Miskolc Mathematical Notes
Vol. 19 (2018), No. 1, pp. 569–579
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2018.1972

SHARP PARTIAL CLOSURE OPERATOR

BRANIMIR ŠEŠELJA, ANNA SLIVKOVÁ, AND ANDREJA TEPAVČEVIĆ

Received 18 April, 2016

Abstract. As an improvement of existing relationships among collections of sets, closure operators and posets, a particular, so called sharp partial closure operator (SPCO) is introduced. It is proved that there is always a unique SPCO corresponding to a given partial closure system. Moreover, an SPCO has the greatest domain among all partial operators corresponding to a given system. If it is a function, an SCPO is a classical closure operator. Dealing with partial closure systems, we introduce principal ones, corresponding to principal ideals of a poset and accordingly, we define principal SPCO's. Finally, we prove a representation theorem for posets in terms of principal SPCO's and principal partial closure systems.

2010 *Mathematics Subject Classification:* 06A15; 06A06

Keywords: partial closure operator, partial closure system, centralized system

1. INTRODUCTION

Connection among closure systems (Moore's families), closure operators and complete lattices is a well known topic in basics of order theory and lattices.

There is an analogue relationship among partial closure (centralized) systems, partial closure operators and posets. Still, the analogy is not full, since the correspondence among partial closure systems and partial closure operators is not unique, as in the case of lattices.

Closure operators and systems appear as a well known basic tool in the research of ordered sets, topology, universal algebra, logic, ... Among numerous relevant books, we mention [1, 11] as related to our work. Some particular papers dealing more closely with various aspects of closure systems and connections to ordered structures are given in References. Namely, papers [3, 4] give surveys on closure systems over finite sets, their properties and properties of the corresponding lattices. In [2], the lattice of particular completions of a finite poset is analyzed. Completions of a poset are also a subject of [8]. Extensive research of generalizations of closure systems and related posets, together with the corresponding properties is done by M. Erné (e.g., [5, 6]). In [7], the lattice of all Dedekind-MacNeille completions of posets with

Research supported by Ministry of Education, Science and Technical Development, Republic of Serbia, Grant No. 174013.

the fixed join-irreducibles is investigated. The number of closure systems on particular cardinalities is investigated in [9], and in [10], the lattice itself of such systems is described. Apart from implicit analysis connected with the corresponding collections of sets, there is not much research of closure operators which are not functions. In [12] a partial closure operator is defined as a special case of the definition in this work: there it is defined on downward-closed subsets (order-ideals) of a lattice (in our work we deal generally with arbitrary sets). In the mentioned paper, partial closure operators are used in developing the semantic foundations of concurrent constraint computing. Our present work is based on our paper [13], moreover it is an extension of this previous one.

In this paper, we present an improvement of existing relationships among collections of sets, closure operators and posets. In order to make the correspondence among partial closure systems and partial closure operators unique, we introduce a particular, so called sharp partial closure operator (SPCO). This is a partial operation on the power set, fulfilling closure like axioms, plus additional one, called sharpness. We prove that there is always a unique SPCO corresponding to a given partial closure system. Moreover, an SPCO has the greatest domain among all partial operators corresponding to a given system. In addition, if an SCPO is a function, then it is a classical closure operator. Dealing with collections of subsets as a counterpart of operators, we analyze partial closure, or centralized systems (point closures). Among these we introduce so called principal ones, corresponding to principal ideals of a poset. Accordingly, we define principal SPCO's. Finally, we prove a representation theorem for posets in terms of principal SPCO's and principal partial closure systems.

By these results we establish bijective correspondences among posets, principal SPCO's and principal partial closure operators.

2. Preliminaries

We start with well known notions and basic properties of closure systems and closure operators, pointing to our notation.

As it is known, a **closure system** (Moore's family) $\mathcal{F}$ on a nonempty set S is a collection of subsets of S, which is closed under arbitrary set intersections.

A **closure operator** on a nonempty set S is a unary operation $X \mapsto \overline{X}$ on the power set $P(\mathcal{S})$, which for all $X, Y \subseteq S$ fulfils properties

$$X \subseteq \overline{X}, \quad X \subseteq Y \text{ implies } \overline{X} \subseteq \overline{Y}, \quad \overline{\overline{X}} = \overline{X}.$$

As usual, if $X \subseteq S$ and $\overline{X} = X$, then X is a **closed** set. The family of closed sets $\mathcal{F}_C$ is the **range** of a closure operator.

Recall that the range of a closure operator on S is a closure system on the same set. On the other hand, if $\mathcal{F}$ is a closure system on S, then the map $x \mapsto \bigcap\{Y \in \mathcal{F} \mid$

$x \in Y\}$, for $x \in S$, is a closure operator on S. This correspondence among closure systems and corresponding operators is unique.

A closure system is a complete lattice under inclusion, and as a converse, the collection of principal ideals of a lattice is a closure system, which is, when equipped by inclusion, order isomorphic with the lattice itself. Still, the closure system of principal ideals is not the only closure system isomorphic to a given lattice.

3. Partial closure operators and systems

Our aim is to establish a particular relationship among collections of sets, operators and posets. This relationship should be analogue (as much as possible) to the one among closure operators, closure systems and complete lattices. We use the relevant known results in this field, and the basic definitions and properties are those given in [13]. Still, our present approach brings some new requirements, which enable essential improvements of the mentioned relationship.

For a nonempty set S, let $C : \mathcal{P}(S) \to \mathcal{P}(S)$ be a partial mapping satisfying:

Pc_1: If $C(X)$ is defined, then $X \subseteq C(X)$.
Pc_2: If $C(X)$ and $C(Y)$ are defined, then $X \subseteq Y$ implies $C(X) \subseteq C(Y)$.
Pc_3: If $C(X)$ is defined, then $C(C(X))$ is also defined and $C(C(X)) = C(X)$.
Pc_4: $C(\{x\})$ is defined for every $x \in S$.

As defined in [13], a partial mapping C fulfilling properties Pc_1–Pc_4 is a **partial closure operator** on S.

As usual, if $X \subseteq S$ and $C(X) = X$, then we call X a **closed** set. The family of closed sets $\mathcal{F}_C$ is called the **range** of a partial closure operator C. The **exact domain** of a partial closure operator C on S is denoted by $Dom(C)$:

$$Dom(C) := \{X \mid X \subseteq S \text{ and } C(X) \text{ is defined}\}.$$

Let C be a partial closure operator on S. If $C(X)$ is defined, then it is straightforward to check that, equivalently to the same property of a closure operator (which is not partial),

$$C(X) = \bigcap\{Y \in \mathcal{F}_C \mid X \subseteq Y\}. \tag{3.1}$$

We say that a partial closure operator C on S is **sharp**, if it satisfies the condition:

Pc_5: Let $B \subseteq S$. If $\bigcap\{X \in \mathcal{F}_C \mid B \subseteq X\} \in \mathcal{F}_C$, then $C(B)$ is defined and

$$C(B) = \bigcap\{X \in \mathcal{F}_C \mid B \subseteq X\} \quad \text{(sharpness)}.$$

We also say that a partial operator on S, fulfilling properties Pc_1–Pc_5 is an **SPCO** on S.

Notice that if in Pc_5 there does not exist a set $X \in \mathcal{F}_C$ such that $B \subseteq X$, then straightforwardly $C(B)$ is not defined (because of $B \subseteq C(B)$).

Observe also that a closure operator C on S (i.e., an operator which is a function) trivially fulfils condition Pc_5, which reduces to the condition (3.1).

Remark 1. By (3.1) the converse implication in the condition Pc_5 is always valid.

A partial closure operator C on a set S is **complete**, if it satisfies:

Pc_6: If $\{X_i \mid i \in I\}$ is a chain and $C(X_i)$ is defined for every $i \in I$, then also $C(\bigcup X_i)$ is defined.

In addition, C is **algebraic** if it is complete and satisfies the following:

Pc_7: If $C(X)$ is defined, then

$$\{C(Y) \mid Y \subseteq X, C(Y) \text{ is defined and } Y \text{ is finite}\} \text{ is a directed set,} \tag{3.2}$$

$$\text{and } C(X) = \bigcup\{C(Y) \mid Y \subseteq X, C(Y) \text{ is defined and } Y \text{ is finite}\}. \tag{3.3}$$

We note that the condition Pc_5 can not be derived from the conditions Pc_1–Pc_4, as shown by the following example.

Example 1. Let C be a partial mapping defined on $\{a,b,c\}$ with

$$C : \begin{pmatrix} \{a\} & \{b\} & \{c\} & \{a,b,c\} \\ \{a\} & \{b\} & \{a,b,c\} & \{a,b,c\} \end{pmatrix}.$$

It is straightforward to check that C satisfies conditions Pc_1–Pc_4, but the property Pc_5 does not hold because $C(\{a,b\})$ is not defined.

From the same example, it follows that Pc_5 can neither be derived from the above conditions, to which Pc_6 and Pc_7 are added.

Further, neither of the conditions Pc_6 and Pc_7 can be derived from Pc_1–Pc_5, as shown by the following example.

Example 2. Let C be a partial mapping defined on $\mathbb{N}$ by

$$C(X) = \begin{cases} X, & \text{if } X \text{ is a finite subset of } \mathbb{N}; \\ E_1, & \text{if } X \text{ is an infinite subset of } E_1; \end{cases}$$

where E is the set of all even natural numbers and $E_1 = E \cup \{1\}$. This is a sharp partial closure operator, but it is not complete. Indeed, consider the family $\{X_i \mid i \in \mathbb{N}\}$, where $X_i = \{1, 2, \ldots, i\}$. This family is a chain and $C(X_i)$ is defined for every $i \in \mathbb{N}$, but $C(\bigcup_{i \in \mathbb{N}} X_i) = C(\mathbb{N})$ is not defined.

The constructed example does not satisfy Pc_7 either. Indeed, $C(E) = E_1$, but there does not exist a finite subset of even numbers that contains 1, hence we cannot represent $C(E)$ as the union of closures of all finite subsets of E.

Next we deal with the set counterpart of partial closure operators.

A **partial closure system** (in the literature known also as a *centralized system*, e. g. [5,6]) is a family $\mathcal{F}$ of subsets of a nonempty set S satisfying:

$$\text{For every } x \in S,\ \bigcap(X \in \mathcal{F} \mid x \in X) \in \mathcal{F}. \tag{3.4}$$

We say that the set $\bigcap(X \in \mathcal{F} \mid x \in X)$ is a **centralized intersection** for $x \in S$. Observe that from the above definition it follows that a partial closure system $\mathcal{F}$ on S also satisfies $\bigcup \mathcal{F} = S$.

The following is a refinement of a theorem from [13].

Theorem 1. *The range of a partial closure operator on a set S is a partial closure system.*

Conversely, for every partial closure system $\mathcal{F}$ on S, there is a unique sharp partial closure operator on S such that its range is $\mathcal{F}$.

Proof. The first part of this theorem was proved in [13], still we repeat it here for the sake of completeness.

Let C be a partial closure operator on a set S and $\mathcal{F}_C$ be its range.

For $x \in S$, let $\mathcal{F}_x := \{X \in \mathcal{F}_C \mid x \in X\}$. We need to show that $\bigcap \mathcal{F}_x \in \mathcal{F}_C$. If $X \in \mathcal{F}_x$, then $X = C(X)$ and $x \in X$, therefore $\{x\} \subseteq X$ and $C(\{x\}) \subseteq C(X) = X$, hence $C(\{x\}) \subseteq \bigcap \mathcal{F}_x$. Since we have $C(\{x\}) \in \mathcal{F}_x$, it follows that $C(\{x\}) = \bigcap \mathcal{F}_x$ and condition (3.4) holds.

For the other direction, let $\mathcal{F}$ be a partial closure system on a set S. We define partial mapping $C : \mathcal{P}(S) \to \mathcal{P}(S)$ as follows:

$$C(X) := \bigcap \{Y \in \mathcal{F} \mid X \subseteq Y\}$$

if intersection on the right-hand side is in $\mathcal{F}$, otherwise $C(X)$ is not defined.

If, for some $X \subseteq S$, the closure $C(X)$ is defined, then it is easy to see that C has properties Pc_1–Pc_3. The property Pc_4 holds because, for $x \in S$, $C(\{x\})$ is defined by (3.4). Hence, C is a partial closure operator on S. Now we show that C is sharp, i.e., that also Pc_5 holds. Let $B \subseteq S$ and assume that

$$\bigcap \{X \in \mathcal{F}_C \mid B \subseteq X\} \in \mathcal{F}_C.$$

Then, by the definition of C, this partial operator fulfills Pc_5 and the range of C is $\mathcal{F}$.

It remains to show that the SPCO defined in this way is the unique partial mapping with the range $\mathcal{F}$ satisfying properties Pc_1–Pc_5. Assume that there exists another partial mapping $K : \mathcal{P}(S) \to \mathcal{P}(S)$ satisfying Pc_1–Pc_5 and that the range of K is also $\mathcal{F}$. We prove that $\mathcal{F}_C = \mathcal{F}_K$. Namely, we show that for $X \subseteq S$, $K(X)$ is defined if and only if $C(X)$ is defined. If $K(X)$ is defined, by Remark 1, $\bigcap\{Y \in \mathcal{F}_K \mid X \subseteq Y\} \in \mathcal{F}_K$ and $K(X) = \bigcap\{Y \in \mathcal{F}_K \mid X \subseteq Y\}$. Since $\mathcal{F}_K = \mathcal{F}_C$, $\bigcap\{Y \in \mathcal{F}_K \mid X \subseteq Y\} = \bigcap\{Y \in \mathcal{F}_C \mid X \subseteq Y\}$ and hence $\bigcap\{Y \in \mathcal{F}_C \mid X \subseteq Y\} \in \mathcal{F}_C$ and by the condition Pc_5, $C(X)$ is defined and $C(X) = K(X)$. If we suppose that $C(X)$ is defined, similarly we obtain that $K(X)$ is defined and $C(X) = K(X)$.

defined, we need to prove that $\bigcap\{Y \in \mathcal{F} \mid X \subseteq Y\}$ belongs to $\mathcal{F}$. Since the range of operator K is $\mathcal{F}$, for every $Y \in \mathcal{F}$ there exists $U \in \mathcal{P}(S)$ such that $K(U) = Y$. Therefore, we need to show that such that $C(Y)$ is defined (that is, $C(Y) \in \mathcal{F}$) there exists $U \subseteq S$ such that $C(Y) = K(U)$, we have $\{C(Y) \mid X \subseteq C(Y)\} = \{K(U) \mid X \subseteq K(U)\}$ and then K(U)}=K(X).

S, and they agree on these subsets. This completes the proof. □

Example 3. Let C_s be a partial mapping defined on $\{a,b,c\}$ with

$$C_s : \begin{pmatrix} \{a\} & \{b\} & \{c\} & \{a,b\} & \{a,c\} & \{b,c\} & \{a,b,c\} \\ \{a\} & \{b\} & \{a,b,c\} & \{a,b,c\} & \{a,b,c\} & \{a,b,c\} & \{a,b,c\} \end{pmatrix}.$$

This partial mapping is an SPCO on the set $\{a,b,c\}$. Note that the range $\mathcal{F}_{C_s}$ here is equal to the range $\mathcal{F}_C$ of the partial closure operator from Example 1. This implies that there is no 1-1 correspondence between partial closure operators and partial closure systems. However, as proven in Theorem 1, there is a bijective correspondence between SPCO's and partial closure systems.

By the above, it is clear that for a given partial closure system $\mathcal{F}$ on S, there is a collection of partial closure operators on S whose range is $\mathcal{F}$, among which, by Theorem 1, precisely one is sharp. In addition, the latter is maximal in the following sense.

Proposition 1. *Let $\mathcal{F}$ be a partial closure system on S. The sharp partial closure operator has the greatest domain among all partial closure operators whose range is $\mathcal{F}$. In addition, if D is a partial closure operator and C the sharp closure operator with the same domain, then $C(A) = D(A)$, for all $A \subseteq S$ for which D is defined.*

Proof. Let D be an arbitrary partial closure operator whose range is $\mathcal{F}$, and let C be the sharp one with the same range $\mathcal{F}$. Now, if $A \subseteq S$ and $D(A)$ is defined, i.e., $A \in Dom(D)$, then $C(D(A)) = D(A)$, since the ranges of C and D coincide by assumption.

We have that

$$D(A) = \bigcap(X \in \mathcal{F} \mid A \subseteq X) \in \mathcal{F}.$$

By the Pc_5, it directly follows that $C(A)$ is defined and $C(A) = \bigcap(X \in \mathcal{F} \mid A \subseteq X)$. Hence, $C(A) = D(A)$. □

The sharp partial closure operator is a natural generalization of the closure operator, as follows.

Theorem 2. *If the range $\mathcal{F}$ of a sharp partial closure operator C on a set S forms a complete lattice with respect to set inclusion, then C is a function. Conversely, if C is a closure operator on S, then it is sharp.*

Proof. Let $X \subseteq S$. We have $X \subseteq \bigcup\{C(\{x\}) \mid x \in X\}$, and since the range $\mathcal{F}$ is a complete lattice, the supremum of the collection $\{C(\{x\}) \mid x \in X\}$ exists and contains its union, which implies that $\bigvee\{C(\{x\}) \mid x \in X\} \in \mathcal{F}$. If $X \subseteq Y$ for a set Y such that $Y \subseteq \mathcal{F}$, then $\bigvee\{C(\{x\}) \mid x \in X\} \subseteq Y$. Indeed, for every $x \in X$, $C(\{x\}) \subseteq Y$. Hence,

$\bigcap\{Y \in \mathcal{F} \mid X \subseteq Y\} = \bigvee\{C(\{x\}) \mid x \in X\}$. By Pc_5 we have that $C(X)$ is defined, so C is a function and $C(X) = \bigvee\{C(\{x\}) \mid x \in X\}$.

Suppose now that C is a closure operator. Then its range forms a complete lattice with respect to a set inclusion [1]. Let $B \subseteq S$. The closure $C(B)$ is defined because C is a function, and it is obvious that it satisfies Pc_5. □

As shown in paper [13], a completion of a partial closure system to a closure system is equivalent to Dedekind MacNeille completion. Here we present a completion of any nonempty collection of subset of S to a partial closure system. Clearly, by adding all singletons of S, we get a partial closure system, but then the existing centralized intersections may not be preserved. Therefore, we introduce another completion, as follows.

For an arbitrary nonempty collection $\mathcal{F}$ of subsets of a set S, we define an extension $\widehat{\mathcal{F}} \subseteq \mathcal{P}(S)$ as follows:

$$\widehat{\mathcal{F}} := \mathcal{F} \cup \{\bigcap_{x \in Y} Y \in \mathcal{F} \mid x \in S\}.$$

Example 4. Let
$S = \{a,b,c,d,e,f,g\}$ and
$\mathcal{F} = \{\{b\},\{c\},\{e\},\{a,b,c\},\{b,c,d,e,f\},\{e,f,g\}\}$.
Then $\widehat{\mathcal{F}} = \mathcal{F} \cup \{\{e,f\}\}$.

The following is a straightforward consequence of the definition of $\widehat{\mathcal{F}}$.

Proposition 2. *For an arbitrary nonempty collection $\mathcal{F}$ of a set S, the extension $\widehat{\mathcal{F}}$ is a partial closure system on S which preserves all intersections and centralized intersections existing in $\mathcal{F}$.*

Recall that the collection of all principal ideals of a complete lattice L is a closure system which is, when ordered by inclusion, order isomorphic with L under the mapping $i(x) = \downarrow x$, $x \in L$. In addition, this closure system consists of closed sets under the corresponding closure operator.

However, it is clear that not every closure system is isomorphic with a collection of all principal ideals of a complete lattice L.

The analogue statement is true for posets and related partial closure operators and partial closure systems.

In the following we introduce a special type of partial closure systems which are isomorphic to collections of all principal ideals in posets.

We say that a partial closure system $\mathcal{F}$ on a nonempty set S is **principal** if $\varnothing \notin \mathcal{F}$ and for every $X \in \mathcal{F}$ we have

$$\left| X \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X\} \right| = 1. \tag{3.5}$$

Our main motivation for the above definition, as already mentioned, are principal ideals in a poset.

Proposition 3. *Let $(S,\leqslant)$ be a poset. Then the family $\{\downarrow x \mid x \in S\}$ of principal ideals is a principal partial closure system.*

Proof. It is easy to see that condition (3.4) holds and that $\varnothing \notin \{\downarrow x \mid x \in S\} = \mathcal{F}$. Let us show that for every $\downarrow x \in \mathcal{F}$ we have $|\downarrow x \setminus \bigcup\{\downarrow y \in \mathcal{F} \mid \downarrow y \subsetneq \downarrow x\}| = 1$.

Obviously, $x \in \downarrow x \setminus \bigcup\{\downarrow y \in \mathcal{F} \mid \downarrow y \subsetneq \downarrow x\}$. Suppose that there is element an $z \neq x$ such that $z \in \downarrow x \setminus \bigcup\{\downarrow y \in \mathcal{F} \mid \downarrow y \subsetneq \downarrow x\}$. It follows that $z < x$, therefore $\downarrow z \in \{\downarrow y \in \mathcal{F} \mid \downarrow y \subsetneq \downarrow x\}$, which is a contradiction with $z \notin \bigcup\{\downarrow y \in \mathcal{F} \mid \downarrow y \subsetneq \downarrow x\}$. □

Let $\mathcal{F}$ be a principal partial closure system on a set S. In order to prove the opposite connection of principal partial closure systems and principal ideals in a poset, we introduce a mapping:

$G : \mathcal{F} \to S$ defined by

$$G(X) = x, \text{ where } x \in X \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X\}. \tag{3.6}$$

The mapping is well defined by the definition of the principal partial closure system.

Proposition 4. *If $\mathcal{F}$ is a principal partial closure system on a set S then the mapping $G : \mathcal{F} \to S$ defined by (3.6) is a bijection.*

Proof. First, let $X_1, X_2 \in \mathcal{F}$ such that $G(X_1) = G(X_2)$. Therefore, there exists $x \in S$ such that $\{x\} = X_1 \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X_1\} = X_2 \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X_2\}$. Since $\mathcal{F}$ is a partial closure operator, a set $T = \bigcap\{Z \in \mathcal{F} \mid x \in Z\}$ is in $\mathcal{F}$. Hence, $T \subseteq X_1 \cap X_2$. Since $x \in T$, we have that $T \notin \{Y \in \mathcal{F} \mid Y \subsetneq X_1\}$. By $T \subseteq X_1 \cap X_2 \subseteq X_1$, it follows that $T = X_1$. Similarly, we have $T = X_2$ and then $X_1 = X_2$, which implies that the mapping G is injective.

Now, let $x \in S$ and denote $X_x = \bigcap\{X \in \mathcal{F} \mid x \in X\}$. Since $\mathcal{F}$ is a partial closure system, we have $X_x \in \mathcal{F}$, and we shall show that $G(X_x) = x$. We have $x \in X_x$ and $x \notin \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X_x\}$ because X_x is the smallest set (with respect to set inclusion) in $\mathcal{F}$ that contains x. Since $|X_x \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X_x\}| = 1$, it follows that $\{x\} = X_x \setminus \bigcup\{Y \in \mathcal{F} \mid Y \subsetneq X_x\}$. Hence G is also a surjective mapping. □

Using the introduced bijection G, an order on S can be naturally induced by the set inclusion in a principal partial closure system $\mathcal{F}$ on S, as follows: for all $x, y \in S$,

$$x \leqslant y \text{ if and only if } G^{-1}(x) \subseteq G^{-1}(y). \tag{3.7}$$

It is straightforward to check that $\leqslant$ is an order on S. Therefore, as a consequence of Proposition 4, we get the following.

Corollary 1. *Let $\mathcal{F}$ be a principal partial closure system on a set S, and $\leqslant$ the order on S, defined by (3.7). Then, the function G defined by (3.6) is an order isomorphism from $(\mathcal{F},\subseteq)$ to $(S,\leqslant)$. In addition, the collection of principal ideals in $(S,\leqslant)$ is $\mathcal{F}$.*

Proof. The function G is a bijection by Proposition 4, which is, by the definition of $\leqslant$ on S, compatible with the corresponding orders. In other words, if $X, Y \in \mathcal{F}$, we have that $X \subseteq Y$ if and only if $G(X) \leqslant G(Y)$. To prove that subsets in $\mathcal{F}$ are principal ideals, for $x \in S$, we use the denotation from Proposition 4, $G^{-1}(x) = X_x$. We will prove that $\downarrow x = X_x$. If $y \leqslant x$, then $G^{-1}(y) \subseteq G^{-1}(x)$ and since $y \in G^{-1}(y)$, we have that $y \in G^{-1}(x)$. On the other hand, suppose that $y \in X_x$. Then, $X_y \subseteq X_x$ and hence, $y \leqslant x$. by the definition of $\leqslant$ on S, if $y \in X$, then either $y = x$ or $y \in X \setminus \{x\}$. Therefore, $X = \downarrow x$ with respect to the order $\leqslant$. Since G is a bijection, all the elements from $\mathcal{F}$ are in the form X_x for $x \in X$, so all of them coincides with the principal ideals of $(S, \leqslant)$. □

We can also start from a poset, and via principal ideals we get a partial closure system, which induces the starting order, as follows.

Corollary 2. *Let $(S, \leqslant)$ be a poset and $\mathcal{F}$ a partial closure system consisting of its principal ideals. Then, the order on S defined by (3.7) coincides with $\leqslant$.*

Proof. By Proposition 3, principal ideals make a principal partial closure system. The function G defined by (3.6) associates to every principal ideal its generator, and by (3.7), inclusion among principal ideals induces the existing order $\leqslant$ from the poset. □

Finally, we introduce a partial closure operator which corresponds to a principal partial closure system.

A partial closure operator C on S is **principal** if it satisfies

Pc_8: If $X = C(X)$, then there exists unique $x \in X$ such that
$x \notin \bigcup\{Y \in \mathcal{F}_C \mid Y \subsetneq X\}$.

It is easy to see that *the axioms Pc_5 and Pc_8 are independent.*

A connection among these notions can be explained as follows.

The range of a principal closure operator is a principal partial closure system and the sharp partial closure operator obtained from a principal partial closure system, as defined in Theorem 1, is principal.

Obviously, the empty set can not be closed under a principal partial closure operator. As an additional property, we prove that the range of a principal partial closure operator consists of closures of singletons.

Proposition 5. *Let C be a principal partial closure operator on S. If $X \in \mathcal{F}_C$, then there exists $x \in X$ such that $C(\{x\}) = X$.*

Proof. If X is a closed set, then by Pc_8 there exists a unique $x \in X$ such that $x \notin \bigcup\{Y \in \mathcal{F}_C \mid Y \subsetneq X\}$. From $x \in C(\{x\}) \subseteq C(X) = X$ it follows that $C(\{x\}) = X$. □

The following is a *Representation theorem* of posets by SPCO's and by the corresponding partial closure systems.

Theorem 3. *Let $(S,\leqslant)$ be a poset. The partial mapping $C:\mathcal{P}(S)\to\mathcal{P}(S)$ defined by*

$$C(X)=\downarrow(\bigvee X),\ \textit{if there exists } \bigvee X,$$

otherwise not defined, is a principal SPCO. The corresponding partial closure system is principal and it is isomorphic with S.

Proof. It is straightforward to check that C is a partial closure operator. In order to prove that it is sharp, suppose that $B\subseteq S$ and that

$$\bigcap\{X\in\mathcal{F}_C \mid B\subseteq X\}\in\mathcal{F}_C.$$

Then, there is a set $Z\subseteq S$, such that $\bigcap\{X\in\mathcal{F}_C\mid B\subseteq X\}=\downarrow(\bigvee Z)$. Consequently, for every $b\in B$, $b\leq\downarrow(\bigvee Z)$. Suppose there is another upper bound of B, say x. Then $B\subseteq\downarrow x$ and $C(\downarrow x)=\downarrow x$. Hence, $\downarrow(\bigvee Z)\subseteq\downarrow x$ and $\bigvee Z\leq x$. Therefore, $C(B)=\downarrow(\bigvee Z)=C(Z)$.

It is easy to see that C is principal by the definition.

Closed elements are principal ideals of S, hence the corresponding partial closure system is isomorphic with S. □

To sum up, we have bijective correspondences among:

- posets
- principal sharp partial closure operators
- principal partial closure systems.

Indeed, correspondences are witnessed by Theorem 3; they are bijective by Theorem 1, Propositions 3, 4 and Corollaries 1, 2.

In particular, if we deal with posets which are complete lattices, then the bijective correspondence already exists among closure systems and closure operators. As mentioned, every closure operator fulfils the sharpness property. Still, to every lattice there correspond more closure operators and systems. If the closure operators and systems are principal, then we get bijective correspondences as for posets.

References

[1] H. P. B.A. Davey, *Introduction to Lattices and Order.* Cambridge: Cambridge University Press, 1992.

[2] G. Bordalo and B. Monjardet, "The lattice of strict completions of a finite poset." *Algebra univers.*, vol. 47, pp. 183–200, 2002.

[3] N. Caspard and B. Monjardet, "The lattices of Moore families and closure operators on a finite set : a survey." *Electronic Notes in Discrete Mathematics*, vol. 2, pp. 25–50, 1999.

[4] N. Caspard and B. Monjardet, "The lattices of closure systems, closure operators, and implicational systems on a finite set: a survey." *Discrete Applied Mathematics.*, vol. 127, pp. 241 – 269, 2003.

[5] M. Erné, "Compact generation in partially ordered sets." *J. Austral. Math. Soc. (Series A).*, vol. 42, pp. 69–83, 1987.

[6] M. Erné, *Algebraic ordered sets and their generalizations, in: Algebras and orders* . Netherlands: Springer, 1993.
[7] M. Erné, B. Šešelja, and A. Tepavčević, "Posets generated by irreducible elements." *Order.*, vol. 20, pp. 79–89, 2003.
[8] M. Gehrke, R. Jansana, and A. Palmigiano, "Δ_1-completions of a Poset." *Order.*, vol. 30, pp. 39–64, 2013.
[9] M. Habib and L. Nourine, "The number of Moore families on n=6." *Discrete Math.*, vol. 294, pp. 291 – 296, 2005.
[10] A. Higuchi, "Lattices of closure operators." *Discrete Math.*, vol. 179, pp. 267–272, 1998.
[11] S. W. K. Denecke, *Universal Algebra and Applications in Theorethical Computer Science.* Boca Raton: Chapman and Hall, 2002.
[12] V. Saraswat, M. Rinard, and P. Panangaden, "Semantic foundations of concurrent constraint programming." *Proc. 18th ACM Symp. on Principles of Programming Languages.*, pp. 333–352, 1991.
[13] B. Šešelja and A. Tepavčević, "Posets via partial closure operators." *Contributions to General Algebra 12, Verlag Johannes Heyn, Klagenfurt.*, vol. 12, pp. 371–375, 2000.

Authors' addresses

Branimir Šešelja
University of Novi Sad, Faculty of Sciences, Department of Mathematics and Informatics, Faculty of Sciences, Trg Dositeja Obradovića 4, Novi Sad 21000, Serbia,
E-mail address: seselja@dmi.uns.ac.rs

Anna Slivková
University of Novi Sad, Faculty of Sciences, Department of Mathematics and Informatics, Faculty of Sciences, Trg Dositeja Obradovića 4, Novi Sad 21000, Serbia,
E-mail address: anna.slivkova@dmi.uns.ac.rs

Andreja Tepavčević
University of Novi Sad, Faculty of Sciences, Department of Mathematics and Informatics, Faculty of Sciences, Trg Dositeja Obradovića 4, Novi Sad 21000, Serbia,
E-mail address: andreja@dmi.uns.ac.rs