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SOME NEW CLASSES OF CONVEX FUNCTIONS AND INEQUALITIES

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Abstract. In this article, we introduce some new class of convex functions involving two arbitrary auxiliary functions $h_1, h_2 : I \to \mathbb{R}$, which are called (h_1, h_2)-convex functions. We derive some new integral inequalities for these classes of convex functions. We also discuss some special cases which can be deduced from our main results. Results obtained in this paper may be viewed as a significant refinement and improvement of the previously known results. The ideas and techniques of this work may be a starting point for future research.

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1. Introduction

A set $\mathcal{C} \subset \mathbb{R}$ is said to be a convex set, if

$$tx + (1-t)y \in \mathcal{C}, \quad \forall x, y \in \mathcal{C},\ t \in [0,1].$$

A function $f : \mathcal{C} \to \mathbb{R}$ is said to be a convex function in the classical sense, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad \forall x, y \in \mathcal{C},\ t \in [0,1].$$

In recent years several new generalizations of classical convexity have been given, for example see [1–3, 8, 17, 18]. Varosanec [18] introduced the notion of h-convexity which along with classical convex functions generalizes several other class of convex functions. The formal definition of h-convex functions is given as:

Definition 1 ([18])**.** Let $h : (0,1) \subseteq J \to \mathbb{R}$ be a non-negative function, $h \not\equiv 0$. We say that $f : \mathcal{C} \to \mathbb{R}$ is an h-convex function, if f is non-negative and for all $x, y \in \mathcal{C}$, $t \in (0,1)$, we have

$$f(tx + (1-t)y) \leq h(t)f(x) + h(1-t)f(y).$$

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For the different choices of the auxiliary function $h(.)$, we have different other classes of convex functions such as: Breckner type of s-convex functions [1], Godunova-Levin-Dragomir type of s-convex functions [4], Godunova-Levin type of functions [8] and P-functions [7]. Since the appearance of this definition many researchers shown their special interest in studying this class of convex functions. Sarikaya et al. [15] has improved the Hermite-Hadamard's inequality for this class of convex functions. Recent trend of the research in this field has shown that theory of convexity and theory of inequalities have a close relationship. Many inequalities can be obtained via convex functions and naturally they can be extended for generalizations of convex functions, see [5, 6, 10–12, 14, 16].

Motivated by the research going on in this field, we introduce the notion of so-called (h_1, h_2)-convex functions, which is the main motivation of this paper. These classes involves two auxiliary functions namely $h_1, h_2 : J \to \mathbb{R}$. We show that these classes include several new and known classes of convex functions as special cases. We also derive some new estimates for Hermite-Hadamard type of inequalities and for the integral $\int_a^b (u-a)^\alpha (b-u)^\beta f(u)\mathrm{d}u$ via (h_1, h_2)-convex functions. Some new and known special cases which can be deduced from our main results, are also discussed.

2. Preliminaries

We now define the new classes of convex functions involving two arbitrary functions.

Definition 2. Let $h_1, h_2 : (0,1) \subseteq J \to \mathbb{R}$ be two real functions, $h_1, h_2 \not\equiv 0$. We say that $f : \mathcal{C} \to \mathbb{R}$ is an (h_1, h_2)-convex function, if

$$f(tx+(1-t)y) \leq h_1(t)h_2(1-t)f(x)+h_1(1-t)h_2(t)f(y), \quad \forall x, y \in \mathcal{C}, t \in (0,1).$$

We now discuss several special cases.

I. If $h_2(t) = 1$, then Definition 2 reduces to the definition for h-convex functions [18].

II. If $h_1(t) = 1 = h_2(t)$, then Definition 2 reduces to the definition for P-functions [7].

III. If $h_1(t) = t^s$ and $h_2(t) = t^s$ in Definition 2, then we have the class of s-convex functions of third kind.

Definition 3. Let $s \in [0,1]$ be a real number. We say that $f : \mathcal{C} \to \mathbb{R}$ is an s-tgs-convex functions, if

$$f(tx+(1-t)y) \leq t^s(1-t)^s[f(x)+f(y)], \quad \forall x, y \in \mathcal{C}, t \in [0,1].$$

Note that if we take $s=1$ in Definition 3, then we have the definition of tgs-convex functions [17].
IV. If $h_1(t)=t^{-s}$ and $h_2(t)=t^{-s}$ in Definition 2, then we have the class of s-Godunova-Levin-Dragomir tgs-convex functions.

Definition 4. Let $s\in(0,1]$ be a real number. We say that $f:\mathcal{C}\to\mathbb{R}$ is an s-Godunova-Levin-Dragomir tgs-convex function, if

$$f(tx+(1-t)y)\leq\frac{1}{t^s}\frac{1}{(1-t)^s}[f(x)+f(y)],\quad\forall x,y\in\mathcal{C},t\in[0,1].$$

Note that if we take $s=1$ in Definition 4, then we have the definition of Godunova-Levin type of tgs-convex functions, which appears to be a new definition.
V. If $h_1(t)=t^{s_1}$ and $h_2(t)=t^{s_2}$ in Definition 2, then we have a new class of convex functions which is called as Breckner type of (s_1,s_2)-convex functions.

Definition 5. Let $s_1,s_2\in(0,1]$ be two real numbers. We say that $f:\mathcal{C}\to\mathbb{R}$ is an (s_1,s_2)-convex function, if

$$f(tx+(1-t)y)\leq t^{s_1}(1-t)^{s_2}f(x)+(1-t)^{s_1}t^{s_2}f(y),\quad\forall x,y\in\mathcal{C},t\in[0,1].$$

VI. If $h_1(t)=t^{-s_1}$ and $h_2(t)=t^{-s_2}$ in Definition 2, then we have a new class of convex functions which is called as Godunova-Levin-Dragomir type of (s_1,s_2)-convex functions.

Definition 6. Let $s_1,s_2\in(0,1]$ be two real numbers. We say that $f:\mathcal{C}\to\mathbb{R}$ is an (s_1,s_2)-convex function, if

$$f(tx+(1-t)y)\leq\frac{1}{t^{s_1}(1-t)^{s_2}}f(x)+\frac{1}{(1-t)^{s_1}t^{s_2}}f(y),\quad\forall x,y\in\mathcal{C},t\in[0,1].$$

It is clear that these new classes of convex functions are quite general and include several new and previously known classes of convex functions as special cases.

The following results will be helpful in deriving our main results in this paper.

Lemma 1 ([13]). *Let $f:\mathcal{I}=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f\in\mathscr{L}[a,b]$, then*

$$\int_a^b(u-a)^\alpha(b-u)^\beta f(u)\mathrm{d}u=(b-a)^{\alpha+\beta+1}\int_0^1 t^\alpha(1-t)^\beta f((1-t)a+tb)\mathrm{d}t.$$

Lemma 2 ([9]). *Let $\mathcal{I}^\circ\subset\mathbb{R}\to\mathbb{R}$, $a,b\in\mathcal{I}^\circ$ with $a<b$ where $\mathcal{I}^\circ$ is the interior of $\mathcal{I}^\circ$. If $f^{(n)}$ exists on $\mathcal{I}^\circ$ and $f^{(n)}\in\mathscr{L}[a,b]$, then for $n\geq1$, we have*

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)$$

$$= \frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)\,f^{(n)}(ta+(1-t)b)\mathrm{d}t.$$

3. Main results

In this section we derive our main results.

Theorem 1. *Let $f : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a (h_1,h_2)-convex function. If $f \in \mathscr{L}[a,b]$, then for $h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right) \neq 0$, we have*

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq [f(a)+f(b)]\int_0^1 h_1(t)h_2(1-t)\mathrm{d}t.$$

Proof. Since f is an (h_1,h_2)-convex function, for $x = ta+(1-t)b$, $y = (1-t)a+tb$ and $t = \frac{1}{2}$, we have

$$f\left(\frac{a+b}{2}\right) \leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)[f(ta+(1-t)b)+f((1-t)a+tb)].$$

Integrating both sides of the above inequality with respect to t on $[0,1]$, we have

$$\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x. \tag{3.1}$$

Also

$$f(ta+(1-t)b) \leq h_1(t)h_2(1-t)f(x)+h_1(1-t)h_2(t)f(y).$$

Integrating both sides of the above inequality with respect to t on $[0,1]$, we have

$$\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq [f(a)+f(b)]\int_0^1 h_1(t)h_2(1-t)\mathrm{d}t. \tag{3.2}$$

On summation of inequalities (3.1) and (3.2) the proof is complete. □

We now discuss a new special case of Theorem 1.

If $h_1(t) = t^{s_1}$ and $h_2 = t^{s_2}$ in Theorem 1, then we have a result for Brecker type of (s_1,s_2)-convex functions.

Corollary 1. *Let $f : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a Brecker type of (s_1,s_2)-convex function. If $f \in \mathscr{L}[a,b]$, then for $s_1,s_2 \in [0,1]$ we have*

$$\frac{1}{2^{1-s_1-s_2}} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq [f(a)+f(b)]\mathbb{B}(s_1+1,s_2+1).$$

If $h_1(t)=t^{-s_1}$ and $h_2=t^{-s_2}$ in Theorem 1, then we have a result for Godunova-Levin-Dragomir type of (s_1,s_2)-convex functions.

Corollary 2. *Let $f:\mathcal{I}=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a Godunova-Levin-Dragomir type of (s_1,s_2)-convex function. If $f\in\mathcal{L}[a,b]$, then for $s_1,s_2\in[0,1]$, we have*

$$\frac{1}{2^{1+s_1+s_2}}f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\leq[f(a)+f(b)]\mathbb{B}(1-s_1,1-s_2).$$

Our next result is a lower bound for Hermite-Hadamard's inequality via product of two (h_1,h_2)-convex functions.

Theorem 2. *Let $f,g:\mathcal{I}=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be two (h_1,h_2)-convex functions such that $h_1^2(\frac{1}{2})h_2^2(\frac{1}{2})\neq 0$. If $fg\in\mathcal{L}[a,b]$, then, we have*

$$\begin{aligned}&\frac{1}{2h_1^2(\frac{1}{2})h_2^2(\frac{1}{2})}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\&\quad-\left[M(a,b)\int_0^1[h_1(t)h_2(t)h_1(1-t)h_2(1-t)]\mathrm{d}t+N(a,b)\int_0^1 h_1^2(t)h_2^2(1-t)\mathrm{d}t\right]\\&\leq\frac{1}{b-a}\int_0^1 f(x)g(x)\mathrm{d}x.\end{aligned}$$

where

$$M(a,b)=f(a)g(a)+f(b)g(b),\tag{3.3}$$

and

$$N(a,b)=f(a)g(b)+f(b)g(a),\tag{3.4}$$

respectively.

Proof. Since f and g are (h_1,h_2)-convex functions, so

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\&\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)[f(ta+(1-t)b)+f((1-t)a+tb)]\\&\quad\times h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)[g(ta+(1-t)b)+g((1-t)a+tb)]\\&=h_1^2\left(\frac{1}{2}\right)h_2^2\left(\frac{1}{2}\right)[f(ta+(1-t)b)g(ta+(1-t)b)\\&\quad f((1-t)a+tb)g((1-t)a+tb)+f(ta+(1-t)b)g((1-t)a+tb)\\&\quad+f((1-t)a+tb)g(ta+(1-t)b)]\end{aligned}$$

$$\leq h_1^2\left(\frac{1}{2}\right)h_2^2\left(\frac{1}{2}\right)[f(ta+(1-t)b)g(ta+(1-t)b)$$
$$+f((1-t)a+tb)g((1-t)a+tb)$$
$$+[2h_1(t)h_2(t)h_1(1-t)h_2(1-t)][f(a)g(a)+f(b)g(b)]$$
$$+[h_1^2(1-t)h_2^2(t)+h_1^2(t)h_2^2(1-t)][f(a)g(b)+f(b)g(a)]\Big].$$

Integrating the above inequality with respect to t on $[0,1]$, we have

$$f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$
$$\leq 2h_1^2\left(\frac{1}{2}\right)h_2^2\left(\frac{1}{2}\right)\left[\frac{1}{b-a}\int_0^1 f(x)g(x)\mathrm{d}x\right.$$
$$\left.+M(a,b)\int_0^1[h_1(t)h_2(t)h_1(1-t)h_2(1-t)]\mathrm{d}t+N(a,b)\int_0^1 h_1^2(t)h_2^2(1-t)\mathrm{d}t\right].$$

This implies

$$\frac{1}{2h_1^2(\frac{1}{2})h_2^2(\frac{1}{2})}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$
$$\leq \frac{1}{b-a}\int_0^1 f(x)g(x)\mathrm{d}x$$
$$+M(a,b)\int_0^1[h_1(t)h_2(t)h_1(1-t)h_2(1-t)]\mathrm{d}t+N(a,b)\int_0^1 h_1^2(t)h_2^2(1-t)\mathrm{d}t.$$

This completes the proof. □

Next we discuss a new special case of Theorem 2.

If $h_1(t)=t^{s_1}$ and $h_2=t^{s_2}$ in Theorem 2, then we have a result for Breckner type of (s_1,s_2)-convex functions.

Corollary 3. *Let $f,g:\mathcal{J}=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be two (s_1,s_2)-convex functions such that $s_1,s_2\in[0,1]$. If $fg\in\mathscr{L}[a,b]$, then we have*

$$\frac{1}{2^{1-2s_1-2s_2}}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)$$
$$-\left[M(a,b)\mathbb{B}(s_1+s_2+1,s_1+s_2+1)+N(a,b)\mathbb{B}(2s_1+1,2s_2+1)\right]$$

$$\leq \frac{1}{b-a}\int_0^1 f(x)g(x)\mathrm{d}x$$

where $M(a,b)$ and $N(a,b)$ are given in (3.3) and (3.4) respectively.

If $h_1(t)=t^{-s_1}$ and $h_2=t^{-s_2}$ in Theorem 2, then we have a result for Godunova-Levin-Dragomir type of (s_1,s_2)-convex functions.

Corollary 4. *Let $f,g:\mathcal{I}=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be two Godunova-Levin-Dragomir type of (s_1,s_2)-convex functions such that $s_1,s_2\in[0,1]$. If $fg\in\mathscr{L}[a,b]$, then we have*

$$\begin{aligned}&\frac{1}{2^{1+2s_1+2s_2}}f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\&\quad-\left[M(a,b)\mathbb{B}(1-s_1-s_2,1-s_1-s_2)+N(a,b)\mathbb{B}(1-2s_1,1-2s_2)\right]\\&\leq\frac{1}{b-a}\int_0^1 f(x)g(x)\mathrm{d}x\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given in (3.3) and (3.4) respectively.

Our next result is the extension of the upper bound of Hermite-Hadamard type inequality via product of two (h_1,h_2)-convex functions.

Theorem 3. *Left $f,g:\mathcal{I}=[a,b]\to\mathbb{R}$ be two (h_1,h_2)-convex functions. If $fg\in\mathscr{L}[a,b]$, then, we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\\&\leq M(a,b)\int_0^1 h_1^2(t)h_2^2(1-t)\mathrm{d}t+N(a,b)\int_0^1 h_1(t)h_2(1-t)h_1(1-t)h_2(t)\mathrm{d}t,\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given by (3.3) and (3.4) respectively.

Proof. Since f and g are (h_1,h_2)-convex functions, then

$$f(ta+(1-t)b)\leq h_1(t)h_2(1-t)f(a)+h_1(1-t)h_2(t)f(b),$$

and

$$g(ta+(1-t)b)\leq h_1(t)h_2(1-t)g(a)+h_1(1-t)h_2(t)g(b).$$

Multiplying both sides of the above inequality and then integrating it with respect to t on $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)\mathrm{d}t \\
&\leq [f(a)g(a)+f(b)g(b)]\int_0^1 h_1^2(t)h_2^2(1-t)\mathrm{d}t \\
&\quad +[f(a)g(b)+f(b)g(a)]\int_0^1 h_1(t)h_2(1-t)h_1(1-t)h_2(t)\mathrm{d}t.
\end{aligned}$$

This implies

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \\
&\leq M(a,b)\int_0^1 h_1^2(t)h_2^2(1-t)\mathrm{d}t + N(a,b)\int_0^1 h_1(t)h_2(1-t)h_1(1-t)h_2(t)\mathrm{d}t.
\end{aligned}$$

This completes the proof. □

The next result is a special case of Theorem 3.

If $h_1(t)=t^{s_1}$ and $h_2=t^{s_2}$ in Theorem 3, then we have a result for Breckner type of (s_1,s_2)-convex functions.

Corollary 5. *Let $f,g:\mathcal{I}=[a,b]\to\mathbb{R}$ be two Breckner type of (s_1,s_2)-convex functions where $s_1,s_2\in[0,1]$. If $fg\in\mathscr{L}[a,b]$, then we have*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x \\
&\leq M(a,b)\mathbb{B}(2s_1+1,2s_2+1)+N(a,b)\mathbb{B}(s_1+s_2+1,s_1+s_2+1),
\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given by (3.3) and (3.4) respectively.

If $h_1(t)=t^{-s_1}$ and $h_2=t^{-s_2}$ in Theorem 3, then we have a result for Godunova-Levin-Dragomir type of (s_1,s_2)-convex functions.

Corollary 6. *Let $f,g:\mathcal{I}=[a,b]\to\mathbb{R}$ be two Godunova-Levin-Dragomir type of (s_1,s_2)-convex functions where $s_1,s_2\in[0,1]$. If $fg\in\mathscr{L}[a,b]$, then we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)g(x)\mathrm{d}x\\&\leq M(a,b)\mathbb{B}(1-2s_1,1-2s_2)+N(a,b)\mathbb{B}(1-s_1-s_2,1-s_1-s_2),\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given by (3.3) and (3.4) respectively.

Theorem 4. *Let $f:\mathcal{I}=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f\in\mathscr{L}[a,b]$. If f is an (h_1,h_2)-convex function, then we have*

$$\int_a^b (u-a)^{\alpha}(b-u)^{\beta}f(u)\mathrm{d}u\leq(b-a)^{\alpha+\beta+1}\left[\psi_1(t)f(a)+\psi_2(t)f(b)\right],$$

where

$$\psi_1(t):=\int_0^1 t^{\alpha}(1-t)^{\beta}h_1(1-t)h_2(t)\mathrm{d}t, \tag{3.5}$$

and

$$\psi_2(t):=\int_0^1 t^{\alpha}(1-t)^{\beta}h_1(t)h_2(1-t)\mathrm{d}t, \tag{3.6}$$

respectively.

Proof. Using Lemma 1 and the fact that f is an (h_1,h_2)-convex function, we have

$$\begin{aligned}&\int_a^b (u-a)^{\alpha}(b-u)^{\beta}f(u)\mathrm{d}u\\&=(b-a)^{\alpha+\beta+1}\int_0^1 t^{\alpha}(1-t)^{\beta}f((1-t)a+tb)\mathrm{d}t\\&\leq(b-a)^{\alpha+\beta+1}\int_0^1 t^{\alpha}(1-t)^{\beta}[h_1(1-t)h_2(t)f(a)+h_1(t)h_2(1-t)f(b)]\mathrm{d}t\\&=(b-a)^{\alpha+\beta+1}\left[\psi_1(t)f(a)+\psi_2(t)f(b)\right].\end{aligned}$$

This completes the proof. □

If $h_1(t)=t^{s_1}$ and $h_2(t)=t^{s_2}$ in Theorem 4, then we have

Corollary 7. *Let $f : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a,b]$. If f is a Breckner type of (s_1,s_2)-convex function, where $s_1,s_2 \in [0,1]$, then we have*

$$\int_a^b (u-a)^{\alpha}(b-u)^{\beta} f(u)\mathrm{d}u \leq (b-a)^{\alpha+\beta+1}\left[\psi_1'(t)f(a)+\psi_2'(t)f(b)\right],$$

where

$$\psi_1'(t) := \mathbb{B}(\alpha+s_2+,\beta+s_1+1), \tag{3.7}$$

and

$$\psi_2'(t) := \mathbb{B}(\alpha+s_1+1,\beta+s_2+1), \tag{3.8}$$

respectively.

If $h_1(t) = t^{-s_1}$ and $h_2(t) = t^{-s_2}$ in Theorem 4, then we have

Corollary 8. *Let $f : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a,b]$. If f is a Godunova-Levin-Dragomir type of (s_1,s_2)-convex function, where $s_1,s_2 \in [0,1]$, then we have*

$$\int_a^b (u-a)^{\alpha}(b-u)^{\beta} f(u)\mathrm{d}u \leq (b-a)^{\alpha+\beta+1}\left[\psi_1''(t)f(a)+\psi_2''(t)f(b)\right],$$

where

$$\psi_1''(t) := \mathbb{B}(\alpha-s_2+1,\beta-s_1+1), \tag{3.9}$$

and

$$\psi_2''(t) := \mathbb{B}(\alpha-s_1+1,\beta-s_2+1), \tag{3.10}$$

respectively.

Theorem 5. *Let $f : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a,b]$. If $|f|^{\frac{r}{r-1}}$ is a (h_1,h_2)-convex function, then we have*

$$\int_a^b (u-a)^{\alpha}(b-u)^{\beta} f(u)\mathrm{d}u$$

$$\leq (b-a)^{\alpha+\beta+1}\mathbb{B}(r\alpha+1,r\beta+1)\left[\left\{|f(a)|^{\frac{r}{r-1}}+|f(b)|^{\frac{r}{r-1}}\right\}\int_0^1 h_1(t)h_2(1-t)\mathrm{d}t\right]^{\frac{r-1}{r}}.$$

Proof. Using Lemma 1, Holder's inequality and the fact that $|f|^{\frac{r}{r-1}}$ is an (h_1,h_2)-convex function, then

$$\int_a^b (u-a)^{\alpha}(b-u)^{\beta} f(u)\mathrm{d}u$$

$$\leq (b-a)^{\alpha+\beta+1}\left[\int_0^1 t^{r\alpha}(1-t)^{r\beta}\mathrm{d}t\right]^{\frac{1}{r}}\left[\int_0^1 |f((1-t)a+tb)|^{\frac{r}{r-1}}\,\mathrm{d}t\right]^{\frac{r-1}{r}}$$

$$\leq (b-a)^{\alpha+\beta+1}\mathbb{B}(r\alpha+1,r\beta+1)$$

$$\left[\int_0^1 \left\{h_1(1-t)h_2(t)|f(a)|^{\frac{r}{r-1}}+h_1(t)h_2(1-t)|f(b)|^{\frac{r}{r-1}}\right\}\mathrm{d}t\right]^{\frac{r-1}{r}}$$

$$\leq (b-a)^{\alpha+\beta+1}\mathbb{B}(r\alpha+1,r\beta+1)$$

$$\left[\left\{|f(a)|^{\frac{r}{r-1}}+|f(b)|^{\frac{r}{r-1}}\right\}\int_0^1 h_1(t)h_2(1-t)\mathrm{d}t\right]^{\frac{r-1}{r}}.$$

This completes the proof. □

If $h_1(t)=t^{s_1}$ and $h_2(t)=t^{s_2}$ in Theorem 5, then we have

Corollary 9. *Let $f:\mathcal{I}=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f\in\mathscr{L}[a,b]$. If $|f|^{\frac{r}{r-1}}$ is a Breckner type of (s_1,s_2)-convex function, where $s_1,s_2\in[0,1]$, then we have*

$$\int_a^b (u-a)^{\alpha}(b-u)^{\beta} f(u)\mathrm{d}u$$

$$\leq (b-a)^{\alpha+\beta+1}\mathbb{B}(r\alpha+1,r\beta+1)\left[\left\{|f(a)|^{\frac{r}{r-1}}+|f(b)|^{\frac{r}{r-1}}\right\}\mathbb{B}(\alpha+s_1+1,\beta+s_2+1)\right]^{\frac{r-1}{r}}.$$

If $h_1(t)=t^{-s_1}$ and $h_2(t)=t^{-s_2}$ in Theorem 5, then we have

Corollary 10. *Let $f:\mathcal{I}=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a continuous function such that $f\in\mathscr{L}[a,b]$. If $|f|^{\frac{r}{r-1}}$ is a Godunova-Levin-Dragomir type of (s_1,s_2)-convex function, where $s_1,s_2\in[0,1]$, then we have*

$$\int_a^b (u-a)^{\alpha}(b-u)^{\beta} f(u)\mathrm{d}u$$

$$\leq (b-a)^{\alpha+\beta+1}\mathbb{B}(1-r\alpha,1-r\beta)\left[\left\{|f(a)|^{\frac{r}{r-1}}+|f(b)|^{\frac{r}{r-1}}\right\}\mathbb{B}(1-\alpha-s_1,1-\beta-s_2)\right]^{\frac{r-1}{r}}.$$

Theorem 6. *Let $f : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a,b]$. If $|f|^r$ is an (h_1,h_2)-convex function, then we have*

$$\int_a^b (u-a)^\alpha (b-u)^\beta f(u)\mathrm{d}u$$
$$\leq (b-a)^{\alpha+\beta+1} [\mathbb{B}(\alpha+1,\beta+1)]^{\frac{r-1}{r}} \left[\psi_1(t)|f(a)|^r + \psi_2(t)|f(b)|^r\right]^{\frac{1}{r}},$$

where $\psi_1(t)$ and $\psi_2(t)$ are given by (3.5) and (3.6) respectively.

Proof. Using Lemma 1, Holder's inequality and the fact that $|f|^r$ is an (h_1,h_2)-convex function, then

$$\int_a^b (u-a)^\alpha (b-u)^\beta f(u)\mathrm{d}u$$
$$\leq (b-a)^{\alpha+\beta+1} \left[\int_0^1 (1-t)^\alpha t^\beta \mathrm{d}t\right]^{\frac{r-1}{r}} \left[\int_0^1 t^\alpha (1-t)^\beta |f((1-t)a+tb)|^r \mathrm{d}t\right]^{\frac{1}{r}}$$
$$\leq (b-a)^{\alpha+\beta+1} [\mathbb{B}(\alpha+1,\beta+1)]^{\frac{r-1}{r}}$$
$$\times \left[\int_0^1 t^\alpha (1-t)^\beta \left[h_1(1-t)h_2(t)|f(a)|^r + h_1(t)h_2(1-t)|f(b)|^r\right]\mathrm{d}t\right]^{\frac{1}{r}}$$
$$= (b-a)^{\alpha+\beta+1} [\mathbb{B}(\alpha+1,\beta+1)]^{\frac{r-1}{r}} \left[\psi_1(t)|f(a)|^r + \psi_2(t)|f(b)|^r\right]^{\frac{1}{r}}.$$

This completes the proof. □

If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$ in Theorem 6, then we have

Corollary 11. *Let $f : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathscr{L}[a,b]$. If $|f|^r$ is a Breckner type of (s_1,s_2)-convex function, where $s_1,s_2 \in [0,1]$, then we have*

$$\int_a^b (u-a)^\alpha (b-u)^\beta f(u)\mathrm{d}u$$
$$\leq (b-a)^{\alpha+\beta+1} [\mathbb{B}(\alpha+1,\beta+1)]^{\frac{r-1}{r}} \left[\psi_1'(t)|f(a)|^r + \psi_2'(t)|f(b)|^r\right]^{\frac{1}{r}},$$

where $\psi_1'(t)$ and $\psi_2'(t)$ are given by (3.7) and (3.8) respectively.

If $h_1(t) = t^{-s_1}$ and $h_2(t) = t^{-s_2}$ in Theorem 6, then we have

Corollary 12. *Let $f : \mathcal{I} = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f \in \mathcal{L}[a,b]$. If $|f|^r$ is a Godunova-Levin-Dragomir type of (s_1,s_2)-convex function, where $s_1, s_2 \in [0,1]$, then we have*

$$\int_a^b (u-a)^{\alpha}(b-u)^{\beta} f(u)\mathrm{d}u$$

$$\leq (b-a)^{\alpha+\beta+1} [\mathbb{B}(1-\alpha,1-\beta)]^{\frac{r-1}{r}} \left[\psi_1''(t)|f(a)|^r + \psi_2''(t)|f(b)|^r\right]^{\frac{1}{r}},$$

where $\psi_1''(t)$ and $\psi_2''(t)$ are given by (3.9) and (3.10) respectively.

Now using Lemma 2, we derive some new inequalities for n-times differentiable (h_1,h_2)-convex functions.

Theorem 7. *Let $f : \mathcal{I}^\circ \subset \mathbb{R} \to \mathbb{R}$, $a,b \in \mathcal{I}^\circ$ with $a < b$ where $\mathcal{I}^\circ$ is the interior of $\mathcal{I}^\circ$. Also suppose $f^{(n)}$ exists on $\mathcal{I}^\circ$ and $f^{(n)} \in \mathcal{L}[a,b]$. If $|f^{(n)}|^q$ is an (h_1,h_2)-convex function, then for $n,q \geq 1$, we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a)\right|$$

$$\leq \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}} \left(\theta_1(t)|f^n(a)|^q + \theta_2(t)|f^n(b)|^q\right)^{\frac{1}{q}},$$

where

$$\theta_1(t) := \int_0^1 t^{n-1}(n-2t)h_1(t)h_2(1-t)\mathrm{d}t, \tag{3.11}$$

and

$$\theta_2(t) := \int_0^1 t^{n-1}(n-2t)h_1(1-t)h_2(t)\mathrm{d}t, \tag{3.12}$$

respectively.

Proof. Using Lemma 2, property of modulus, power means inequality and the fact that $|f^{(n)}|^q$ is an (h_1,h_2)-convex function, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a)\right|$$

$$= \left|\frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t) f^{(n)}(ta+(1-t)b)\mathrm{d}t\right|$$

$$\leq \frac{(b-a)^n}{2n!}\int_0^1 t^{n-1}(n-2t)|f^{(n)}(ta+(1-t)b)|\mathrm{d}t$$

$$\leq \frac{(b-a)^n}{2n!}\left(\int_0^1 t^{n-1}(n-2t)\mathrm{d}t\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{n-1}(n-2t)|f^{(n)}(ta+(1-t)b)|^q\mathrm{d}t\right)^{\frac{1}{q}}$$

$$\leq \frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 t^{n-1}(n-2t)[h_1(t)h_2(1-t)|f^n(a)|^q+h_1(1-t)h_2(t)|f^n(b)|^q]\mathrm{d}t\right)^{\frac{1}{q}}$$

$$=\frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left(\theta_1(t)|f^n(a)|^q+\theta_2(t)|f^n(b)|^q\right)^{\frac{1}{q}}.$$

This completes the proof. □

If $h_1(t)=t^{s_1}$ and $h_2(t)=t^{s_2}$ in Theorem 7, then we have

Corollary 13. *Let $f:\mathcal{I}^\circ\subset\mathbb{R}\to\mathbb{R}$, $a,b\in\mathcal{I}^\circ$ with $a<b$ where $\mathcal{I}^\circ$ is the interior of $\mathcal{I}^\circ$. Also suppose $f^{(n)}$ exists on $\mathcal{I}^\circ$ and $f^{(n)}\in\mathscr{L}[a,b]$. If $|f^{(n)}|^q$ is a Breckner type of (s_1,s_2)-convex function, then for $s_1,s_2\in[0,1]$ and $n,q\geq 1$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$

$$\leq\frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left(\theta_1'(t)|f^n(a)|^q+\theta_2'(t)|f^n(b)|^q\right)^{\frac{1}{q}},$$

where

$$\theta_1'(t):=n\mathbb{B}(n+s_1,s_2+1)-2\mathbb{B}(n+s_1+1,s_2+1),$$

and

$$\theta_2'(t):=n\mathbb{B}(n+s_2,s_1+1)-2\mathbb{B}(n+s_2+1,s_2+1),$$

respectively.

If $h_1(t)=t^{-s_1}$ and $h_2(t)=t^{-s_2}$ in Theorem 7, then we have

Corollary 14. *Let $f:\mathcal{I}^\circ\subset\mathbb{R}\to\mathbb{R}$, $a,b\in\mathcal{I}^\circ$ with $a<b$ where $\mathcal{I}^\circ$ is the interior of $\mathcal{I}^\circ$. Also suppose $f^{(n)}$ exists on $\mathcal{I}^\circ$ and $f^{(n)}\in\mathscr{L}[a,b]$. If $|f^{(n)}|^q$ is a Godunova-Levin-Dragomir type of (s_1,s_2)-convex function, then for $s_1,s_2\in[0,1]$ and $n,q\geq 1$,*

we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\leq\frac{(b-a)^n}{2n!}\left(\frac{n-1}{n+1}\right)^{1-\frac{1}{q}}\left(\theta_1''(t)|f^n(a)|^q+\theta_2''(t)|f^n(b)|^q\right)^{\frac{1}{q}},$$

where

$$\theta_1''(t):=n\mathbb{B}(n-s_1,1-s_2)-2\mathbb{B}(n-s_1+1,1-s_2),$$

and

$$\theta_2''(t):=n\mathbb{B}(n-s_2,1-s_1)-2\mathbb{B}(n-s_2+1,1-s_2),$$

respectively.

Theorem 8. *Let $f:\mathcal{I}^\circ\subset\mathbb{R}\to\mathbb{R}$, $a,b\in\mathcal{I}^\circ$ with $a<b$ where $\mathcal{I}^\circ$ is the interior of $\mathcal{I}^\circ$. Also suppose $f^{(n)}$ exists on $\mathcal{I}^\circ$ and $f^{(n)}\in\mathscr{L}[a,b]$. If $|f^{(n)}|^P$ is a (h_1,h_2)-convex function, then for $n,q\geq 1$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$
$$\leq\frac{(b-a)^n}{2n(n-1)^{\frac{1}{q}}}\left(\phi_1(t)|f^n(a)|^q+\phi_2(t)|f^n(b)|^q\right)^{\frac{1}{q}},$$

where

$$\phi_1(t):=\int_0^1 t^{q(n-1)}(n-2t)h_1(t)h_2(1-t)\mathrm{d}t, \tag{3.13}$$

and

$$\phi_2(t):=\int_0^1 t^{q(n-1)}(n-2t)h_1(1-t)h_2(t)\mathrm{d}t, \tag{3.14}$$

respectively.

Proof. Using Lemma 2, property of modulus, Holder's inequality and the fact that $|f^{(n)}|^q$ is an (h_1,h_2)-convex function, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$

$$
\begin{aligned}
&= \left| \frac{(b-a)^n}{2n!} \int_0^1 t^{n-1}(n-2t) f^{(n)}(ta+(1-t)b)\mathrm{d}t \right| \\
&\leq \frac{(b-a)^n}{2n!} \int_0^1 t^{n-1}(n-2t) |f^{(n)}(ta+(1-t)b)|\mathrm{d}t \\
&\leq \frac{(b-a)^n}{2n!} \left(\int_0^1 (n-2t)\mathrm{d}t \right)^{1-\frac{1}{q}} \left(\int_0^1 t^{q(n-1)}(n-2t) |f^{(n)}(ta+(1-t)b)|^q \mathrm{d}t \right)^{\frac{1}{q}} \\
&\leq \frac{(b-a)^n}{2n!} (n-1)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1 t^{q(n-1)}(n-2t)[h_1(t)h_2(1-t)|f^n(a)|^q + h_1(1-t)h_2(t)|f^n(b)|^q]\mathrm{d}t \right)^{\frac{1}{q}} \\
&= \frac{(b-a)^n}{2n(n-1)^{\frac{1}{q}}} \left(\phi_1(t)|f^n(a)|^q + \phi_2(t)|f^n(b)|^q \right)^{\frac{1}{q}}.
\end{aligned}
$$

This completes the proof. □

If $h_1(t) = t^{s_1}$ and $h_2(t) = t^{s_2}$ in Theorem 8, then we have

Corollary 15. *Let $f : \mathcal{I}^\circ \subset \mathbb{R} \to \mathbb{R}$, $a, b \in \mathcal{I}^\circ$ with $a < b$ where $\mathcal{I}^\circ$ is the interior of $\mathcal{I}^\circ$. Also suppose $f^{(n)}$ exists on $\mathcal{I}^\circ$ and $f^{(n)} \in \mathscr{L}[a,b]$. If $|f^{(n)}|^P$ is a Breckner type of (s_1,s_2)-convex function, then for $s_1, s_2 \in [0,1]$ and $n, q \geq 1$, we have*

$$
\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\mathrm{d}x - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right| \\
&\leq \frac{(b-a)^n}{2n(n-1)^{\frac{1}{q}}} \left(\phi_1'(t)|f^n(a)|^q + \phi_2'(t)|f^n(b)|^q \right)^{\frac{1}{q}},
\end{aligned}
$$

where

$$\phi_1'(t) := n\mathbb{B}(nq+s_1-q+1, s_2+1) - 2\mathbb{B}(nq+s_1-q+2, s_2+1),$$

and

$$\phi_2'(t) := n\mathbb{B}(nq+s_2-q+1, s_1+1) - 2\mathbb{B}(nq+s_2-q+2, s_1+1),$$

respectively.

If $h_1(t) = t^{-s_1}$ and $h_2(t) = t^{-s_2}$ in Theorem 8, then we have

Corollary 16. *Let $f : \mathcal{I}^\circ \subset \mathbb{R} \to \mathbb{R}$, $a,b \in \mathcal{I}^\circ$ with $a < b$ where $\mathcal{I}^\circ$ is the interior of $\mathcal{I}^\circ$. Also suppose $f^{(n)}$ exists on $\mathcal{I}^\circ$ and $f^{(n)} \in \mathscr{L}[a,b]$. If $|f^{(n)}|^P$ is a Godunova-Levin-Dragomir type of (s_1,s_2)-convex function, then for $s_1,s_2 \in [0,1]$ and $n,q \geq 1$, we have*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x - \sum_{k=2}^{n-1} \frac{(k-1)(b-a)^k}{2(k+1)!} f^{(k)}(a) \right|$$
$$\leq \frac{(b-a)^n}{2n(n-1)^{\frac{1}{q}}} \left(\phi_1''(t)|f^n(a)|^q + \phi_2''(t)|f^n(b)|^q\right)^{\frac{1}{q}},$$

where

$$\phi_1''(t) := n\mathbb{B}(nq-s_1-q+1,1-s_2) - 2\mathbb{B}(nq-s_1-q+2,1-s_2),$$

and

$$\phi_2''(t) := n\mathbb{B}(nq-s_2-q+1,1-s_1) - 2\mathbb{B}(nq-s_2-q+2,1-s_1),$$

respectively.

CONCLUSION

In this paper, we have introduced a new extension of convex functions which is called as (h_1,h_2)-convex functions. We have noticed that it contains some new and known classes of convex functions among one of those is h-convex functions. We have derived several new generalizations of Hermite-Hadamard type inequalities via (h_1,h_2)-convex functions. We have also discussed some of new special cases which can be easily deduced from our main results. It is expected that the interested readers may further explore the property of (h_1,h_2)-convexity of functions.

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