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SOME RESULTS FOR q-POLY-BERNOULLI POLYNOMIALS WITH A PARAMETER

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Abstract. The main object of this paper is to investigate a new class of the generalized q-poly-Bernoulli numbers and polynomials with a parameter. We give explicit formulas and a recursive method for the calculation of the q-poly-Bernoulli numbers and polynomials. As a consequence, we derive a method for the calculation of the special values at negative integral points of the Arakawa–Kaneko zeta function also known as generalized Hurwitz zeta function.

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1. Introduction

Let q be an indeterminate with $0 \le q < 1$. The q-analogue of x is defined by

$$[x]_q = \frac{1-q^x}{1-q}$$

with $[0]_q = 0$ and $\lim_{q\to 1}[x]_q = x$. Recently Komatsu in [12] introduced and studied a new family of polynomials, called q-poly-Bernoulli polynomials $B_{n,\rho,q}^{(k)}(z)$ with a real parameter ρ which are defined by the following generating function:

$$F_{q,\rho}(t;z) := \frac{\rho}{1-e^{-\rho t}}\mathrm{Li}_{k,q}\left(\frac{1-e^{-\rho t}}{\rho}\right)e^{-tz} = \sum_{n=0}^{\infty} B_{n,\rho,q}^{(k)}(z)\frac{t^n}{n!}, \tag{1.1}$$

$$(n \ge 0; k \in \mathbb{Z}; \rho \ne 0)$$

where $\mathrm{Li}_{k,q}(z)$ is the q-polylogarithm function [11] defined by

$$\mathrm{Li}_{k,q}(z) = \sum_{n=1}^{\infty} \frac{z^n}{[n]_q^k}.$$

Clearly, we have

$$\lim_{q\to 1} B_{n,\rho,q}^{(k)}(z) = B_{n,\rho}^{(k)}(z),$$

which is the poly-Bernoulli polynomial with a ρ parameter [7], and

$$\lim_{q\to 1} \mathrm{Li}_{k,q}(z) = \mathrm{Li}_k(z),$$

which is the ordinary polylogarithm function, defined by

$$\mathrm{Li}_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}. \tag{1.2}$$

In addition, when $z = 0$, $B_{n,\rho}^{(k)}(0) = B_{n,\rho}^{(k)}$ is the poly-Bernoulli number with a ρ parameter. When $z = 0$ and $\rho = 1$, $B_{n,1}^{(k)}(0) = B_n^{(k)}$ is the poly-Bernoulli number [1–3, 10] defined by

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n^{(k)} \frac{t^n}{n!}, \tag{1.3}$$

In this paper, we propose to investigate a new class of the generalized q-poly-Bernoulli numbers and polynomials with a parameter which we call (m,q)-poly-Bernoulli polynomials with a parameter ρ. We establish several properties of these polynomials. The study of (m,q)-poly-Bernoulli polynomials with a parameter yields an interesting algorithm for calculating $\mathbb{B}_{n,m}^{(k)}(z;\rho,q)$. As an application, we derive a recursive method for the calculation of the special values at negative integral points of the Arakawa-Kaneko zeta function.

We first recall some basic definitions and some results [8, 16] that will be useful in the rest of the paper. The (signed) Stirling numbers $s(n,i)$ of the first kind are the coefficients in the following expansion:

$$x(x-1)\cdots(x-n+1) = \sum_{i=0}^{n} s(n,i)\, x^i, \quad n \geq 1$$

and satisfy the recurrence relation given by

$$s(n+1,i) = s(n,i-1) - n s(n,i) \qquad (1 \leq i \leq n). \tag{1.4}$$

The Stirling numbers of the second kind, denoted $S(n,i)$ are the coefficients in the expansion

$$x^n = \sum_{i=0}^{n} S(n,i) x(x-1)\cdots(x-i+1), \quad n \geq 1.$$

These numbers count the number of ways to partition a set of n elements into exactly i nonempty subsets.

The exponential generating functions for $s(n,i)$ and $S(n,i)$ are given by

$$\sum_{n=i}^{\infty} s(n,i) \frac{z^n}{n!} = \frac{1}{i!} [\ln(1+z)]^i$$

and

$$\sum_{n=i}^{\infty} S(n,i)\,\frac{z^n}{n!} = \frac{1}{i!}\left(e^z-1\right)^i,$$

respectively.

The weighted Stirling numbers $\mathcal{S}_n^i(x)$ of the second kind are defined by (see [5,6])

$$\begin{aligned}\mathcal{S}_n^i(x) &= \frac{1}{i!}\Delta^i x^n \\ &= \frac{1}{i!}\sum_{j=0}^{i}(-1)^{i-j}\binom{i}{j}(x+j)^n,\end{aligned}$$

where Δ denotes the forward difference operator. The exponential generating function of $\mathcal{S}_n^k(x)$ is given by

$$\sum_{n=i}^{\infty}\mathcal{S}_n^i(x)\,\frac{z^n}{n!} = \frac{1}{i!}e^{xz}\left(e^z-1\right)^i \tag{1.5}$$

and weighted Stirling numbers $\mathcal{S}_n^i(x)$ satisfy the following recurrence relation:

$$\mathcal{S}_{n+1}^i(x) = \mathcal{S}_n^{i-1}(x) + (x+i)\,\mathcal{S}_n^i(x) \qquad (1 \le i \le n).$$

In particular, we have for nonnegative integer r

$$\mathcal{S}_n^i(0) = S(n,i) \text{ and } \mathcal{S}_n^i(r) = \left\{ {n+r \atop i+r} \right\}_r .$$

where $\left\{ {n \atop i} \right\}_r$ denotes the r-Stirling numbers of the second kind [4].

2. The (m,q)-poly-Bernoulli numbers with a parameter ρ

In order to compute $B_{n,\rho,q}^{(k)} := B_{n,\rho,q}^{(k)}(0)$, we define (m,q)-poly-Bernoulli numbers $\mathbb{B}_{n,m}^{(k)}(\rho,q)$ with a parameter ρ in terms of m-Stirling numbers of the second kind by:

$$\mathbb{B}_{n,m}^{(k)}(\rho,q) = \frac{(-\rho)^n[m+1]_q^k}{m!}\sum_{i=0}^{n}\frac{(m+i)!\,\mathcal{S}_n^i(m)}{(-\rho)^i[m+i+1]_q^k},\ m \ge 0 \tag{2.1}$$

with $\mathbb{B}_{0,m}^{(k)}(\rho,q) = 1$ and $\mathbb{B}_{n,0}^{(k)}(\rho,q) = B_{n,\rho,q}^{(k)}$.

By direct computation from (2.1), we find

$$\begin{aligned}\mathbb{B}_{0,m}^{(k)}(\rho,q) &= 1, \\ \mathbb{B}_{1,m}^{(k)}(\rho,q) &= (m+1)\left(\frac{q^{m+1}-1}{q^{m+2}-1}\right)^k - m,\end{aligned}$$

$$\mathbb{B}_{2,m}^{(k)}(\rho,q) = (m+1)(m+2)\left(\frac{q^{m+1}-1}{q^{m+3}-1}\right)^k - \left(2m^2+3m+1\right)\left(\frac{q^{m+1}-1}{q^{m+2}-1}\right)^k + m^2.$$

The following Theorem gives us a relation between the (m,q)-poly-Bernoulli numbers $\mathbb{B}_{n,m}^{(k)}(\rho,q)$ and q-poly-Bernoulli numbers $B_n^{(k)}(\rho,q) := B_{n,\rho,q}^{(k)}$.

Theorem 1. *For $m \geq 0$, we have*

$$\mathbb{B}_{n,m}^{(k)}(\rho,q) = \frac{(-\rho)^m [m+1]_q^k}{m!} \sum_{i=0}^{m} s(m,i) \frac{B_{n+i}^{(k)}(\rho,q)}{(-\rho)^i}. \tag{2.2}$$

Proof. The explicit formula (2.2) can be derived from a known result in [14, p. 681, Corollary 1] for the Stirling transform upon specializing the initial sequence

$$a_{0,m} = \frac{m!}{(-\rho)^m [m+1]_q^k}.$$

□

The next Theorem contains the exponential generating function for (m,q)-poly-Bernoulli numbers with a parameter ρ.

Theorem 2. *The exponential generating function for $\mathbb{B}_{n,m}^{(k)}(\rho,q)$ is given by*

$$\sum_{n=0}^{\infty} \mathbb{B}_{n,m}^{(k)}(\rho,q) \frac{t^n}{n!} = \frac{(-\rho)^{m+n}[m+1]_q^k}{m!} e^{mt} \left(e^{-t}\frac{d}{dt}\right)^m F_{q,\rho}(t;z).$$

Proof. We have

$$\sum_{n=0}^{\infty} \mathbb{B}_{n,m}^{(k)}(\rho,q) \frac{t^n}{n!} = \frac{(-\rho)^m [m+1]_q^k}{m!} \sum_{i=0}^{m} s(m,i) \sum_{n=0}^{\infty} \frac{B_{n+i}^{(k)}(\rho,q)}{(-\rho)^i} \frac{t^n}{n!}$$
$$= \frac{(-\rho)^{m+n}[m+1]_q^k}{m!} \sum_{i=0}^{m} s(m,i) \frac{d^i}{dt^i}\left(\frac{\rho}{1-e^t}\mathrm{Li}_{k,q}\left(\frac{1-e^t}{\rho}\right)\right).$$

Since [13]

$$\sum_{i=0}^{m} s(m,i)\left(\frac{d}{dt}\right)^i = e^{mt}\left(e^{-t}\frac{d}{dt}\right)^m,$$

we get the desired result. □

Next, we propose an algorithm, which is based on a three-term recurrence relation, for calculating the $(m,q)-$poly-Bernoulli numbers $\mathbb{B}_{n,m}^{(k)}(\rho,q)$ with a parameter ρ.

Theorem 3. *For every integer k, the $\mathbb{B}_{n,m}^{(k)}(\rho,q)$ satisfies the following three-term recurrence relation:*

$$\mathbb{B}_{n+1,m}^{(k)}(\rho,q)=(m+1)\left(\frac{q^{m+1}-1}{q^{m+2}-1}\right)^{k}\mathbb{B}_{n,m+1}^{(k)}(\rho,q)-\rho m\mathbb{B}_{n,m}^{(k)}(\rho,q) \tag{2.3}$$

with the initial sequence given by $\mathbb{B}_{0,m}^{(k)}(\rho,q)=1$.

Proof. From (2.2) and (1.4), we have

$$\mathbb{B}_{n,m+1}^{(k)}(\rho,q)=\frac{(-\rho)^{m+1}[m+2]_q^k}{(m+1)!}\sum_{i=0}^{m+1}(s(m,i-1)-ms(m,i))\frac{B_{n+i}^{(k)}(\rho,q)}{(-\rho)^i}.$$

After some simplifications, we find that

$$\mathbb{B}_{n,m+1}^{(k)}(\rho,q)=\frac{1}{[m+1]_q^k}\frac{(-\rho)[m+2]_q^k}{m+1}\left(\frac{1}{(-\rho)}\mathbb{B}_{n+1,m}^{(k)}(\rho,q)-m\mathbb{B}_{n,m}^{(k)}(\rho,q)\right).$$

This evidently equivalent to (2.3). □

Remark 1. If we set $\rho=1,k=1$ and $q\to 1$, in (2.3), we get

$$B_{n+1,m}=\frac{(m+1)^2}{(m+2)}B_{n,m+1}-mB_{n,m}, \tag{2.4}$$

an algorithm for the classical Bernoulli numbers with $B_1=\frac{1}{2}$. See [15] for the case $B_1=-\frac{1}{2}$.

3. The (m,q)-poly-Bernoulli polynomials with a parameter ρ

For $m\geq 0$, let us consider the (m,q)-poly-Bernoulli polynomials with a parameter $\mathbb{B}_{n,m}^{(k)}(z;\rho,q)$ as follows:

$$\mathbb{B}_{n,m}^{(k)}(z;\rho,q)=\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\mathbb{B}_{i,m}^{(k)}(\rho,q)z^{n-i}. \tag{3.1}$$

It is easy to show that the generating function of $\mathbb{B}_{n,m}^{(k)}(z;\rho,q)$ is given by

$$\begin{aligned}\sum_{n\geq 0}\mathbb{B}_{n,m}^{(k)}(z;\rho,q)\frac{t^n}{n!}&=e^{-zt}\sum_{n\geq 0}\mathbb{B}_{n,m}^{(k)}(\rho,q)\frac{t^n}{n!}\\&=\frac{1}{m!}(-\rho)^{m+n}[m+1]_q^k e^{(m-z)t}\left(e^{-t}\frac{d}{dt}\right)^m F_{q,\rho}(t;z).\end{aligned}$$

Next, we show an explicit formula about $\mathbb{B}_{n,m}^{(k)}(z;\rho,q)$.

Theorem 4. *The following formula holds true*

$$\mathbb{B}_{n,m}^{(k)}(z;\rho,q)=\frac{[m+1]_q^k}{m!}\sum_{i=0}^{n}(-\rho)^{n-i}\frac{(m+i)!}{[m+i+1]_q^k}\mathcal{S}_n^i\left(\frac{z}{\rho}+m\right).$$

Proof. From (3.1), we have

$$\begin{aligned}\mathbb{B}_{n,m}^{(k)}(z;\rho,q)&=\frac{[m+1]_q^k}{m!}\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\sum_{j=0}^{i}\frac{(-\rho)^{i-j}(m+j)!\mathcal{S}_i^j(m)}{[m+j+1]_q^k}z^{n-i}\\&=\frac{[m+1]_q^k}{m!}\sum_{j=0}^{n}\frac{(-\rho)^{n-j}(m+j)!}{[m+j+1]_q^k}\sum_{i=0}^{n}\binom{n}{i}\mathcal{S}_i^j(m)\left(\frac{z}{\rho}\right)^{n-i}.\end{aligned}$$

By using the relation

$$\mathcal{S}_n^i(x+y)=\sum_{l=0}^{n}\binom{n}{l}\mathcal{S}_l^i(x)\,y^{n-l},$$

we obtain

$$\mathbb{B}_{n,m}^{(k)}(z;\rho,q)=\frac{[m+1]_q^k(-\rho)^n}{m!}\sum_{j=0}^{n}\frac{(m+j)!}{(-\rho)^j[m+j+1]_q^k}\mathcal{S}_n^j\left(\frac{z}{\rho}+m\right),$$

we arrive at the desired result. □

Theorem 5. *The polynomials* $\mathbb{B}_{n,m}^{(k)}(z;\rho,q)$ *satisfy the following three-term recurrence relation:*

$$\begin{aligned}\mathbb{B}_{n+1,m}^{(k)}(z;\rho,q)=(m+1)\left(\frac{q^{m+1}-1}{q^{m+2}-1}\right)^k\mathbb{B}_{n,m+1}^{(k)}(z;\rho,q)&\\+(z-\rho m)\,\mathbb{B}_{n,m}^{(k)}(z;\rho,q),&\end{aligned}\qquad(3.2)$$

with the initial sequence given by

$$\mathbb{B}_{0,m}^{(k)}(z;\rho,q)=1.$$

Proof. From (3.1), we get

$$\begin{aligned}\frac{d}{dz}\mathbb{B}_{n,m}^{(k)}(z;\rho,q)&=\sum_{i=0}^{n}(n-i)(-1)^{n-i}\binom{n}{i}\mathbb{B}_{i,m}^{(k)}(\rho,q)\,z^{n-i-1}\\&=n\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}\mathbb{B}_{i,m}^{(k)}(\rho,q)\,z^{n-i-1}\\&-n\sum_{i=1}^{n}(-1)^{n-i}\binom{n-1}{i-1}\mathbb{B}_{i,m}^{(k)}(\rho,q)\,z^{n-i-1}.\end{aligned}$$

Then

$$z\frac{d}{dz}\mathbb{B}_{n,m}^{(k)}(z;\rho,q)=n\mathbb{B}_{n,m}^{(k)}(z;\rho,q)\\ -n\sum_{i=0}^{n-1}(-1)^{n-i-1}\binom{n-1}{i}\mathbb{B}_{i+1,m}^{(k)}(\rho,q)z^{n-i-1}.$$

Now, using (2.3), we have

$$z\frac{d}{dz}\mathbb{B}_{n,m}^{(k)}(z;\rho,q)=n\mathbb{B}_{n,m}^{(k)}(z;\rho,q)+n\rho m\sum_{i=0}^{n-1}\binom{n-1}{i}\mathbb{B}_{i,m}^{(k)}(\rho,q)(-z)^{n-i-1}\\ -n(m+1)\left(\frac{q^{m+1}-1}{q^{m+2}-1}\right)^{k}\sum_{i=0}^{n-1}\binom{n-1}{i}\mathbb{B}_{i,m+1}^{(k)}(\rho,q)(-z)^{n-i-1},$$

which, after simplification, yields

$$z\mathbb{B}_{n-1,m}^{(k)}(z;\rho,q)=\mathbb{B}_{n,m}^{(k)}(z;\rho,q)-(m+1)\left(\frac{q^{m+1}-1}{q^{m+2}-1}\right)^{k}\mathbb{B}_{n-1,m+1}^{(k)}(z;\rho,q)\\ +\rho m\mathbb{B}_{n-1,m}^{(k)}(z;\rho,q),$$

which is obviously equivalent to (3.2) and the proof is complete. □

As consequence of Theorem 5, one can deduce a three-term recurrence relation for (m,q)-poly-Bernoulli polynomials with a parameter ρ and negative upper indices $\mathbb{B}_{n,m}^{(-k)}(z;\rho,q)$.

Corollary 1. *The* $\mathbb{B}_{n,m}^{(-k)}(z;\rho,q)$ *satisfies the following three-term recurrence relation:*

$$\mathbb{B}_{n+1,m}^{(-k)}(z;\rho,q)=(m+1)\left(\frac{q^{m+2}-1}{q^{m+1}-1}\right)^{k}\mathbb{B}_{n,m+1}^{(-k)}(z;\rho,q)\\ +(z-\rho m)\mathbb{B}_{n,m}^{(-k)}(z;\rho,q),$$

with the initial sequence given by

$$\mathbb{B}_{0,m}^{(-k)}(z;\rho,q)=1.$$

The next result gives a method for the calculation of the special values at negative integral points of the Arakawa–Kaneko zeta function. Recall that the Arakawa-Kaneko zeta function $\xi_k(s,x)$, for $s\in\mathbb{C}$, $x>0$, $k\in\mathbb{Z}$, is defined by [9]

$$\xi_k(s,x)=\frac{1}{\Gamma(s)}\int_0^{+\infty}t^{s-1}\frac{\mathrm{Li}_k\left(1-e^{-t}\right)}{1-e^{-t}}e^{-xt}\,dt$$

It is well-known that the special values at negative integral points are given in terms of poly-Bernoulli polynomials

$$\xi_k(-n,x) = (-1)^n B_n^{(k)}(x), n \geq 0.$$

We now present the following algorithm for $\xi_k(-n,x)$. We start with the sequence $\mathcal{K}_{0,m} = 1$ as the first row of the matrix $(\mathcal{K}_{n,m})_{n,m\geq 0}$. Each entry is determined recursively by

$$\mathcal{K}_{n+1,m}(k,x) = \frac{(m+1)^{k+1}}{(m+2)^k}\mathcal{K}_{n,m+1}(k,x) + (x-m)\mathcal{K}_{n,m}(k,x).$$

Then

$$\xi_k(-n,x) = (-1)^n \mathcal{K}_{n,0}(k,x)$$

where $\mathcal{K}_{n,0}(k,x)$ are the first column of the matrix $(\mathcal{K}_{n,m})_{n,m\geq 0}$.

4. Conclusion

In our present research, we have investigated a new class of the generalized q-poly-Bernoulli numbers and polynomials with a parameter. As a consequence, we derive a method for the calculation of the special values at negative integral points of the Arakawa-Kaneko zeta function. We have also given a recursive method for the calculation of q-poly-Bernoulli numbers and polynomials with parameter.

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