# STABILITY OF TRIVIAL TORUS OF NONLINEAR MULTIFREQUENCY SYSTEMS VIA SIGN-DEFINITE QUADRATIC FORMS

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*Abstract.* This paper studies stability properties of the trivial invariant torus of a class of nonlinear extensions of dynamical systems on torus. Two theorems on exponential stability and instability of the invariant torus in terms of quadratic forms that are sign-definite in nonwandering set of the dynamical system have been proven.

## 1. INTRODUCTION

Fundamental results of the qualitative theory of multifrequency oscillations, in particular, the problems of the existence and stability of invariant sets of dynamical systems defined in the direct product of $m$-dimensional torus and $n$-dimensional Euclidean space, have been developed by A. M. Samoilenko and summarized in [12]. In [6], the stability properties of invariant tori have been studied in terms of sign-definite quadratic forms. In this paper, we establish new conditions for exponential stability and instability of the trivial invariant torus of nonlinear extension of dynamical system on torus which are formulated in terms of quadratic forms that are sign-definite not on the entire surface of the torus, but in nonwandering set [7] of dynamical system on torus only. The corresponding results for linear extensions of dynamical systems on torus have been obtained in [1, 3, 8–11].

## 2. MOTIVATION AND PROBLEM STATEMENT

We consider a system of ordinary differential equations defined in the direct product of $m$-dimensional torus $\mathcal{T}_m$ and $n$-dimensional Euclidean space $\mathbb{R}^n$

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = P(\varphi, x)x, \qquad (2.1)$$

where $\varphi = (\varphi_1, \ldots, \varphi_m)^T \in \mathcal{T}_m$, $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$, function $P$ is continuous in $\mathcal{T}_m \times \mathbb{R}^n$ and for every $x \in \mathbb{R}^n$ $P(\cdot, x)$, $a(\cdot) \in C(\mathcal{T}_m)$; $C(\mathcal{T}_m)$ is a space of continuous $2\pi$-periodic with respect to each of the component $\varphi_v$, $v = 1, \ldots, m$ functions defined on $\mathcal{T}_m$.

We assume that the following conditions hold:

$$\exists\, M > 0 \text{ such that } \forall\, (\varphi, x) \in \mathcal{T}_m \times \mathbb{R}^n \quad \|P(\varphi, x)\| \leq M; \tag{2.2}$$

$$\forall\, r > 0\; \exists\, L = L(r) > 0 \text{ such that } \forall\, x', x'',\, \|x'\| \leq r, \|x''\| \leq r,\, \forall\, \varphi \in \mathcal{T}_m$$
$$\left\| P(\varphi, x'') - P(\varphi, x') \right\| \leq L \left\| x'' - x' \right\|; \tag{2.3}$$

$$\exists\, A > 0 \;\; \forall\, \varphi', \varphi'' \in \mathcal{T}_m \;\; \left\| a(\varphi'') - a(\varphi') \right\| \leq A \left\| \varphi'' - \varphi' \right\|. \tag{2.4}$$

Condition (2.4) guarantees that the system

$$\frac{d\varphi}{dt} = a(\varphi) \tag{2.5}$$

generates a dynamical system on $\mathcal{T}_m$, which we will denote by $\varphi_t(\varphi)$.

**Definition 1** ([7]). A point $\varphi \in \mathcal{T}_m$ is called a nonwandering point of dynamical system (2.5) if there exist a neighbourhood $U(\varphi)$ and a moment of time $T = T(\varphi) > 0$ such that

$$U(\varphi) \cap \varphi_t(U(\varphi)) = \varnothing \quad \forall\, t \geq T.$$

Let us denote by $\Omega$ a set of all nonwandering points of dynamical system (2.5). Since $\mathcal{T}_m$ is a compact set, the set $\Omega$ is nonempty, invariant, and compact subset of $\mathcal{T}_m$ [12]. Additionally, the following holds:

**Lemma 1** ([7]). *For any $\varepsilon > 0$ there exist $T(\varepsilon) > 0$ and $N(\varepsilon) > 0$ such that for any $\varphi \notin \Omega$ the corresponding trajectory $\varphi_t(\varphi)$ spends only a finite time that is bounded by $T(\varepsilon)$ outside the $\varepsilon$-neighbourhood of the set $\Omega$, and leaves this set not more than $N(\varepsilon)$ times.*

**Definition 2** ([12]). Trivial invariant torus

$$x = 0, \quad \varphi \in \mathcal{T}_m$$

of the system (2.1) is called exponentially stable if there exist constants $K > 0$, $\gamma > 0$, and $\delta > 0$ such that for all $\varphi \in \mathcal{T}_m$ and for all $x^0 \in \mathbb{R}^n$, $\|x^0\| \leq \delta$ it holds that

$$\forall\, t \geq 0 \quad \|x(t, \varphi, x^0)\| \leq K \|x^0\| e^{-\gamma t}, \tag{2.6}$$

where $x(t, \varphi, x^0)$ is a solution to the Cauchy problem

$$\frac{dx}{dt} = P(\varphi_t(\varphi), x)x, \quad x(0) = x^0. \tag{2.7}$$

In [5], the conditions for the exponential stability of the trivial invariant torus of the system (2.1) have been established in terms of the properties of function $\varphi \mapsto P(\varphi, 0)$ in the nonwandering set $\Omega$ of dynamical system (2.5):

**Lemma 2** ([5]). *Let*

$$\forall\, \varphi \in \Omega \quad \lambda(\varphi, 0) < 0, \tag{2.8}$$

*where $\lambda(\varphi, x)$ is the largest eigenvalue of the matrix*

$$\hat{P}(\varphi, x) = \frac{1}{2}\left(P(\varphi, x) + P^T(\varphi, x)\right).$$

*Then, the trivial invariant torus of system (2.1) is exponentially stable.*

The following example demonstrates the case when the trivial invariant torus is exponentially stable (this will be proven in Theorem 1), however the condition (2.8) does not hold.

*Example* 1. Consider a system defined in $\mathcal{T}_1 \times \mathbb{R}^2$

$$\frac{d\varphi}{dt} = -\sin^2\left(\frac{\varphi}{2}\right), \tag{2.9}$$

$$\begin{pmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{pmatrix} = \begin{pmatrix} \sin(\varphi + x_1 + x_2)x_1 & -x_2 \\ x_1 & -\sin(x_1 - x_2 - \varphi)x_2 \end{pmatrix}. \tag{2.10}$$

Dynamical system on torus $\mathcal{T}_1$ that are generated by (2.9) has a nonwandering set

$$\Omega = \{\varphi = 0\}.$$

However, the matrix

$$\hat{P}(0, \bar{0}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

does not fulfill condition (2.8).

The main results of this paper are theorems on exponential stability and instability of the trivial torus of the systems of a class (2.9) with conditions given in terms of quadratic forms that are sign-definite in nonwandering set $\Omega$ of dynamical system on torus only.

## 3. Main results

For any $\varphi \in \mathcal{T}_m$, $x \in \mathbb{R}^n$ let us denote

$$\begin{aligned} \hat{S}(\varphi, x) &= \frac{\partial S(\varphi, x)}{\partial \varphi} a(\varphi) + \frac{\partial S(\varphi, x)}{\partial x}\left(P(\varphi, x)x\right) \\ &\quad + S(\varphi, x)P(\varphi, x) + P^T(\varphi, x)S(\varphi, x), \end{aligned} \tag{3.1}$$

where $S = S(\varphi, x)$ is a symmetric matrix of a class $C^1(\mathcal{T}_m \times \mathbb{R}^n)$.

**Theorem 1.** *Let there exist a symmetric matrix $S = S(\varphi, x)$ of the class $C^1(\mathcal{T}_m \times \mathbb{R}^n)$ such that*

$$\forall\, \varphi \in \Omega \quad S(\varphi, 0) > 0, \quad \hat{S}(\varphi, 0) < 0. \tag{3.2}$$

*Then, the trivial torus of system (2.1) is exponentially stable.*

*Proof.* Due to the conditions (3.2) and continuous dependence of the polynomialFLs roots on the coefficients [4], we get that for some $r > 0$, $\gamma > 0$ the following inequalities hold

$$\forall \, \varphi \in O_r(\Omega) \ \ \forall \, x \in \mathbb{R}^n, \|x\| < r \quad S(\varphi, x) \geq \gamma E, \quad \hat{S}(\varphi, x) \leq -\gamma E. \qquad (3.3)$$

Also, there exists a constant $C = C(r) > 0$ such that for all $\varphi \in T_m$ and for all $x \in \mathbb{R}^n$, $\|x\| \leq r$

$$\|S(\varphi, x)\| + \|\hat{S}(\varphi, x)\| \leq C. \qquad (3.4)$$

Denote by

$$V(\varphi, x) = (S(\varphi, x)x, x). \qquad (3.5)$$

Let

$$\varphi \in O_r(\Omega) \quad \text{and} \quad \forall \, s \geq 0 \quad \varphi_s(\varphi) \in O_r(\Omega).$$

Then, for the solution to (2.7) $x(t) = x(t, \varphi, x^0)$ with $\|x^0\| < r$ the following estimate $\|x(t)\| < r$ holds for $t \in [0, T)$, $0 < T \leq +\infty$. Hence, from (3.3),

$$\gamma \|x(t)\|^2 \leq V(\varphi_t(\varphi), x(t)) \leq C\|x(t)\|^2,$$

$$\frac{d}{dt} V(\varphi_t(\varphi), x(t)) \leq -\gamma \|x(t)\|^2$$

hold for $t \in [0, T)$. From the last inequalities we get that

$$V(\varphi_t(\varphi), x(t)) \leq V(\varphi, x^0) e^{-\frac{\gamma}{C}t}.$$

Hence, there exist constants $K_1 > 0$ and $\gamma_1 > 0$ such that $\forall \, t \in [0, T)$

$$\|x(t)\| \leq K_1 \|x^0\| e^{-\gamma_1 t}. \qquad (3.6)$$

For $\|x^0\| < \frac{r}{K_1}$ we obtain that $T = +\infty$ and the inequality (3.6) holds for all $t \geq 0$.

Now, let $\varphi \in O_r(\Omega)$, but there exists $t_1 > 0$ such that

$$\forall \, t \in [0, t_1) \quad \varphi_t(\varphi) \in O_r(\Omega), \ \varphi_{t_1}(\varphi) \notin O_r(\Omega).$$

From Lemma 1, there exist

$$N(\varphi, r) \leq N(r), \quad \{\tau_i(\varphi, r)\}_{i=1}^{N(\varphi,r)+1}, \quad \{t_i(\varphi, r)\}_{i=1}^{N(\varphi,r)}$$

$$\sum_{i=1}^{N(\varphi,r)+1} \tau_i(\varphi, r) =: T(\varphi, r) \leq T(r)$$

such that

$$\varphi_t(\varphi) \in O_r(\Omega)$$

$$\forall \, t \in (0, t_1) \cup \bigcup_{k=1}^{N(\varphi,r)-1} \left( \sum_{i=1}^{k}(\tau_i + t_i), \sum_{i=1}^{k}(\tau_i + t_{i+1}) \right) \bigcup \left( \sum_{i=1}^{N(\varphi,r)}(\tau_i + t_i), +\infty \right). \qquad (3.7)$$

Then, for $t \in [0, t_1]$

$$\|x(t)\| \leq K_1 \|x^0\| e^{-\gamma_1 t} < r \quad \text{if} \quad \|x^0\| < \frac{r}{K_1}.$$

For $t \in [t_1, t_1 + \tau_1]$, from (2.2) and Wazewski inequality [2],

$$\|x(t)\| \leq \|x(t_1)\| e^{M(t-t_1)}$$
$$\leq K_1 \|x^0\| e^{(\gamma_1 + M)\tau_1} e^{-\gamma_1 t}$$
$$< r \quad \text{if} \quad \|x^0\| < \frac{r}{K_1 e^{(\gamma_1 + M)\tau_1}}.$$

For $t \in [t_1 + \tau_1, t_1 + \tau_1 + t_2]$,

$$\|x(t)\| \leq K_1^2 \|x^0\| e^{(\gamma_1 + M)\tau_1} e^{-\gamma_1 t} < r \quad \text{if} \quad \|x^0\| < \frac{r}{K_1^2 e^{(\gamma_1 + M)\tau_1}}.$$

Continuing this process, due to (3.7), finally we get:

$$\text{for} \quad K := K_1^{N(r)} e^{(\gamma + M)T(r)}, \quad \delta := \frac{r}{K_1^{N(r)} e^{(\gamma + M)T(r)}}, \quad \|x^0\| < \delta,$$

$$\forall \, t \geq 0 \quad \|x(t)\| \leq K \|x^0\| e^{-\gamma_1 t}. \tag{3.8}$$

Now, let us consider the case of $\varphi \notin O_r(\Omega)$. In this case,

$$\forall \, \tau_0 \in (0, T(r)) \quad \varphi_{\tau_0}(\varphi) \in O_r(\Omega)$$

and

$$\forall \, t \in [0, \tau_0] \quad \|x(t)\| \leq \|x^0\| e^{(M+\gamma_1)T(r)} \|e^{-\gamma_1 t}.$$

Then, for

$$\hat{K} = K e^{(\gamma + M)T(r)}, \quad \hat{\delta} = \frac{\delta}{e^{(\gamma + M)T(r)}}$$

we obtain the required estimate (2.6). This completes the proof. $\square$

*Example* 2 (revisited). Let us illustrate the usage of Theorem 1 for the system (2.9), (2.10). Let

$$S = S(\varphi, x) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} > 0$$

Then,

$$\hat{S}(0, \bar{0}) = \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix} < 0$$

which guarantee exponential stability of the trivial invariant torus.

The following theorem provides sufficient conditions for instability of the trivial torus of system (2.1) in terms of sign-definite on the set $\Omega$ quadratic forms.

**Theorem 2.** *Let there exist a symmetric matrix* $S = S(\varphi, x)$ *of the class* $C^1(\mathcal{T}_m \times \mathbb{R}^n)$ *such that for the matrix* (3.1) *and for the quadratic form* (3.5) *the following conditions hold:*

$$\forall \varphi \in \Omega \quad \hat{S}(\varphi, 0) > 0, \tag{3.9}$$

$$\forall \delta > 0 \; \exists x_0 \in \mathbb{R}^n, \; \|x_0\| < \delta, \; \exists \varphi_0 \in \Omega \quad such \; that \quad V(\varphi_0, x_0) > 0. \tag{3.10}$$

*Then, the trivial torus of system* (2.1) *is unstable.*

*Proof.* From (3.9) and (3.3), $\exists \beta > 0$ and $\exists r > 0$ such that

$$\forall \varphi \in \Omega \quad \forall x \in \mathbb{R}^n, \|x\| < r \quad \hat{S}(\varphi, x) \geq \beta E.$$

Let for arbitrary $\delta > 0 \; (\delta < r)$, the corresponding $x_0, \varphi_0$ are from the condition (3.10), and $x(t) = x(t, \varphi_0, x_0)$ is the solution to the Cauchy problem (2.7). Next, we show that $\exists t_1 > 0$ such that $\|x(t_1)\| = r$. This will be sufficient to prove instability.

Suppose the opposite: Let

$$\forall t \geq 0 \quad \|x(t)\| < r.$$

Then, for the function $v(t) = V(\varphi_t(\varphi_0), x(t))$ we have:

$$\frac{d}{dt} v(t) \geq \beta \|x(t)\|^2. \tag{3.11}$$

From the condition (3.10), $v(0) = V(\varphi_0, x_0) = \alpha > 0$. Hence,

$$\forall t \geq 0 \quad v(t) \geq \alpha. \tag{3.12}$$

Then, $\exists \varepsilon > 0$ such that $\forall t \geq 0 \; \|x(t)\| \geq \varepsilon$. Really, if it is not so then there exists a sequence $t_k \to \infty$ such that $\|x(t_k)\| \to 0$. Finally,

$$v(t_k) = \left( S(\varphi_{t_k}(\varphi_0), x(t_k)) x(t_k), x(t_k) \right) \leq C \|x(t_k)\|^2 \to 0,$$

where constant $C > 0$ is from (3.4), which contradicts (3.12).

Then, from (3.11):

$$v(t) \geq \alpha + \varepsilon^2 t \quad \forall t \geq 0.$$

Hence, $\forall t \geq 0$

$$\|x(t)\|^2 \geq \frac{1}{C} \left( \alpha + \varepsilon^2 t \right).$$

The latter estimate implies the existence of $t_1 > 0$ such that $\|x(t_1)\| = r$. This completes the proof. □

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