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# ON STRONGLY STARLIKE FUNCTIONS OF ORDER $(\alpha,\beta)$

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*Abstract.* We consider the class $\mathcal{SS}^*(\alpha,\beta)$ of analytic functions which satisfy the condition $-\pi\beta/2 < \arg\{zf'(z)/f(z)\} < \pi\alpha/2$ for all $z$ in the unit disc $\mathbb{E}$ on the complex plane, where $0 \le \alpha < 1$ and $0 \le \beta < 1$. For $\alpha = \beta$ the class $\mathcal{SS}^*(\alpha,\beta)$ is equal to the well known class $\mathcal{SS}^*(\beta)$ of strongly starlike functions of order $\beta$. In this work we derive a sufficient condition for analytic function to be in the class $\mathcal{SS}^*(\alpha,\beta)$ strongly starlike functions of order $(\alpha,\beta)$.

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## 1. INTRODUCTION

Let $\mathcal{A}$ denote the class of analytic functions $f$ in the unit disc $\mathbb{E} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$ with the normalization $f(0) = 0$, $f'(0) = 1$. A function $f \in \mathcal{A}$ is said to be starlike of order $\delta$ if

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \delta \quad (z \in \mathbb{E}), \tag{1.1}$$

for some $0 \le \delta < 1$, Robertson [8]. We denote by $\mathcal{S}^*(\delta)$ the class of functions starlike of order $\delta$. We say that a function $f \in \mathcal{A}$ is strongly starlike of order $\beta$ if and only if

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\pi}{2}\beta \quad (z \in \mathbb{E}),$$

for some $\beta$ $(0 < \beta \le 1)$. Let $\mathcal{SS}^*(\beta)$ denote the class of strongly starlike functions of order $\beta$. The class $\mathcal{SS}^*(\beta)$ was introduced independently by Stankiewicz [9, 10] and by Brannan and Kirvan [1]. In [11] Takahashi and Nunokawa defined the following subclass of $\mathcal{A}$:

$$\mathcal{SS}^*(\alpha,\beta) = \left\{ f \in \mathcal{A} : \frac{-\pi\beta}{2} < \arg\frac{zf'(z)}{f(z)} < \frac{\pi\alpha}{2},\ z \in \mathbb{E} \right\},$$

for some $0 < \alpha \le 1$, and for some $0 < \beta \le 1$. We recall here the fact, that in [2] and in [3] a similar class was studied, see also [4, p.141]. Note that $\mathcal{SS}^*(\min\{\alpha,\beta\}) \subset$

$\mathcal{SS}^*(\alpha,\beta) \subset \mathcal{SS}^*(\max\{\alpha,\beta\})$. Of course for $\alpha = \beta$ the class $\mathcal{SS}^*(\alpha,\beta)$ becomes the class $\mathcal{SS}^*(\beta)$. It is easily seen that $\mathcal{SS}^*(\alpha,\beta) \subset \mathcal{S}^*(0)$.

## 2. Preliminary results

**Lemma 1** ([5]). *Let $w(z) = a + w_n z^n + w_{n+1} z^{n+1} + \cdots$ be analytic in $\mathbb{E}$ with $w(z) \not\equiv a$ and $n \geq 1$. If $z_0 = r_0 e^{i\theta}$, $0 < r_0 < 1$ and*

$$|w(z_0)| = \max_{|z| \leq r_0} |w(z)|,$$

*then it follows that*

$$\frac{z_0 w'(z_0)}{w(z_0)} = m,$$

*where*

$$m \geq n \frac{|w(z_0) - a|^2}{|w(z_0)|^2 - |a|^2} \geq n \frac{|w(z_0)| - |a|}{|w(z_0)| + |a|}.$$

**Theorem 1.** *Let $p$ be analytic in $\mathbb{E}$ with $p(0) = 1$ and $p(z) \neq 0$. If there exist two points $z_1 \in \mathbb{E}$ and $z_2 \in \mathbb{E}$ such that $|z_1| = |z_2| = r < 1$ and for $z \in \mathbb{E}_r = \{z : |z| < r\}$*

$$-\frac{\pi\beta}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi\alpha}{2}, \tag{2.1}$$

*with some $0 < \alpha \leq 2$, $0 < \beta \leq 2$, then we have*

$$\frac{z_1 p'(z_1)}{p(z_1)} = -im\left(\frac{\alpha+\beta}{2}\right)\frac{1+s^2}{2s} \tag{2.2}$$

*and*

$$\frac{z_2 p'(z_2)}{p(z_2)} = im\left(\frac{\alpha+\beta}{2}\right)\frac{1+t^2}{2t}, \tag{2.3}$$

*where*

$$m \geq \frac{1-|a|}{1+|a|}, \quad a = i \tan\frac{\pi}{4}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)$$

*and where*

$$s = |p(z_1)|^{2/(\alpha+\beta)}, \quad t = |p(z_2)|^{2/(\alpha+\beta)}.$$

*Proof.* The assumption (2.1) says that the domain $p(\mathbb{E}_r)$ lies in a sector between two rays $\arg\{w\} = -\pi\beta/2$ and $\arg\{w\} = \pi\alpha/2$ and it contacts with the rays at $p(z_1)$ and at $p(z_2)$. The idea of this proof is that we transform this sector into the unit disc and then we will use the Lemma 1. We restrict our considerations to proving (2.3), the proof of (2.2) runs analogously as that of (2.3). The function

$$q(z) = A\{p(z)\}^B, \quad (z \in \mathbb{E}_r), \tag{2.4}$$

where

$$A = \exp\left\{-i\frac{\pi(\alpha-\beta)}{2(\alpha+\beta)}\right\}, \quad B = \frac{2}{\alpha+\beta},$$

maps $\mathbb{E}_r$ onto the set $q(\mathbb{E}_r)$ on the right half-plane $\mathfrak{Re}\{\omega\} > 0$. The boundary $\partial q(\mathbb{E}_r)$ is tangent to the imaginary axis at $q(z_1)$ and at $q(z_2)$ because $\partial p(\mathbb{E}_r)$ is tangent to the sector $-\pi\beta/2 < \arg w < \pi\alpha/2$ at $p(z_1)$ and at $p(z_2)$. Moreover, $q(z_1)$ lies on the negative imaginary axis, while $q(z_2)$ lies on the positive imaginary axis,

$$\arg\{q(z_1)\} = \arg\left\{A\{p(z_1)\}^B\right\} = -\frac{\pi(\alpha-\beta)}{2(\alpha+\beta)} - \frac{\beta\pi}{2}\frac{2}{\alpha+\beta} = -\frac{\pi}{2} \tag{2.5}$$

and

$$\arg\{q(z_2)\} = \arg\left\{A\{p(z_2)\}^B\right\} = -\frac{\pi(\alpha-\beta)}{2(\alpha+\beta)} + \frac{\alpha\pi}{2}\frac{2}{\alpha+\beta} = \frac{\pi}{2}, \tag{2.6}$$

with

$$|\arg\{q(z)\}| = \left|\arg\left\{A\{p(z)\}^B\right\}\right| < \frac{\pi}{2} \quad (z \in \mathbb{E}_r).$$

This shows that

$$\mathfrak{Re}\{q(z)\} = \mathfrak{Re}\left\{A\{p(z)\}^B\right\} > 0 \quad (z \in \mathbb{E}_r).$$

Therefore, the function

$$\phi(z) = \frac{1-q(z)}{1+q(z)} \quad (z \in \mathbb{E}_r) \tag{2.7}$$

maps the $\mathbb{E}_r$ into the unit $\mathbb{E}$ disc and satisfies

$$\max_{|z|\le r} |\phi(z)| = |\phi(z_1)| = |\phi(z_2)| = 1. \tag{2.8}$$

By logarithmic differentiation of (2.7), we have

$$\frac{z\phi'(z)}{\phi(z)} = \frac{zp'(z)}{p(z)}\frac{2ABp^B(z)}{1-A^2p^{2B}(z)}.$$

For the case $\arg\{p(z_2)\} = \alpha\pi/2$, from (2.6), we can put

$$Ap^B(z_2) = it, \ \ 0 < t.$$

Applying Lemma 1 and (2.8), we have

$$\begin{aligned}\frac{z_2\phi'(z_2)}{\phi(z_2)} &= -\frac{z_2p'(z_2)}{p(z_2)}\frac{2ABp^B(z_2)}{1-A^2p^{2B}(z_2)}\\ &= -\frac{z_2p'(z_2)}{p(z_2)}\frac{4it}{\alpha+\beta}\frac{1}{1+t^2}\\ &= m\end{aligned} \tag{2.9}$$

where

$$m \ge n\frac{|\phi(z_2)-\phi(0)|^2}{|\phi(z_2)|^2-|\phi(0)|^2} \ge \frac{1-|\phi(0)|}{1+|\phi(0)|} = \frac{1-|a|}{1+|a|}$$

and

$$a = i\tan\frac{\pi}{4}\left(\frac{\alpha-\beta}{\alpha+\beta}\right).$$

From (2.9), we obtain (2.3) and for the case $\arg\{p(z_1)\} = -\alpha\pi/2$, applying the same method as the above and from (2.5) and (2.8), we obtain (2.2). It completes the proof of Theorem 1. □

The above result is a stronger form of Theorem 1 in [7]. For $\alpha = \beta$ Theorem 1 becomes Nunokawa's lemma [6].

## 3. Main theorem

Our main result is contained in:

**Theorem 2.** *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in $\mathbb{E}$. Assume that $0 < \alpha \le 1$, $0 < \beta \le 1$ and*

$$-\frac{\pi\beta}{2} - \mathfrak{B}(\alpha,\beta) < \arg\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{\pi\alpha}{2} + \mathfrak{A}(\alpha,\beta) \quad z \in \mathbb{E}, \tag{3.1}$$

*where $\mathfrak{A}(1,1) = 0$, $\mathfrak{B}(1,1) = 0$, while for $\alpha + \beta < 2$*

$$\mathfrak{B}(\alpha,\beta) = \tan^{-1} \frac{m(\alpha+\beta)g(t_0)\cos(\pi\beta/2)}{2 + m(\alpha+\beta)g(t_0)\sin(\pi\beta/2)},$$

$$\mathfrak{A}(\alpha,\beta) = \tan^{-1} \frac{m(\alpha+\beta)g(t_0)\cos(\pi\alpha/2)}{2 + m(\alpha+\beta)g(t_0)\sin(\pi\alpha/2)},$$

*and where,*

$$m \ge \frac{1-|a|}{1+|a|}, \quad a = i\tan\left(\frac{\alpha-\beta}{\alpha+\beta}\right),$$

$$g(t) = \frac{1+t^2}{2t^{(2+\alpha+\beta)/2}}, \quad t_0 = \sqrt{\frac{2+\alpha+\beta}{2-\alpha-\beta}}.$$

*Then we have*

$$-\frac{\pi\beta}{2} < \arg\left\{\frac{zf'(z)}{f(z)}\right\} < \frac{\pi\alpha}{2} \quad z \in \mathbb{E}, \tag{3.2}$$

*Proof.* Let us define the function

$$p(z) = \frac{zf'(z)}{f(z)}, \quad p(0) = 1,$$

then it follows that

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)}.$$

If there exists two points $z_1, z_2 \in \mathbb{E}$, $|z_1| = |z_2| = r$ such that

$$-\frac{\pi\beta}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi\alpha}{2},$$

with some $0<\alpha\le 1$, $0<\beta\le 1$, then for all $z\in\mathbb{E}_r$, then from Theorem 1

$$\frac{z_1 p'(z_1)}{p(z_1)} = -im\left(\frac{\alpha+\beta}{2}\right)\frac{1+s^2}{2s}$$

and

$$\frac{z_2 p'(z_2)}{p(z_2)} = im\left(\frac{\alpha+\beta}{2}\right)\frac{1+t^2}{2t},$$

where

$$m\ge\frac{1-|a|}{1+|a|},\quad a=i\tan\frac{\pi}{4}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)$$

and where

$$s=|p(z_1)|^{2/(\alpha+\beta)},\quad t=|p(z_2)|^{2/(\alpha+\beta)}.$$

For the case $\arg\{p(z_2)\}=\alpha\pi/2$, from Theorem 1 we have

$$\begin{aligned}\arg\left\{1+\frac{z_2 f''(z_2)}{f'(z_2)}\right\} &= \arg\left\{p(z_2)+\frac{z_2 p'(z_2)}{p(z_2)}\right\}\\ &= \arg\{p(z_2)\}+\arg\left\{1+\frac{z_2 p'(z_2)}{p(z_2)}\frac{1}{p(z_2)}\right\}\\ &= \frac{\alpha\pi}{2}+\arg\left\{1+im\left(\frac{\alpha+\beta}{2}\right)\left(\frac{1+t^2}{2t}\right)\frac{1}{t^{(\alpha+\beta)/2}e^{i\pi\alpha/2}}\right\}\\ &= \frac{\alpha\pi}{2}+\arg\left\{1+m\left(\frac{\alpha+\beta}{2}\right)\left(\frac{1+t^2}{2t^{(2+\alpha+\beta)/2}}\right)e^{i\pi(1-\alpha)/2}\right\}.\end{aligned}$$

Hence, if $\alpha=1$ then $\arg\{1+z_2 f''(z_2)/f'(z_2)\}=\alpha\pi/2$, which contradicts (3.1) because in this case $\mathfrak{A}=0$. For $0<\alpha<1$, it easily confirm that

$$\arg\left\{1+m\left(\frac{\alpha+\beta}{2}\right)\left(\frac{1+t^2}{2t^{(2+\alpha+\beta)/2}}\right)e^{i\pi(1-\alpha)/2}\right\}.$$

takes its minimum value when the function

$$g(t)=\frac{1+t^2}{2t^{(2+\alpha+\beta)/2}},\quad t>0,$$

takes its minimum value at $t_0=\sqrt{(2+\alpha+\beta)/(2-\alpha-\beta)}$ and so, we have

$$\begin{aligned}&\arg\left\{1+\frac{z_2 f''(z_2)}{f'(z_2)}\right\}\\ &\ge\frac{\alpha\pi}{2}+\tan^{-1}\frac{m(\alpha+\beta)g(t_0)\sin(\pi(1-\alpha)/2)}{2+m(\alpha+\beta)g(t_0)\cos(\pi(1-\alpha)/2)}\end{aligned}$$

This contradicts (3.1) and for the case $\arg\{p(z_1)\}=\beta\pi/2$, applying the same method as the above we obtain for $\beta=1$ a contradiction with (3.1), while for $0<\beta<1$ we have

$$\arg\left\{1+\frac{z_1 f''(z_1)}{f'(z_1)}\right\} \\ \leq -\frac{\beta\pi}{2} - \tan^{-1}\frac{m(\alpha+\beta)g(t_0)\sin(\pi(1-\beta)/2)}{2+m(\alpha+\beta)g(t_0)\cos(\pi(1-\beta)/2)}.$$

This also contradicts (3.1) and therefore, it completes the proof. □

If $\alpha = \beta$ in the above theorem then we get the following corollary.

**Corollary 1.** *Assume that*

$$-\frac{\pi}{4} - \tan^{-1}\frac{\sqrt{2}}{2\sqrt[4]{27}+\sqrt{2}} < \arg\left\{1+\frac{zf''(z)}{f'(z)}\right\} < \frac{\pi}{4} + \tan^{-1}\frac{\sqrt{2}}{2\sqrt[4]{27}+\sqrt{2}} \quad z \in \mathbb{E}.$$

*Then we have*

$$-\frac{\pi}{4} < \arg\left\{\frac{zf'(z)}{f(z)}\right\} < \frac{\pi}{4} \quad z \in \mathbb{E},$$

*this means that* $f$ *is strongly starlike of order* $1/2$.

Note that

$$\tan^{-1}\frac{\sqrt{2}}{2\sqrt[4]{27}+\sqrt{2}} = 0.23\ldots.$$

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