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A SHORT NOTE ON NESTED SUMS

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Abstract. We express a set of nested sums as a 1-index weighted sum using Stars and Bars Theorem.

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1. Introduction

Nested sums appear in various fields of research. Here, we consider the following summation:

$$\sum_{a_1=0}^{N}\sum_{a_2=0}^{a_1}\sum_{a_3=0}^{a_2}\cdots\sum_{a_n=0}^{a_{n-1}} f(\alpha_1 a_1+\alpha_2 a_2+\cdots+\alpha_n a_n) \tag{1.1}$$

where $f(m)$ may be any map defined for $0 \le m \le 2N$. Under certain constraints on the coefficients α_i, $i \in \{1,\dots,n\}$, we simplify the expression to the form $\sum_{\gamma} f(\gamma)P(\gamma)$ where $P(\gamma)$ is the number of occurrences of γ within the sum. Surprisingly, the authors could not find any formula concerning such expressions. Consider for example the sum:

$$S=\sum_{a_1=0}^{N}\sum_{a_2=0}^{a_1}\sum_{a_3=0}^{a_2}\sum_{a_4=0}^{a_3} f(2a_3-a_4)\ ,$$

applying Theorem 1 below, simplifies the expression to:

$$S=\sum_{\gamma=0}^{2N}\sum_{\phi=\phi_{min}(\gamma)}^{\phi_{max}(\gamma)}\binom{N-\gamma+\phi+2}{2} f(\gamma)$$

where, $\phi_{min}(\gamma)=\max\{0,\gamma-N\}$ and $\phi_{max}(\gamma)=\min\{N,\lfloor\gamma/2\rfloor\}$.

2. Discussion

Denote $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)^T$ and $\underline{a} = (a_1, \ldots, a_n)$ and $\gamma = \underline{\alpha} \cdot \underline{a}$ we set:

$$\begin{aligned} x_1 &= N - a_1, \\ x_2 &= a_1 - a_2, \\ &\vdots \\ x_n &= a_{n-1} - a_n, \\ x_{n+1} &= a_n \end{aligned}$$

so

$$N = x_1 + x_2 + \cdots + x_{n+1} \tag{2.1}$$

where $0 \leq x_i \leq N, i \in \{1, \ldots, n+1\}$. This implies that:

$$\gamma = N\beta_1 - x_1\beta_1 - x_2\beta_2 \cdots - x_n\beta_n$$

where $\beta_j = \sum_{i=j}^{n} \alpha_i$.

Substituting into Eq. 2.1 we get the *Frobenius Equations* (cf. [2]):

$$N\beta_1 - \gamma = x_1\beta_1 + \ldots + x_n\beta_n \tag{2.2}$$

The number of integer solutions to this equation yields the following:

Theorem 1. *Consider the sum:*

$$S = \sum_{a_1=0}^{N} \sum_{a_2=0}^{a_1} \sum_{a_3=0}^{a_2} \cdots \sum_{a_n=0}^{a_{n-1}} f(\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n)$$

Denote by $\mathcal{I}^+$ the set of indices of coefficients $\beta_j = \sum_{i=j}^{n} \alpha_i$ that are equal to $+1$, by $\mathcal{I}^0$ the set of coefficients' indices that vanish and by $\mathcal{I}^-$ the set of coefficients' indices that are equal -1. Moreover we assume that $\beta_1 = 1$. Then

$$S = \sum_{\gamma=0}^{2N} P(\gamma, \underline{\alpha}, N) f(\gamma)$$

where the occurrence function is:

$$P(\gamma, \underline{\alpha}, N) = \begin{cases} \binom{N-\gamma+\|\mathcal{I}^+\|-1}{\|\mathcal{I}^+\|-1} & \text{if } \|\mathcal{I}^-\| = 0 \\ \sum_{\varphi=\varphi_{min}(\gamma)}^{\varphi_{max}(\gamma)} \binom{\varphi+\|\mathcal{I}^-\|-1}{\|\mathcal{I}^-\|-1} \binom{N-\gamma+\varphi+\|\mathcal{I}^+\|-1}{\|\mathcal{I}^+\|-1} & \text{otherwise} \end{cases}$$

and

$$\varphi_{min}(\gamma) = \max\{0,\ \gamma - N\},\quad \varphi_{max}(\gamma) = \min\{N\|\mathcal{I}^-\|, \lfloor\frac{\gamma}{2}\rfloor\}.$$

Proof. Recall the "Stars and Bars Theorem" by W. Feller [1]: *"For any pair of positive integers n and k, the number of k-tuples of non-negative integers whose sum is n is equal to* $\binom{n+k-1}{k-1}$*".*

So in the case where $\|\mathcal{I}^-\| = 0$, the number of solutions for Eq. 2.2 takes the desired form. (Here $\| \ \|$ denotes the cardinality of a set). In the general case we have $\|\mathcal{I}^+\|, \|\mathcal{I}^-\|, \|\mathcal{I}^0\| > 0$, and Eq. 2.2 takes the form:

$$N - \gamma + \sum_{k\in\mathcal{I}^-} x_k = \sum_{i\in\mathcal{I}^+} x_i \tag{2.3}$$

Equation $\varphi = \sum\limits_{k\in\mathcal{I}^-} x_k$ have $\binom{\varphi+\|\mathcal{I}^-\|-1}{\|\mathcal{I}^-\|-1}$ roots. The number of solutions to the equation $N - \gamma + \varphi = \sum\limits_{i\in\mathcal{I}^+} x_i$ (thinking of the left hand side as a constant value) is $\binom{N-\gamma+\varphi+\|\mathcal{I}^+\|-1}{\|\mathcal{I}^+\|-1}$. Summing over φ yields:

$$P(\gamma, \underline{\alpha}, N) = \sum_{\varphi=\varphi_{min}}^{\varphi_{max}} \binom{\varphi + \|\mathcal{I}^-\| - 1}{\|\mathcal{I}^-\| - 1}\binom{N - \gamma + \varphi + \|\mathcal{I}^+\| - 1}{\|\mathcal{I}^+\| - 1} \tag{2.4}$$

To produce the upper and lower bound of φ for the summation above note that $N \geq x_i \geq 0$ for all x_i's so $N\|\mathcal{I}^-\| \geq \varphi \geq 0$. Eq. 2.3 along with the fact that $\beta_1 = 1$ (since it must be positive by definition) then implies $\varphi \leq N(\|\mathcal{I}^+\| - 1) + \gamma$. Substituting Eq. 2.1 into Eq. 2.2 yields:

$$\gamma = (\beta_1 - \beta_2)x_2 + (\beta_1 - \beta_3)x_3 + \cdots + (\beta_1 - \beta_n)x_n + \beta_1 x_{n+1}$$
$$= 2\sum_{k\in\mathcal{I}^-} x_k + \sum_{k\in\mathcal{I}^-} x_k x_j + x_{n+1} \tag{2.5}$$

where we used:

$$\beta_1 - \beta_i = \begin{cases} 0 & i \in \mathcal{I}^+ \\ 1 & i \in \mathcal{I}^0 \\ 2 & i \in \mathcal{I}^- \end{cases}$$

Note that the maximal γ occurs when $\sum_{k\in\mathcal{I}^-} x_k = N$ so that $\sum_{k\in\mathcal{I}^-} x_k = 0$ (see 2.1). Taking the two last terms of Eq. 2.5 to zero, it yields $\varphi \leq \frac{\gamma}{2}$. Since $\binom{a}{b}$ vanishes when $b > a$ we also require $\varphi \geq \gamma - N$. Finally, we obtain:

$$\varphi_{min}(\gamma) = \max\{0,\ \gamma - N\},\quad \varphi_{max}(\gamma) = \min\{N\|\mathcal{I}^-\|, \lfloor\frac{\gamma}{2}\rfloor\}$$

□

Example 1. Consider the case where n is odd and $\|\mathcal{I}^-\| = 0$ (e.g. γ is the alternating sum: $a_1 - a_2 + a_3 - \cdots + a_n$. Let $f(k,n,p)$ be the probability mass binomial function $\binom{n}{k} p^k (1-p)^{n-k}$. Applying Theorem 1 yields :

$$\sum_{a_1=0}^{N} \sum_{a_2=0}^{a_1} \cdots \sum_{a_n=0}^{a_{n-1}} f(\|\mathcal{I}^+\|, N-\gamma, p)$$

$$= \frac{p^{\|\mathcal{I}^+\|-1}}{(\|\mathcal{I}^+\|!)^2} \left[(1-p)^{-N-\|\mathcal{I}^+\|-1} - (1-p)^{N-\|\mathcal{I}^+\|} \right]$$

where we used the identity $\binom{a-b}{b}\binom{a}{b} = (b!)^{-2}$.

References

[1] W. Feller, *An Introduction to Probability Theory and Its Applications*. Wiley, Vol 1, 2nd ed., 1950.

[2] S. Sertöz, "On the number of solutions of a Diophantine equation of Frobenius," *Discrete Mathematics and Applications*, vol. 8, pp. 153–162, 1998, doi: 10.1515/dma.1998.8.2.153.

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