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FINE LIMITS OF GENERALIZED POTENTIAL-TYPE INTEGRAL OPERATORS WITH NON-ISOTROPIC KERNEL

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Abstract. This paper deals with the fine limits of generalized potential-type operators with non-isotropic kernels defined for functions on $\mathbb{R}^n$ satisfying appropriate conditions.

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1. INTRODUCTION

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be positive numbers with $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_n$ and $\|x\|_\lambda = (|x_1|^{\frac{1}{\lambda_1}} + \dots + |x_n|^{\frac{1}{\lambda_n}})^{\frac{|\lambda|}{n}}$, $x \in \mathbb{R}^n$. The expression $\|x-y\|_\lambda$, where $x,\ y \in \mathbb{R}^n$, is called the λ-distance or non-isotropic distance between x and y. This distance is an important concept in the theory of partial differential equations and imbedding theorems. Some problems with the λ-distance were examined in [6, 7].

It can be seen that λ-distance becomes the ordinary Euclidean distance $|x-y|$ for $\lambda_j = \frac{1}{2}$, $j = 1, 2, \dots, n$. The λ-distance has the following properties.

Using the inequality $(a+b)^m \le 2^m (a^m + b^m)$, $m > 1$, we obtain

$$\|x-y\|_\lambda \le M_\lambda (\|x\|_\lambda + \|y\|_\lambda), \tag{1.1}$$

where $M_\lambda = 2^{\left(1+\frac{1}{\lambda_{\min}}\right)\frac{|\lambda|}{n}}$ and $\lambda_{\min} = \min(\lambda_1, \lambda_2, \dots \lambda_n)$.

Several authors have investigated the properties of classical Riesz potentials and their generalizations. For example, taking some appropriate conditions on the kernel depending on Euclidean distance type of $K(|x-y|)$, Gadjiev [3] proved a variant of the Hardy–Littlewood–Sobolev theorem. He also gave the properties of convergence almost everywhere. In [1], a theorem similar to results of [3] was proved for potential-type integrals with kernel depending on the λ-distance.

Some results on potential-type integral operators and Riesz potentials given by generalized shift operators can be found in [2, 4, 5]. Various generalizations of the Riesz potentials are given in [10].

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A potential-type integral operator depending on the λ-distance and defined for non-negative measurable functions f on $\mathbb{R}^n$ is given by the equality

$$(Lf)(x) = \int_{\mathbb{R}^n} K(\|x-y\|_\lambda)\, f(y)dy,$$

where K is the kernel function satisfying the following conditions (see [1]):

(K_1) *K is a non-negative continuous and decreasing function on semiaxis* $[0,\infty)$ *and* $\lim_{t\to 0} K(t) = \infty$*;*

(K_2) $L(r) = \int_r^a K\big(2\beta M_\lambda t^{\frac{2|\lambda|}{n}}\big) t^{2|\lambda|-\delta-1} dt < \infty$ *for* $0 < \delta < 2|\lambda|$, $\beta \in (0,1)$ *and* $0 \le r < a$.

We know that $(Lf)(x) \neq \infty$ if and only if

$$\int_{\mathbb{R}^n} K(\beta(1+\|y\|_\lambda))\, f(y)dy, \tag{1.2}$$

where $\beta \in (0,1)$. Hence it is seen that $(Lf)(x) \neq \infty$ when f is integrable on $\mathbb{R}^n$. Note that (1) is equivalent to

$$\int_{\mathbb{R}^n - B_\lambda(x,1)} K(\beta\|x-y\|_\lambda)\, f(y)dy$$

for every $x \in \mathbb{R}^n$, and $\beta \in (0,1)$, where $B_\lambda(x,1)$ is λ-ball centered at x with radius 1. That is $B_\lambda(x,1) = \{y \in \mathbb{R}^n : \|x-y\|_\lambda < 1\}$.

In what follows, we investigate the fine limits of generalized potential-type integral operators with non-isotropic kernels Lf at $x_0 \in \mathbb{R}^n$. Our results are generalizations of the corresponding results for classical Riesz potentials given in [9, 11].

To obtain a general result, we assume the condition

$$\int_{\mathbb{R}^n} \phi_p(f(y))\, w\Big(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\Big) dy < \infty. \tag{1.3}$$

where $x_0 \in \mathbb{R}^n$ and $\phi_p(r)$ is positive monotone function on interval $(0,\infty)$ having the following properties:

(φ_1) *$\phi_p(r)$ is of the form $r^p\varphi(r)$, where $1 \le p < \infty$ and φ is a positive non-decreasing function on interval* $(0,\infty)$.

(φ_2) *There exists A_1 such that $\varphi(2r) \le A_1\varphi(r)$ whenever $r > 0$.*

Throughout this paper, let $w(r)$ be a positive non-increasing function on $(0,\infty)$ satisfying the condition:

(w_1) *There exists $A_2 > 0$ such that $A_2^{-1}w(r) \le w(2r) \le A_2 w(r)$ whenever $r > 0$.*

In this paper we will use some ideas from [9, 11]. By the symbol M, we denote a positive constant whose value may change depending on the context.

2. Preliminary lemmas

First we collect properties which follow from conditions (φ_1) and (φ_2).

Lemma 2.1. *The function φ satisfies the doubling condition, that is, there exists $A_3 > 1$ such that*

$$\varphi(r) \le \varphi(2r) \le A_3\varphi(r) \quad \textit{for } r > 0.$$

Lemma 2.2. *For any $\gamma > 0$, there exists $A_4(\gamma) > 1$ such that*

$$A_4^{-1}(\gamma)\varphi(r) \le \varphi(r^\gamma) \le A_4(\gamma)\varphi(r), \textit{ whenever } r > 0.$$

3. The estimate of Lf

We write $(Lf)(x) = L_1(x) + L_2(x) + L_3(x)$ for $x \in \mathbb{R}^n - \{x_0\}$, where

$$L_1(x) = \int_{\mathbb{R}^n - B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda)} K(\|x-y\|_\lambda) f(y)dy,$$
$$L_2(x) = \int_{B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda) - B_\lambda(x, \|x-x_0\|_\lambda/2M_\lambda)} K(\|x-y\|_\lambda) f(y)dy,$$
$$L_3(x) = \int_{B_\lambda(x, \|x-x_0\|_\lambda/2M_\lambda)} K(\|x-y\|_\lambda) f(y)dy.$$

Using (1.1), then for any $x, y \in \mathbb{R}^n$

$$\|x-y\|_\lambda \ge \frac{1}{M_\lambda}\|y-x_0\|_\lambda - \|x-x_0\|_\lambda.$$

It is obvious that, if $y \in \mathbb{R}^n - B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda)$, then $\|x-y\|_\lambda \ge \frac{1}{2M_\lambda}\|y-x_0\|_\lambda$. Taking into account $L_1(x)$, we have the inequality

$$L_1(x) \le M \int_{\mathbb{R}^n - B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda)} K(\beta\|y-x_0\|_\lambda) f(y)dy \tag{3.1}$$

for any $\beta = \frac{1}{2M_\lambda} \in (0,1)$. For $y \in B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda) - B_\lambda(x, \|x-x_0\|_\lambda/2M_\lambda)$, since $\|y-x\|_\lambda \ge \frac{1}{2M_\lambda}\|x-x_0\|_\lambda$, we have similarly

$$L_2(x) \le K(\beta\|x-x_0\|_\lambda) \int_{B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda) - B_\lambda(x, \|x-x_0\|_\lambda/2M_\lambda)} f(y)dy \tag{3.2}$$

for any $\beta = \frac{1}{2M_\lambda} \in (0,1)$.

Let us begin with the Hölder type inequality.

Lemma 3.1. *Let $p>1$, $\delta>0$, and f be a non-negative measurable function on $\mathbb{R}^n$. If $0\le 2M_\lambda\|x-x_0\|_\lambda<2M_\lambda a^{\frac{2|\lambda|}{n}}<1$, then*

$$\begin{aligned}&\int_{\mathbb{R}^n-B_\lambda(x_0,2M_\lambda\|x-x_0\|_\lambda)}K\left(\beta\|y-x_0\|_\lambda\right)f(y)dy\\&\quad\le\int_{\mathbb{R}^n-B_\lambda(x_0,2M_\lambda a^{\frac{2|\lambda|}{n}})}K\left(\beta\|y-x_0\|_\lambda\right)f(y)dy+ML\left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right)\\&\quad+MR_1\left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right)\left(\int_{B_\lambda(x_0,2M_\lambda a^{\frac{2|\lambda|}{n}})}\phi_p(f(y))w\left(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right)dy\right)^{\frac{1}{p}},\end{aligned}$$

where $R_1(r)=\left(\int_r^a K^{p'}\left(2\beta M_\lambda t^{\frac{2|\lambda|}{n}}\right)\left[\varphi(t^{-1})w(t)\right]^{\frac{p'}{p}}t^{2|\lambda|-1}dt\right)^{\frac{1}{p'}}$ if $0<2M_\lambda r^{\frac{2|\lambda|}{n}}<1$ and $R_1(r)=R_1\left((2M_\lambda)^{-\frac{n}{2|\lambda|}}\right)$ in the other cases.

Proof. Without loss of generality we assume that $f=0$ outside of $B_\lambda(x_0,2M_\lambda a^{\frac{2|\lambda|}{n}})$. We have

$$\begin{aligned}&\int_{\mathbb{R}^n-B_\lambda(x_0,2M_\lambda\|x-x_0\|_\lambda)}K\left(\beta\|y-x_0\|_\lambda\right)f(y)dy=\int_{A(y)}K\left(\beta\|y-x_0\|_\lambda\right)f(y)dy\\&\qquad\le\int_{\left\{y\in A(y);\,f(y)>\|y-x_0\|_\lambda^{-\frac{\delta n}{2|\lambda|}}\right\}}K\left(\beta\|y-x_0\|_\lambda\right)f(y)dy\\&\qquad+\int_{\left\{y\in A(y);\,f(y)\le\|y-x_0\|_\lambda^{-\frac{\delta n}{2|\lambda|}}\right\}}K\left(\beta\|y-x_0\|_\lambda\right)f(y)dy=:L_{11}+L_{12},\end{aligned}$$

where $A(y)=B_\lambda(x_0,2M_\lambda a^{\frac{2|\lambda|}{n}})-B_\lambda(x_0,2M_\lambda\|x-x_0\|_\lambda)$. Consider the integral L_{11}. From Hölder's inequality, we obtain

$$\begin{aligned}L_{11}(x)\le&\left(\int_{U(y)}f^p(y)\varphi(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})dy\right)^{\frac{1}{p}}\\&\times\left(\int_{U(y)}K\left(\beta\|y-x_0\|_\lambda\right)^{p'}\left[\varphi(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})\right]^{-\frac{P'}{p}}dy\right)^{\frac{1}{p'}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$ and $U\left(y\right)=\left\{y\in A(y);\ f(y)>\|y-x_0\|_\lambda^{-\frac{\delta n}{2|\lambda|}}\right\}$.

Since φ is a non-decreasing function, we have $\varphi(f(y)) \geq \varphi(\|y-x_0\|_\lambda^{-\frac{\delta n}{2|\lambda|}})$ and therefore, Lemma 2.2 implies $\varphi(\|y-x_0\|_\lambda^{-\frac{\delta n}{2|\lambda|}}) \geq M\varphi(\|y-x_0\|_\lambda^{-\frac{n}{2|\lambda|}})$. Thus,

$$
\begin{aligned}
L_{11}(x) \leq M &\left(\int_{U(y)} \phi_p(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})dy\right)^{\frac{1}{p}} \\
&\times \left(\int_{U(y)} K(\beta\|y-x_0\|_\lambda)^{p'}\left[\varphi(\|y-x_0\|_\lambda^{-\frac{n}{2|\lambda|}})w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})\right]^{-\frac{P'}{p}} dy\right)^{\frac{1}{p'}}.
\end{aligned}
\tag{3.3}
$$

The right hand side integral with respect to y may be easily calculated. Namely, passing to generalized spherical coordinates by transformation

$$
\begin{aligned}
y_1 &= x_{01} + (t\cos\theta_1)^{2\lambda_1}, \\
y_2 &= x_{02} + (t\sin\theta_1\cos\theta_2)^{2\lambda_2}, \\
&\vdots \\
y_n &= x_{0n} + (t\sin\theta_1\sin\theta_2\ldots\sin\theta_{n-1})^{2\lambda_n},
\end{aligned}
$$

where θ_j, $j = 1,2,\ldots,n$, are the coordinates of the point θ on unit sphere. We can see that the Jacobian of this transformation $t^{2|\lambda|-1}\Omega_\lambda(\theta)$, where $\Omega_\lambda(\theta)$ depends on angles $\theta_1,\theta_2,\ldots,\theta_{n-1}$ only $0 \leq \theta_1,\ldots,\theta_{n-2} \leq \pi$, $0 \leq \theta_{n-1} \leq 2\pi$ and

$$
\Omega_\lambda(\theta) = 2^n \prod_{j=1}^{n-1} \left(\cos\theta_j\right)^{2\lambda_j-1} \left(\sin\theta_j\right)^{2|\lambda|-\sum_{k=1}^{j}\lambda_k-1}.
$$

Here the integral $\int_{S^{n-1}} \Omega_\lambda(\theta)d\theta$ is finite, where S^{n-1} is the unit ball in $\mathbb{R}^n$. Consequently, from (3.3) we have

$$
\begin{aligned}
L_{11}(x) \leq M &\left(\int_{(2M_\lambda)^{\frac{n}{2|\lambda|}}\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}}^{(2M_\lambda)^{\frac{n}{2|\lambda|}}a} K^{p'}\left(\beta t^{\frac{2|\lambda|}{n}}\right)\left[\varphi(t^{-1})w(t)\right]^{-\frac{p'}{p}} t^{2|\lambda|-1}dt\right)^{\frac{1}{p'}} \\
&\times \left(\int_{U(y)} \phi_p(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})dy\right)^{\frac{1}{p}}.
\end{aligned}
\tag{3.4}
$$

Let us now consider the integral L_{12}. By passing to generalized spherical coordinates, we get

$$L_{12}(x) \le \int_{A(y)} K\left(\|y-x_0\|_\lambda\right) \|y-x_0\|_\lambda^{-\frac{n}{2|\lambda|}\delta} dy$$
$$= ML\left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right), \tag{3.5}$$

where $L(r)$ is defined in the condition (K_2). Relations (3.4) and (3.5) give the desired conclusion. □

Lemma 3.2. *Let f be a non-negative measurable function on $\mathbb{R}^n$. If $0 < 2M_\lambda \|x - x_0\|_\lambda < 1$ and $0 < \delta < 2|\lambda|$, then there exists a positive M such that*

$$L_2(x) \le MR_2(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}) \left(\int_{B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda)} \phi_p(f(y)) w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}}) dy\right)^{\frac{1}{p}}$$
$$+ M\|x-x_0\|_\lambda^{2|\lambda|-\delta},$$

where

$$R_2(r) = K\left(\beta t^{\frac{2|\lambda|}{n}}\right) r^{\frac{2|\lambda|}{n}\frac{2|\lambda|}{p'}} \left[\varphi\left(r^{-\frac{2|\lambda|}{n}}\right) w(r)\right]^{-\frac{1}{p}}.$$

Proof. It follows from (3.2) that

$$L_2(x) \le K\left(\beta\|x-x_0\|_\lambda\right) \int_{B(x)} f(y) dy$$
$$\le K\left(\beta\|x-x_0\|_\lambda\right) \left\{\int_{\{y\in B(x);\, f(y) > \|x-x_0\|_\lambda^{-\delta}\}} f(y) dy\right.$$
$$\left. + \int_{\{y\in B(x);\, f(y)\le \|x-x_0\|_\lambda^{-\delta}\}} f(y) dy\right\} =: L_{21}(x) + L_{22}(x),$$

where $B(x) = B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda) - B_\lambda(x, \|x-x_0\|_\lambda / 2M_\lambda)$.

Let us first consider L_{21}. Since φ is a non-decreasing function, by Lemma 3.1, we get

$$L_{21}(x) \le M\left[\varphi\left(\|x-x_0\|_\lambda^{-1}\right)\right]^{-\frac{1}{p}} K\left(\beta\|x-x_0\|_\lambda\right) \int_{B(x)} f(y) \left[\varphi(f(y))\right]^{\frac{1}{p}} dy.$$

From Hölder's inequality, we obtain

$$L_{21}(x) \le M\left[\varphi\left(\|x-x_0\|_\lambda^{-1}\right)\right]^{-\frac{1}{p}} K\left(\beta\|x-x_0\|_\lambda\right)$$
$$\times \left(\int_{B_\lambda(x_0, 2M_\lambda\|x-x_0\|_\lambda)} dy\right)^{\frac{1}{p'}} \left(\int_{B(x)} f(y)^p \varphi(f(y)) dy\right)^{\frac{1}{p}},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Therefore, because w is a non-increasing function, it follows that

$$L_{21}(x) \leq M\left[\varphi\left(\|x-x_0\|_\lambda^{-1}\right)\right]^{-\frac{1}{p}} K\left(\beta\|x-x_0\|_\lambda\right)\|x-x_0\|_\lambda^{\frac{2|\lambda|}{P'}} \times\left(\int_{B(x)}\phi_p(f(y))dy\right)^{\frac{1}{p}} \tag{3.6}$$

$$\leq M\left(\varphi\left(\|x-x_0\|_\lambda^{-1}\right)w(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}})\right)^{-\frac{1}{p}} K\left(\beta\|x-x_0\|_\lambda\right)\|x-x_0\|_\lambda^{\frac{2|\lambda|}{P'}} \times\left(\int_{B_\lambda(x_0,2M_\lambda\|x-x_0\|_\lambda)}\phi_p(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})dy\right)^{\frac{1}{p}}. \tag{3.7}$$

On the other hand, we have

$$L_{22} \leq K\left(\beta\|x-x_0\|_\lambda\right)\int_{B_\lambda(x_0,2M_\lambda\|x-x_0\|_\lambda)}\|x-x_0\|_\lambda^{-\delta}dy \leq MK\left(\beta\|x-x_0\|_\lambda\right)\|x-x_0\|_\lambda^{2|\lambda|-\delta}. \tag{3.8}$$

We have the desired conclusion from (3.7) and (3.8). □

Lemma 3.3. *Let f be a non-negative measurable function on $\mathbb{R}^n$. If $\delta>0$, then there exists a positive M such that*

$$L_3(x) \leq MR_3\left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right)\left(\int_{B_\lambda(x,\|x-x_0\|_\lambda/2M_\lambda)}\phi_p(f(y))w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}})dy\right)^{\frac{1}{p}} + M\int_0^{(\|x-x_0\|_\lambda/2M_\lambda)^{\frac{n}{2|\lambda|}}} K^{p'}\left(2\beta M_\lambda t^{\frac{2|\lambda|}{n}}\right)t^{2|\lambda|-\delta-1}dt$$

where $R_3(r)=\varphi^*(r)\omega(r)^{-\frac{1}{p}}$ *and* $\varphi^*(r)=\left(\int_0^r K^{p'}(t^{\frac{2|\lambda|}{n}})\left[\varphi(t^{-1})\right]^{-\frac{p'}{p}}t^{2|\lambda|-1}dt\right)^{\frac{1}{p'}}$.

Proof. By change of variable, we have

$$L_3(x)=\int_{B_\lambda(0,\|x-x_0\|_\lambda/2M_\lambda)}K\left(\|y\|_\lambda\right)f(x+y)dy.$$

In a way similar to the proof of Lemmas 3.1 and 3.2, we obtain

$$
\begin{aligned}
L_3(x) \le M & \left(\int_0^{(\|x-x_0\|_\lambda/2M_\lambda)^{\frac{n}{2|\lambda|}}} K^{p'}\left(t^{\frac{2|\lambda|}{n}}\right) \left[\varphi(t^{-1})\right]^{-\frac{p'}{p}} t^{2|\lambda|-1} dt \right)^{\frac{1}{p'}} \\
& \times \left(\int_{B_\lambda(0,\|x-x_0\|_\lambda/2M_\lambda)} \phi_p(f(x+y)) dy \right)^{\frac{1}{p}} \\
& + M \int_0^{(\|x-x_0\|_\lambda/2M_\lambda)^{\frac{n}{2|\lambda|}}} K\left(\beta 2M_\lambda t^{\frac{2|\lambda|}{n}}\right) t^{2|\lambda|-\delta-1} dt \\
\le M\varphi^* & \left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}} \right) w\left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}} \right)^{-\frac{1}{p}} \\
& \times \left(\int_{B_\lambda(x,\|x-x_0\|_\lambda/2M_\lambda)} \phi_p(f(y)) w(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}}) dy \right)^{\frac{1}{p}} \\
& + \int_0^{(\|x-x_0\|_\lambda/2M_\lambda)^{\frac{n}{2|\lambda|}}} K\left(\beta 2M_\lambda t^{\frac{2|\lambda|}{n}}\right) t^{2|\lambda|-\delta-1} dt,
\end{aligned}
$$

as required. □

4. Fine limit of $R_\alpha f$

We consider the function

$$
\begin{aligned}
R(r) &= R_1(r) + R_2(r) + R_3(r) \\
&= R_1(r) + K\left(\beta t^{\frac{2|\lambda|}{n}}\right) r^{\frac{2|\lambda|}{n}\frac{2|\lambda|}{p'}} \left(w(r)\varphi(r^{-\frac{2|\lambda|}{n}}) \right)^{-\frac{1}{p}} + \varphi^*(r) w(r)^{-\frac{1}{p}}.
\end{aligned}
$$

Theorem 4.1. *Let $p > 1$ and f be a non-negative measurable function on $\mathbb{R}^n$ satisfying conditions (1.2) and (1.3). If $\varphi^*(1) < \infty$ and $\lim_{r\to 0} R(r) = \infty$, then*

$$
\lim_{x\to x_0} \left[R\left(\|x-x_0\|_\lambda\right)\right]^{-1} (Lf)(x) = 0.
$$

If $R(r)$ is bounded, then $(Lf)(x_0)$ is finite and $(Lf)(x)$ tends to $(Lf)(x_0)$ as $x \to x_0$.

Proof. By condition (1.2), the integral

$$
\int_{\mathbb{R}^n - B_\lambda(x_0, 2M_\lambda a^{\frac{2|\lambda|}{n}})} K\left(\beta \|y-x_0\|_\lambda\right) f(y) dy
$$

is finite. It follows from (3.1), the condition K_2 and Lemma 3.1 that

$$\limsup_{x\to x_0}\left(R\left(\|x-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right)\right)^{-1}L_1(x)$$
$$\leq M\left(\int_{B_\lambda(x_0,2M_\lambda a^{\frac{2|\lambda|}{n}})}\phi_p(f(y))w\left(\|y-x_0\|_\lambda^{\frac{n}{2|\lambda|}}\right)dy\right)^{\frac{1}{p}}.$$

Since a is arbitrary, we see that the integral in the left-hand side of the last estimate is equal to zero.

In view of Lemmas 3.2 and 3.3 and condition (1.3), we have

$$\lim_{x\to x_0}\left[R\left(\|x-x_0\|_\lambda\right)\right]^{-1}\left(L_2(x)+L_3(x)\right)=0.$$

If we combine these results, we have

$$\lim_{x\to x_0}\left[R\left(\|x-x_0\|_\lambda\right)\right]^{-1}(Lf)(x)=0.$$

If $R(r)$ is bounded, then Lemmas 3.2 and 3.3 imply that $L_2(x)+L_3(x)$ tends to zero at $x\to x_0$. Furthermore, in view of Lemma 3.1, we have $\limsup_{x\to x_0}L_1(x)<\infty$. Thus it follows that $(Lf)(x_0)$ is finite. Hence

$$L_1(x)+L_2(x)=\int_{\mathbb{R}^n-B_\lambda(x,\|x-x_0\|_\lambda/2M_\lambda)}K\left(\|x-y\|_\lambda\right)f(y)dy.$$

Since $\|y-x_0\|_\lambda\leq 2M_\lambda^2\|y-x\|_\lambda$ for $y\in\mathbb{R}^n-B_\lambda(x,\|x-x_0\|_\lambda/2M_\lambda)$, we have by Lebesgue's dominated convergence theorem

$$\lim_{x\to x_0}\left(L_1(x)+L_2(x)\right)=(Lf)(x_0).$$

However, we also know that $\lim_{x\to x_0}L_3(x)=0$. The proof of Theorem 4.1 is thus complete. □

Corollary 4.1 ([8,9]). *Let $p=\frac{n}{\alpha}$ and $\varphi^*(1)<\infty$. If f is a non-negative measurable function on $\mathbb{R}^n$ satisfying (1.2) and the condition*

$$\int_{\mathbb{R}^n}\phi_p(f(y))dy<\infty,$$

then $L_\alpha f$ is continuous on $\mathbb{R}^n$ with $K(t)=t^{\alpha-n}$, $0<\alpha<n$, and $\lambda_k=\frac{1}{2}$, $k=1,2,\ldots,n$.

Corollary 4.2 ([1]). *Let f be a non-negative measurable function satisfying conditions (1.2) and the condition*

$$\int_{\mathbb{R}^n}\phi_p(f(y))dy<\infty,$$

then $L_\alpha f$ is continuous on $\mathbb{R}^n$.

Proposition 4.1. *Let $ap = n$, $\varphi^*(1) < \infty$, $x_0 = 0$, $K(t) = t^{\alpha-n}$, and*

$$\lim_{r\to 0} r^{\frac{2|\lambda|}{p'}} (w(r))^{-\frac{1}{p}} \left(\varphi(r^{-1})\right)^{-\frac{1}{p}} = 0.$$

Then for any positive non-decreasing function $a(r)$ on $(0,\infty)$ such that

$$\lim_{r\to 0} a(r) = \infty,$$

there exists a non-negative measurable function f satisfying (1.2) and (1.3) such that

$$\limsup_{x\to x_0} a\left(\|x\|_\lambda^{\frac{n}{2|\lambda|}}\right)\left(w\left(\|x\|_\lambda^{\frac{n}{2|\lambda|}}\right)\varphi\left(\|x\|_\lambda^{-\frac{n}{2|\lambda|}}\right)\right)^{-\frac{1}{p}} R_\alpha f(x) = \infty,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. Let (j_i) be a sequence of positive integers such that $j_i + 2 < J_{i+1}$ and $\sum_i a_i^{-\frac{1}{p}} < \infty$, where $a_i(r_j) = a_i$ and $r_j = 2^{-j_i}$. We set

$$f(y) = a_i^{-\frac{1}{p}} \left(\varphi(r_j^{-1})\right)^{\frac{1}{p'}} (w(r_j))^{-\frac{1}{p}} \|x_i - y\|_\lambda^{-\alpha} \left[\varphi(\|x_i - y\|_\lambda^{-1})\right]^{-1}$$

if $y \in \bigcup_{i=1}^{\infty} B_\lambda(x_i, (2r_j)^{\frac{2|\lambda|}{n}}) - B_\lambda(x_i, (r_j)^{\frac{2|\lambda|}{n}})$, otherwise $f(y) = 0$, where $x_i = (r_j, 0, \dots, 0) \in \mathbb{R}^n$.

Let us now show that f meets all the conditions in the proposition. If we use Lemmas 2.1 and 2.2, then we have

$$\begin{aligned}\int f(y)dy &= \sum_i a_i^{-\frac{1}{p}} (\varphi(r_j^{-1}))^{\frac{1}{p'}} (w(r_j))^{-\frac{1}{p}} \\ &\quad\times \int_{B_\lambda(x_i,(2r_j)^{\frac{2|\lambda|}{n}})-B_\lambda(x_i,(r_j)^{\frac{2|\lambda|}{n}})} \|x_i - y\|_\lambda^{-\alpha} \left[\varphi(\|x_i - y\|_\lambda^{-1})\right]^{-1} dy \\ &\le M \sum_i a_i^{-\frac{1}{p}} \left(\varphi(r_j^{-1})\right)^{\frac{1}{p'}} (w(r_j))^{-\frac{1}{p}} \int_{r_j}^{2r_j} t^{-\frac{2|\lambda|}{n}\alpha} \left(\varphi(t^{-\frac{2|\lambda|}{n}})\right)^{-1} t^{2|\lambda|-1} dt \\ &\le M \sum_i a_i^{-\frac{1}{p}} \left\{ r_j^{\frac{2|\lambda|}{p'}} \left(\varphi(r_j^{-1})\right)^{\frac{1}{p}} (w(r_j))^{-\frac{1}{p}} \right\} \\ &\le M \sum_i a_i^{-\frac{1}{p}} < \infty.\end{aligned}$$

Consequently f satisfies (1.2). On the other hand, since the values $\left(a_i^{-\frac{1}{p}}\right)$ and $\left(r_j^{\frac{2|\lambda|}{p'}}\left(\varphi(r_j^{-1})\right)^{-\frac{1}{p}}\left(w(r_j)\right)^{-\frac{1}{p}}\right)$ are bounded, we have

$$\begin{aligned}
f(y) &\le M\left(\varphi(r_j^{-1})\right)^{\frac{1}{p'}}\left(w(r_j)\right)^{-\frac{1}{p}}\|x_i-y\|_\lambda^{-\alpha}\left[\varphi(\|x_i-y\|_\lambda)\right]^{-1} \\
&\le M\left(\varphi(r_j^{-1})\right)^{-\frac{1}{p}}\left\{r_j^{\frac{2|\lambda|}{p'}}\varphi(r_j^{-1})^{-\frac{1}{p}}\|x_i-y\|_\lambda^{-\alpha}\right\}^{-1} \\
&\le M\|x_i-y\|_\lambda^{-\alpha-\frac{n}{p'}}.
\end{aligned}$$

Thus, the inequality $\varphi(f(y)) \le \varphi(\|x_i-y\|_\lambda^{-1})$ holds.

Now we show that f satisfies (1.3). Using condition (w_1), we get

$$\begin{aligned}
&\int_{\mathbb{R}^n}\phi_p(f(y))w\left(\|y\|_\lambda^{\frac{n}{2|\lambda|}}\right)dy \\
&\quad\le \sum_i a_i^{-1}\left(\varphi(r_j^{-1})\right)^{\frac{p}{p'}}\int_{B_\lambda(x_i,(2r_j)^{\frac{2|\lambda|}{n}})-B_\lambda(x_i,(r_j)^{\frac{2|\lambda|}{n}})}\|x_i-y\|_\lambda^{-\alpha p} \\
&\qquad\times\left[\varphi(\|x_i-y\|_\lambda^{-1})\right]^{-\frac{p}{p'}}dy \\
&\quad\le M\sum_i a_i^{-1}\left(\varphi(r_j^{-1})\right)^{\frac{p}{p'}}\int_{r_j}^{2r_j}t^{-\frac{2|\lambda|}{n}\alpha p}\left(\varphi(t^{-\frac{2|\lambda|}{n}})\right)^{-\frac{p}{p'}}t^{2|\lambda|-1}dt \\
&\quad\le M\sum_i a_i^{-1}\left(\varphi(r_j^{-1})\right)^{\frac{p}{p'}}\int_{r_j}^{2r_j}\left(\varphi(t^{-1})\right)^{-\frac{p}{p'}}t^{-1}dt \\
&\quad\le M\sum_i a_i^{-1}<\infty.
\end{aligned}$$

Finally,

$$\begin{aligned}
R_\alpha f(x_i) &\ge a_i^{-\frac{1}{p}}\left(\varphi(r_j^{-1})\right)^{\frac{1}{p'}}\left(w(r_j)\right)^{-\frac{1}{p}} \\
&\qquad\times\int_{B_\lambda(x_i,(2r_j)^{\frac{2|\lambda|}{n}})-B_\lambda(x_i,(r_j)^{\frac{2|\lambda|}{n}})}\|x_i-y\|_\lambda^{-n}\left[\varphi(\|x_i-y\|_\lambda^{-1})\right]^{-1}dy \\
&\ge Ma_i^{-\frac{1}{p}}\left(\varphi(r_j^{-1})\right)^{\frac{1}{p'}}\left(w(r_j)\right)^{-\frac{1}{p}}\int_{r_j}^{2r_j}\left(\varphi(t^{-1})\right)^{-1}t^{-1}dt \\
&\ge Ma_i^{-\frac{1}{p}}\left(\varphi(r_j^{-1})\right)^{-\frac{1}{p}}\left(w(r_j)\right)^{-\frac{1}{p}}.
\end{aligned}$$

Thus we have

$$a_i\left(\varphi(r_j^{-1})\right)^{\frac{1}{p}}\left(w(r_j)\right)^{\frac{1}{p}}R_\alpha f(x^{(i)}) \ge Ma_i^{\frac{1}{p}}.$$

This proves the proposition for $j \to \infty$. □

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