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CHARACTERIZATIONS OF STRONG WELL-POSEDNESS FOR A CLASS OF MULTI-VALUED VARIATIONAL INEQUALITIES

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Abstract. In this paper, by using the limiting subdifferential we consider the well-posedness for multi-valued variational inequalities and give some equivalence formulations for them. Moreover, we show that the strong well-posedness for a multi-valued variational inequality is equivalent to the existence and uniqueness of its solution.

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1. INTRODUCTION

The classical concept of well-posedness for a minimization problem, which has been known as the Tykhonov well-posedness, was introduced by Tykhonov [7] in 1966. A minimization problem is Tykhonov well-posed if it has a unique solution and every minimizing sequence of the problem converges to the unique solution. In the last decades, various concepts of well-posedness such as α-well-posedness, Hadamard well-posedness, Levitin-Polyak well-posedness and well-posedness by perturbations have been presented and studied for optimization problems, see [3, 4, 10, 12] and references therein. The concept of well-posedness for hemivariational inequality was first introduced by Goeleven and Mentagui [2] to provide some conditions guaranteing the existence and uniqueness of a solution for a hemivariational inequality. Later, Xiao and Huang [10] considered a concept of well-posedness for a variational-hemivariational inequality and obtained the equivalence of well-posedness between the variational-hemivariational inequality and the corresponding inclusion problem. Very recently, Xiao et al. [9] established two kinds of conditions under which the strong and weak well-posedness for the hemivariational inequality are equivalent to the existence and uniqueness of its solutions, respectively.

In this paper, by using the limiting subdifferential we extend the concept of well-posedness to a class of multi-valued variational inequality which include as a special case the classical variational and hemivariational inequalities. Moreover, we establish some equivalence results for them. The paper is organized as follows: Section 2 prepares briefly some preliminary notions and results used in sequel. In Section 3, we show that the strong well-posedness for a multi-valued variational inequality is equivalent to the existence and uniqueness of its solution. Also, a metric characterization for the strong well-posedness of multi-valued variational inequality is obtained.

2. Notations and preliminaries

Let X be a Banach space and X^* its topological dual space. The norm in X and X^* will be denoted by $||.||$. We denote $\langle .,.\rangle$, $[x,y]$ and $]x,y[$ the dual pair between X and X^*, the line segment for $x,y \in X$, and the interior of $[x,y]$, respectively. Now, we recall some concepts of subdifferentials that we need in the next section.

Definition 1 ([6])**.** Let X be a normed vector space, Ω be a nonempty subset of X, $x \in \Omega$ and $\varepsilon \geq 0$. The set of ε-normals to Ω at x is

$$\widehat{N}_{\varepsilon}(x;\Omega) := \{x^* \in X^* | \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u-x\rangle}{||u-x||} \leq \varepsilon\}.$$

Assume that $\overline{x} \in \Omega$, the limiting normal cone to Ω at $\overline{x}$ is

$$N(\bar{x};\Omega) := \limsup_{x \to \bar{x}, \varepsilon \downarrow 0} \widehat{N}_{\varepsilon}(x;\Omega).$$

Let $J : X \to \bar{\mathbb{R}}$ be finite at $\bar{x} \in X$; the limiting subdifferential of J at $\bar{x}$ is defined as follows

$$\partial_M J(\bar{x}) := \{x^* \in X^* | (x^*, -1) \in N((\bar{x}, J(\bar{x})); \text{epi} J)\}.$$

Remark 1 ([6]). The set-valued mapping $x \mapsto \partial_M J(x)$ has closed graph for locally Lipschitz functions.

Definition 2. Let $T : X \to 2^{X^*}$ be a set-valued mapping. T is said to be relaxed invariant monotone with respect to η if there exists a constant α such that for any $x,y \in X$ and any $u \in T(x), v \in T(y)$, one has

$$\langle v, \eta(x,y)\rangle + \langle u, \eta(y,x)\rangle \leq -\alpha(||\eta(x,y)||^2 + ||\eta(y,x)||^2).$$

Remark 2.

(1) When $T : X \to X^*$ is a single-valued operator, we obtain the definition of a relaxed invariant monotone operator.
(2) If $\alpha = 0$, then the Definition 2 reduces to the definition of an invariant monotone map.

Definition 3 ([11])**.** A mapping $T : X \to X^*$ is said to be hemicontinuous if for any $x_1, x_2 \in X$, the function $t \mapsto \langle T(x_1 + tx_2), x_2\rangle$ from $[0,1]$ into $]-\infty, +\infty[$ is continuous at 0_+.

Condition C ([5]). Let $\eta : X \times X \to X$. Then, for any $x,y \in X, \lambda \in [0,1]$

$$\eta(y, y + \lambda\eta(x,y)) = -\lambda\eta(x,y), \qquad \eta(x, y + \lambda\eta(x,y)) = (1-\lambda)\eta(x,y).$$

Remark 3. By some computation, we can see that if Condition C holds, then for any $x_1, x_2 \in X$ and $\lambda \in [0,1]$

$$\eta(x_1 + \lambda\eta(x_2, x_1), x_1) = \lambda\eta(x_2, x_1).$$

Now, suppose that $J : X \to \mathbb{R}$, $\eta : X \times X \to X$, $A : X \to X^*$ is a mapping and $f \in X^*$ is some given element. Consider the following multi-valued variational-like inequality associated with (A, f, J):
$MVLI(A, f, J)$: Find $\bar{x} \in X$ such that for any $x \in X$, there exists $\xi \in \partial_M J(\bar{x})$, that

$$\langle A\bar{x} - f + \xi, \eta(x, \bar{x})\rangle \geq 0.$$

Definition 4. A sequence $\{x_n\} \subset X$ is said to be an approximating sequence for the $MVLI(A, f, J)$, if there exists $\{\epsilon_n\}$ with $\epsilon_n \downarrow 0$ such that for any $x \in X$ there exists $x_n^* \in \partial_M J(x_n)$, that

$$\langle Ax_n - f + x_n^*, \eta(x, x_n)\rangle \geq -\epsilon_n ||\eta(x, x_n)||.$$

Definition 5. The multi-valued variational-like inequality $MVLI(A, f, J)$ is said to be strongly well-posed if it has a unique solution $\bar{x}$ on X and for every approximating sequence $\{x_n\}$, $\eta(\bar{x}, x_n)$ converges strongly to 0.

3. Main results

In this section, we establish some conditions under which the well-posedness for the multi-valued variational-like inequality is equivalent to the existence and uniqueness of its solution. Theorems in this section extend theorems in [8,9] from hemivariational inequalities with Clarke's subdifferential which is convex to multi-valued variational-like inequalities with limiting subdifferential which is not necessarily convex.

Theorem 1. *Assume that operator $A : X \to X^*$ is hemicontinuous and relaxed invariant monotone with constant c and J is locally Lipschitz such that $\partial_M J$ satisfies relaxed invariant monotonicity condition with constant α. Consider the following assertions:*

(i) *$\bar{x}$ is a solution of the $MVLI(A, f, J)$.*
(ii) *$\bar{x}$ is a solution of the following associated multi-valued variational-like inequality:*
$AMVLI(A, f, J)$: Find $\bar{x} \in X$ such that for any $x \in X$ there exists $x^* \in \partial_M J(x)$ such that $\langle Ax - f + x^*, \eta(x, \bar{x})\rangle \geq 0$.

If $c + \alpha \geq 0$ and η is skew, then $(i) \Rightarrow (ii)$. If η satisfies Condition C, then $(ii) \Rightarrow (i)$.

Proof. $(i) \Rightarrow (ii)$. Let $\bar{x} \in X$ be a solution of $MVLI(A, f, J)$. Hence, for any $x \in X$ there exists $\xi \in \partial_M J(\bar{x})$ such that

$$\langle A\bar{x} - f + \xi, \eta(x, \bar{x})\rangle \geq 0. \tag{3.1}$$

By the relaxed invariant monotonicity of $\partial_M J$, for any $x \in X$ and $x^* \in \partial_M J(x)$, one has

$$\langle x^*, \eta(\bar{x}, x)\rangle + \langle \xi, \eta(x, \bar{x})\rangle \leq -\alpha(||\eta(x, \bar{x})||^2 + ||\eta(\bar{x}, x)||^2). \tag{3.2}$$

It follows from relaxed invariant monotonicity of the operator A, (3.1) and (3.2), that

$$\begin{aligned}\langle Ax - f + x^*, \eta(x, \bar{x})\rangle &\geq \langle Ax + x^*, \eta(x, \bar{x})\rangle - \langle A\bar{x} + \xi, \eta(x, \bar{x})\rangle \\ &= -[\langle Ax, \eta(\bar{x}, x)\rangle + \langle A\bar{x}, \eta(x, \bar{x})\rangle + \langle x^*, \eta(\bar{x}, x)\rangle + \langle \xi, \eta(x, \bar{x})\rangle] \\ &\geq (c + \alpha)(||\eta(x, \bar{x})||^2 + ||\eta(\bar{x}, x)||^2),\end{aligned}$$

which shows that $\bar{x}$ is a solution of $AMVLI(A, f, J)$.
$(ii) \Rightarrow (i)$. Conversely, let $\bar{x}$ be a solution to the $AMVLI(A, f, J)$. Hence, for any $x \in X$ there exists $x^* \in \partial_M J(x)$ such that

$$\langle Ax - f + x^*, \eta(x, \bar{x})\rangle \geq 0. \tag{3.3}$$

For any $z \in X$ and $t \in [0, 1]$, set $x(t) = \bar{x} + t\eta(z, \bar{x})$ in inequality (3.3), we have

$$\langle A(\bar{x} + t\eta(z, \bar{x})) - f + x_t^*, \eta(\bar{x} + t\eta(z, \bar{x}), \bar{x})\rangle \geq 0,$$

that $x_t^* \in \partial_M J(x(t))$. It follows Condition C, that

$$\langle A(\bar{x} + t\eta(z, \bar{x})) - f + x_t^*, \eta(z, \bar{x})\rangle \geq 0.$$

Since J is locally Lipschitz, we deduce that $\partial_M J$ is locally bounded (Corollary 1.81 in [6]). Hence, there exists a neighborhood of $\bar{x}$ and a constant $\ell > 0$ such that for each z in this neighborhood and $x^* \in \partial_M J(z)$, we have $||x^*|| \leq \ell$. Since $x(t) \to \bar{x}$ when $t \to 0$ for t to be sufficiently small $||x_t^*|| \leq \ell$, without loss of generality we may assume that $x_t^* \to x^*$ in weak*-topology. Now, hemicontinuity of the operator A on X implies that

$$\langle A\bar{x} - f + x^*, \eta(z, \bar{x})\rangle \geq 0.$$

By the arbitrariness of $z \in X$, we conclude that $\bar{x}$ is a solution of $MVLI(A, f, J)$. □

Proposition 1. *Let $C^* \subset X^*$ be nonempty, closed, convex and bounded, $\varphi : X \to \mathbb{R}$ be proper, convex and lower semi-continuous and $y \in X$ be arbitrary. Assume that η is continuous and affine with respect to the first argument and for each $x \in X$, there exists $x^*(x) \in C^*$ such that*

$$\langle x^*(x), \eta(x, y)\rangle \geq \varphi(y) - \varphi(y + \eta(x, y)).$$

Then, there exists $y^* \in C^*$ such that

$$\langle y^*, \eta(x, y)\rangle \geq \varphi(y) - \varphi(y + \eta(x, y)), \quad \forall x \in X.$$

Proof. With some minor modification in the proof of Proposition 3.3 in [1], we can deduce the proof. □

Theorem 2. *Let $A : X \to X^*$ be relaxed invariant monotone with constant c and J a l.s.c. function that its limiting subdifferential satisfies relaxed invariant monotonicity condition with constant α. If $\alpha + c > 0$ and η is continuous and affine with respect to the second argument and skew, then $MVLI(A, f, J)$ is strongly well-posed if and only if it has a unique solution on X.*

Proof. The necessity is obvious. For the sufficiency, suppose that $MVLI(A, f, J)$ has a unique solution $\bar{x}$. Since $\bar{x}$ is the unique solution of $MVLI(A, f, J)$, for any $x \in X$, there exists $x^* \in \partial_M J(\bar{x})$ such that

$$\langle A\bar{x} - f + x^*, \eta(x, \bar{x})\rangle \geq 0. \tag{3.4}$$

Suppose that $\{x_n\}$ is an approximating sequence for $MVLI(A, f, J)$. It follows that there exists $\epsilon_n \downarrow 0$ such that for any $x \in X$, there exists $\xi_n(x) \in \partial_M J(x_n)$ that

$$\langle Ax_n - f + \xi_n(x), \eta(x, x_n)\rangle \geq -\epsilon_n ||\eta(x, x_n)||.$$

Now, consider the nonempty, convex and bounded set $\mathrm{co}\{Ax_n - f + \xi_n | \xi_n \in \partial_M J(x_n)\}$. Hence, it follows from Proposition 1 with $\varphi(x) = \epsilon_n ||x - x_n||$ that there exists ξ_n which is independent on x, such that

$$\langle Ax_n - f + \xi_n, \eta(x, x_n)\rangle \geq -\epsilon_n ||\eta(x, x_n)||, \quad \forall x \in X.$$

By choosing ξ_n, we can set $\xi_n = \sum_{i=1}^m \lambda_i \xi_n^i$ that $m \in \mathbb{N}$, $\sum_{i=1}^m \lambda_i = 1$ and $\xi_n^i \in \partial_M J(x_n)$. Hence,

$$\sum_{i=1}^m \lambda_i \langle Ax_n - f + \xi_n^i, \eta(x, x_n)\rangle \geq -\epsilon_n ||\eta(x, x_n)||, \quad \forall x \in X.$$

Now, set $x = \bar{x}$ in above inequality, yields

$$\sum_{i=1}^m \lambda_i \langle Ax_n - f + \xi_n^i, \eta(\bar{x}, x_n)\rangle \geq -\epsilon_n ||\eta(\bar{x}, x_n)||.$$

Hence, it follows from relaxed invariant monotonicity of the operator A, relaxed invariant monotonicity of the $\partial_M J$, the skewness of η and above inequality that

$$\begin{aligned}
-\epsilon_n ||\eta(\bar{x}, x_n)|| &\leq \sum_{i=1}^m \lambda_i \langle Ax_n - f + \xi_n^i, \eta(\bar{x}, x_n)\rangle \\
&\leq \sum_{i=1}^m \lambda_i [\langle Ax_n + \xi_n^i, \eta(\bar{x}, x_n)\rangle - \langle A\bar{x} + \zeta_n^i, \eta(\bar{x}, x_n)\rangle] \\
&= \sum_{i=1}^m \lambda_i [-\langle A\bar{x} - Ax_n + \zeta_n^i - \xi_n^i, \eta(\bar{x}, x_n)\rangle \\
&\leq -\sum_{i=1}^m 2\lambda_i (c + \alpha) ||\eta(\bar{x}, x_n)||^2 = -2(c + \alpha) ||\eta(\bar{x}, x_n)||^2,
\end{aligned}$$

where $\zeta_n^i \in \partial_M J(\bar{x})$ is obtained from (3.4) by setting $x = x_n$. Since $c + \alpha > 0$, it follows that

$$||\eta(\bar{x}, x_n)|| \leq \frac{\epsilon_n}{2(c+\alpha)}.$$

Taking limit at both sides of the above inequality, implies that $\eta(\bar{x}, x_n)$ converges strongly to 0. □

Example 1. Let $X = \mathbb{R}$, $f = 0$, A be the identity map and J be defined as

$$J(x) = \begin{cases} x^2 + 2x & \text{if } x > 0, \\ x^2 - x & \text{if } x \leq 0. \end{cases}$$

The limiting subdifferential of J is

$$\partial_M J(x) = \begin{cases} 2x+2 & \text{if } x > 0, \\ \text{[-1,2]} & \text{if } x = 0, \\ 2x-1 & \text{if } x < 0. \end{cases}$$

Let η be defined as $\eta(x, y) := k(x - y)$, such that $0 < k \leq 1$. Then by some computation we can see that $\partial_M J, A$ are relaxed invariant monotone with constants $\alpha = 1$, $c = \frac{1}{2}$, respectively. Hence, all assumptions of Theorem 2 are fulfilled and $\bar{x} = 0$ is a unique solution of $(MVLI)$ and therefore it is strongly well-posed.

For any $\epsilon > 0$, consider the following two sets:

$$\Omega(\epsilon) = \{\bar{x} : \forall x \in X,\ \exists x^* \in \partial_M J(\bar{x}) \text{ s.t. } \langle A\bar{x} - f + x^*, \eta(x, \bar{x})\rangle \geq -\epsilon||\eta(x, \bar{x})||\},$$

$$\Psi(\epsilon) = \{\bar{x} : \forall x \in X,\ \exists x^* \in \partial_M J(\bar{x}) \text{ s.t. } \langle -Ax, \eta(\bar{x}, x)\rangle + \langle -f + x^*, \eta(x, \bar{x})\rangle \geq -\epsilon||\eta(x, \bar{x})||\}.$$

Lemma 1. *Suppose that $A : X \to X^*$ is invariant monotone and hemicontinuous. Then $\Omega(\epsilon) = \Psi(\epsilon)$ for all $\epsilon > 0$.*

Proof. Taking into account the invariant monotonicity of mapping A, it is easy to obtain that $\Omega(\epsilon) \subset \Psi(\epsilon)$. For the other side suppose that $\bar{x} \in \Psi(\epsilon)$. Then, for any $x \in X$, there exists $x^* \in \partial_M J(\bar{x})$ such that

$$\langle -Ax, \eta(\bar{x}, x)\rangle + \langle -f + x^*, \eta(x, \bar{x})\rangle \geq -\epsilon||\eta(x, \bar{x})||. \tag{3.5}$$

Set $x = \bar{x} + t\eta(z, \bar{x})$ in (3.5), that $z \in X$ and $t \in [0, 1]$, yields

$$\langle -A(\bar{x} + t\eta(z, \bar{x})), \eta(\bar{x}, \bar{x} + t\eta(z, \bar{x}))\rangle + \langle -f + x^*, \eta(\bar{x} + t\eta(z, \bar{x}), \bar{x})\rangle \geq -\epsilon||\eta(\bar{x} + t\eta(z, \bar{x}), \bar{x})||.$$

By using Condition C, we obtain

$$\langle -A(\bar{x} + t\eta(z, \bar{x})), -\eta(z, \bar{x})\rangle + \langle -f + x^*, \eta(z, \bar{x})\rangle \geq -\epsilon||\eta(z, \bar{x})||.$$

Now, it follows from the hemicontinuity of mapping A that

$$\langle A\bar{x} - f + x^*, \eta(z, \bar{x})\rangle \geq -\epsilon||\eta(z, \bar{x})||,$$

which shows that $\bar{x} \in \Omega(\epsilon)$. This completes the proof. □

Lemma 2. *Suppose that $A : X \to X^*$ is a hemicontinuous mapping. If J is locally Lipschitz and η is continuous with respect to the second argument, then $\Omega(\epsilon)$ is closed in X for all $\epsilon > 0$.*

Proof. Let $\{x_n\} \subset \Omega(\epsilon)$ be a sequence such that $x_n \to \bar{x}$ in X. Then for any $x \in X$, there exists $x_n^* \in \partial_M J(x_n)$ such that

$$\langle Ax_n - f + x_n^*, \eta(x, x_n)\rangle \geq -\epsilon||\eta(x, x_n)||. \tag{3.6}$$

Since J is locally Lipschitz, there exists a subsequence of x_n^* that convergent to a $x^* \in \partial_M J(\bar{x})$ in weak*-topology. Consider (3.6) with this subsequence, taking limit at both sides of it and using this fact that A is hemicontinuous and η is continuous with respect to the second argument, we obtain

$$\langle A\bar{x} - f + x^*, \eta(x, \bar{x})\rangle \geq -\epsilon||\eta(x, \bar{x})||,$$

which implies that $\bar{x} \in \Omega(\epsilon)$. This completes the proof. □

Theorem 3. *Suppose that $A : X \to X^*$ is hemicontinuous and invariant monotone with respect to η that η is continuous with respect to the second argument, satisfies Condition C and skew. Then $MVLI(A, f, J)$ is strongly well-posed if and only if*

$$\Omega(\epsilon) \neq \varnothing, \forall \epsilon > 0 \quad \text{and} \quad diam(\Omega(\epsilon)) \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Proof. "Necessity" follows similarly from the first part of Theorem 3.1 in [8]. Hence, we prove the "Sufficiency". Suppose that $\{x_n\} \subset X$ is an approximating sequence for $MVLI(A, f, J)$. Then there exist a nonnegative sequence $\epsilon_n \to 0$ such that for any $x \in X$, there exists $\xi_n \in \partial_M J(x_n)$ such that

$$\langle Ax_n - f + \xi_n, \eta(x, x_n)\rangle \geq -\epsilon_n||\eta(x, x_n)||, \tag{3.7}$$

it means that $x_n \in \Omega(\epsilon_n)$. It follows from diam$(\Omega(\epsilon)) \to 0$, that $\{x_n\}$ is a cauchy sequence and so converges strongly to some point $\bar{x} \in X$. Since J is locally lipschitz, there exists a subsequence of ξ_n (e.g. $\{\xi_{n_i}\}$) that is convergent to a $\xi \in \partial_M J(\bar{x})$ in weak*-topology. Taking limit at the both side of (3.7) and using this fact that A is monotone and η is skew and continuous with respect to the second argument, we obtain

$$\begin{aligned}\langle Ax - f + \xi, \eta(x, \bar{x})\rangle &= \lim_{i\to\infty} \langle Ax - f + \xi_{n_i}, \eta(x, x_{n_i})\rangle \\ &\geq \lim_{i\to\infty} \langle Ax_{n_i} - f + \xi_{n_i}, \eta(x, x_{n_i})\rangle \\ &\geq \lim_{i\to\infty} -\epsilon_{n_i}||\eta(x, x_{n_i})|| = 0.\end{aligned}$$

Now, by using Theorem 1, $\bar{x}$ is a solution of $MVLI(A, f, J)$. Since, diam$(\Omega(\epsilon)) \to 0$ when $\epsilon \to 0$, $MVLI(A, f, J)$ has a unique solution. This completes the proof. □

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