xi_3. \end{gathered} \tag{16}$$

For the cooperativeness of system (S_3'), we can choose r, s, ξ_2, and ξ_3 such that

$$r > 0, \qquad a_{12}+r\xi_2 = 0, \qquad a_{13}+r\xi_3 = 0, \tag{17}$$

$$a_{11}-\xi_2 a_{21}-\xi_3 a_{31}-s = 0, \tag{18}$$

$$a_{12}-\xi_2 a_{22}-\xi_3 a_{32}+s\xi_2 = 0, \tag{19}$$

$$a_{13}-\xi_2 a_{23}-\xi_3 a_{33}+s\xi_3 = 0, \tag{20}$$

and then, using (15), we get $\xi_3 = -\frac{a_{13}}{r} > 0$ and $\xi_2 = -\frac{a_{12}}{r} > 0$. Now, system (S_3') satisfies the maximum principle if the matrix

$$\begin{pmatrix} \lambda_q - a_{11}+r & -a_{12}-r\xi_2 & -a_{13}-r\xi_3 & -r \\ -a_{21} & \lambda_q - a_{22} & -a_{23} & 0 \\ -a_{31} & -a_{32} & \lambda_q - a_{33} & 0 \\ -b_1 & -b_2 & -b_3 & \lambda_q - s \end{pmatrix}$$

is a non-singular M-matrix, which means that

$$\lambda_q - a_{11} + r > 0,$$

$$\begin{vmatrix} \lambda_q - a_{11}+r & -a_{12}-r\xi_2 \\ -a_{21} & \lambda_q - a_{22} \end{vmatrix} > 0,$$

$$\begin{vmatrix} \lambda_q - a_{11}+r & a_{12}+r\xi_2 & -a_{13}-r\xi_3 \\ -a_{21} & \lambda_q - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & \lambda_q - a_{33} \end{vmatrix} > 0,$$

and

$$\begin{vmatrix} \lambda_q - a_{11}-r & -a_{12}-r\xi_2 & -a_{13}-r\xi_3 & -r \\ -a_{21} & \lambda_q - a_{22} & -a_{23} & 0 \\ -a_{31} & -a_{32} & \lambda_q - a_{33} & 0 \\ -b_1 & -b_2 & -b_3 & \lambda_q - s \end{vmatrix} > 0.$$

Therefore, using (17)–(20), we obtain

$$\lambda_q - a_{11} + r > 0, \quad (\lambda_q - a_{11} + r)(\lambda_q - a_{22}) > 0,$$

$$(\lambda_q - a_{11} + r)\big[(\lambda_q - a_{22})(\lambda_q - a_{33}) - a_{23}a_{32}\big] > 0,$$

and

$$(\lambda_q - a_{11} + r)\big((\lambda_q - a_{22})(\lambda_q - a_{33})(\lambda_q - s) - a_{23}a_{32}(\lambda_q - s)\big) > 0,$$

which implies that

$$\lambda_q - a_{11} + r > 0, \quad \lambda_q - a_{22} > 0, \quad (\lambda_q - a_{22})(\lambda_q - a_{33}) > a_{23}a_{32}, \quad \lambda_q - s > 0.$$

From (15), we obtain $(\lambda_q - a_{22})(\lambda_q - a_{33}) > 0$ and, thus,

$$a_{11} - r < \lambda_q, \quad a_{22} < \lambda_q, \quad a_{33} < \lambda_q, \quad s < \lambda_q.$$

It follows from (17) and (19) that

$$a_{22} + r + \frac{\xi_3}{\xi_2} a_{32} = s, \tag{21}$$

whereas (17) and (20) yield that

$$a_{33} + r + \frac{\xi_2}{\xi_3} a_{23} = s \tag{22}$$

Since $s < \lambda_q$, from (21) and (22) we obtain

$$\lambda_q > a_{22} + r + \frac{\xi_3}{\xi_2} a_{32}, \qquad \lambda_q > a_{33} + r + \frac{\xi_2}{\xi_3} a_{23},$$

which, in view of (15), guarantee that

$$\lambda_q > a_{22} + r \qquad \lambda_q > a_{33} + r,$$

i. e.,

$$r < \lambda_q - a_{22}, \quad r < \lambda_q - a_{33}, \quad \text{and} \quad r > a_{11} - \lambda_q.$$

Consequently, we have

Theorem 5. *Assume that* (15) *holds. Then for* $0 \le \xi_3 f_3 \le \xi_2 f_2 \le f_1$, *system* ($S_3$) *satisfies the generalized maximum principle if*

$$a_{11} - \lambda_q < r < \lambda_q - a_{22}, \qquad a_{11} - \lambda_q < r < \lambda_q - a_{33}.$$

For non-cooperative system (S), we set the following conditions:

$$\begin{aligned} a_{1j} &< 0 \quad \text{for} \quad j = 2, 3, \ldots, n, \\ a_{ij} &\ge 0 \quad \text{for} \quad i = 2, 3, \ldots, n, \ j = 1, 2, \ldots, n, \ i \ne j. \end{aligned} \tag{23}$$

Similarly, to construct $(n+1) \times (n+1)$ cooperative system, we introduce the following equation of a new variable $y_{n+1} = y_1 - \sum_{i=2}^{n} \xi_i y_i$, where $\xi_2, \ldots, \xi_n$ are positive numbers and thus we have

Theorem 6. *Assume that* (23) *holds. Then for* $0 \le \xi_n f_n \le \cdots \le \xi_2 f_2 \le f_1$, *system* (S) *satisfies the generalized maximum principle if*

$$a_{11} - \lambda_q < r < \lambda_q - a_{ii} \quad \text{for all} \quad i = 2, 3, \ldots, n.$$

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