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A NADLER-TYPE FIXED POINT THEOREM IN DISLOCATED SPACES AND APPLICATIONS

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Abstract. In this paper, we introduce the concept of a Hausdorff dislocated metric . We initiate the study of fixed point theory for multi-valued mappings on dislocated metric space using the Hausdorff dislocated metric and we prove a generalization of the well known Nadler's fixed point theorem. Moreover, we provide some examples and we give an application of our main result.

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1. INTRODUCTION AND PRELIMINARIES

Let (X,d) be a metric space and $CB(X)$ denotes the collection of all nonempty closed and bounded subsets of X. For $A,B \in CB(X)$, define

$$H(A,B) := \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\},$$

where $d(x,A) := \inf\{d(x,a) : a \in A\}$ is the distance of a point x to the set A. It is known that H is a metric on $CB(X)$, called the Hausdorff metric induced by the metric d.

Definition 1. Let X be any nonempty set. An element x in X is said to be a a fixed point of a multi-valued mapping $T : X \to 2^X$ if $x \in Tx$, where 2^X denotes the collection of all nonempty subsets of X.

We recall that a multi-valued mapping $T : X \to CB(X)$ is said to be a contraction if

$$H(Tx,Ty) \le k\,d(x,y)$$

for all $x,y \in X$ and for some k in $[0,1)$.

The study of fixed points for multi-valued contractions using the Hausdorff metric was initiated by Nadler [18] who proved the following theorem.

Theorem 1 ([18]). *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a contraction mapping. Then, there exists $x \in X$ such that $x \in Tx$.*

The notion of dislocated metric space was introduced by Hitzler and Seda [12] (see also [11]). Later, Amini-Harandi [9] re-discovered the notion of dislocated metric under the name of "metric-like". In this paper, the author [9] presented some fixed point results in the class of dislocated metric spaces. Very recently, Karapınar and Salimi [19] established some fixed point theorems for cyclic contractions. For more fixed point results on dislocated metric spaces, see e.g. [1–3, 7, 8, 13, 15, 16, 20–23].

Definition 2. Let X be a nonempty set. A function $\sigma : X\times X \to [0,\infty)$ is said to be a dislocated metric (or a metric-like) on X if for any $x,y,z \in X$, the following conditions hold:

(σ_1) $\sigma(x,y) = 0 \Longrightarrow x = y$;
(σ_2) $\sigma(x,y) = \sigma(y,x)$;
(σ_3) $\sigma(x,z) \le \sigma(x,y)+\sigma(y,z)$.

The pair (X,σ) is then called a dislocated metric (metric-like) space.

It is known that a partial metric [17] is also a dislocated metric. So, a trivial example of a dislocated metric space is the pair $([0,\infty),\sigma)$, where $\sigma : [0,\infty)\times[0,\infty) \to [0,\infty)$ is defined as $\sigma(x,y) = \max\{x,y\}$.

In the sequel, $\mathbb{R}_0^+$ represents the set of all nonnegative reals. In the following example, we give a dislocated metric which is neither a metric nor a partial metric.

Example 1 ([6]). Take $X = \{1,2,3\}$ and consider the dislocated metric $\sigma : X^2 \to \mathbb{R}_0^+$ given by

$$\sigma(1,1) = 0, \quad \sigma(2,2) = 1, \quad \sigma(3,3) = \frac{2}{3},$$

$$\sigma(1,2) = \sigma(2,1) = \frac{9}{10}, \quad \sigma(2,3) = \sigma(3,2) = \frac{4}{5},$$

$$\sigma(1,3) = \sigma(3,1) = \frac{7}{10}.$$

Since $\sigma(2,2) \neq 0$, σ is not a metric and since $\sigma(2,2) > \sigma(1,2)$, σ is not a partial metric [17].

Each dislocated metric σ on X generates a T_0 topology τ_σ on X which has as a base the family open σ-balls $\{B_\sigma(x,\varepsilon) : x \in X, \varepsilon > 0\}$, where $B_\sigma(x,\varepsilon) = \{y \in X : |\sigma(x,y) - \sigma(x,x)| < \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

Observe that a sequence $\{x_n\}$ in a dislocated metric space (X,σ) converges to a point $x \in X$, with respect to τ_σ, if and only if $\sigma(x,x) = \lim_{n\to\infty} \sigma(x,x_n)$.

Definition 3. Let (X,σ) be a dislocated metric space.

(a) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty} \sigma(x_n,x_m)$ exists and is finite.

(b) (X,σ) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_σ to a point $x \in X$ such that $\lim_{n\to\infty} \sigma(x,x_n) = \sigma(x,x) = \lim_{n,m\to\infty} \sigma(x_n,x_m)$.

We need in the sequel the following trivial inequality

$$\sigma(x,x) \leq 2\sigma(x,y) \quad \text{for all } x,y \in X. \tag{1.1}$$

In this paper, we introduce a new concept called a Hausdorff dislocated metric . Using this concept, we establish a fixed point result for multi-valued mappings involving a generalized contraction. We derive many interesting corollaries on existing known results in the literature. Our obtained results are supported by some examples and an application to an integral equation.

2. Hausdorff dislocated metric

Let (X,σ) be a dislocated metric space. Let $CB^\sigma(X)$ be the family of all nonempty, closed and bounded subsets in the dislocated metric space (X,σ), induced by the dislocated metric σ. Note that the boundedness is given as follows: A is a bounded subset in (X,σ) if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_\sigma(x_0, M)$, that is,

$$|\sigma(x_0,a) - \sigma(x_0,x_0)| < M.$$

The Closedness is taken in (X,τ_σ) (where τ_σ is the topology induced by σ). Let $\bar{A}$ be the closure of A with respect to the dislocated metric σ. We have

Definition 4.

$$\begin{aligned} a \in \bar{A} &\Longleftrightarrow B_\sigma(a,\varepsilon) \cap A \neq \varnothing \quad \text{for all } \varepsilon > 0 \\ &\Longleftrightarrow \text{there exists } x_n \in A, \quad x_n \to a \text{ in } (X,\sigma). \end{aligned}$$

If $A \in CB^\sigma(X)$, then $\bar{A} = A$.

For $A, B \in CB^\sigma(X)$ and $x \in X$, define

$$\sigma(x,A) = \inf\{\sigma(x,a),\ a \in A\},\ \delta_\sigma(A,B) = \sup\{\sigma(a,B) : a \in A\} \quad \text{and}$$
$$\delta_\sigma(B,A) = \sup\{\sigma(b,A) : b \in B\}.$$

Lemma 1. *Let (X,σ) be a dislocated metric space and A be any nonempty set in (X,σ), then*

$$\text{if } \sigma(a,A) = 0, \quad \text{then } a \in \bar{A}. \tag{2.1}$$

Also, if $\{x_n\}$ is a sequence in (X,σ) that is τ_σ-convergent to $x \in X$, then

$$\lim_{n\to\infty} |\sigma(x_n,A) - \sigma(x,A)| = \sigma(x,x). \tag{2.2}$$

Proof. If $\sigma(a,A)=0$, so $\inf_{x\in A}\sigma(a,x)=0$, that is, for all $\varepsilon>0$, there exists $x\in A$ such that $\sigma(a,x)<\varepsilon$. Hence, for all $n\geq 1$, there exists $x_n\in A$ such that

$$\sigma(a,x_n)<\frac{1}{n}.$$

Thus, $\lim_{n\to\infty}\sigma(a,x_n)=0$. By (1.1), we have $\sigma(a,a)\leq 2\sigma(a,x_n)$, $\forall n$.
Then, $\sigma(a,a)\leq 2\lim_{n\to\infty}\sigma(a,x_n)=0$. Finally, we obtain $\lim_{n\to\infty}\sigma(a,x_n)=\sigma(a,a)=0$, which means that $\{x_n\}$ converges to a in (X,σ). By Definition 4, $a\in\bar{A}$.

The equality from (2.2) follows from the inequality

$$|\sigma(x_n-A)-\sigma(x,A)|=\sigma(x_n,x).$$

□

Remark 1. It was shown in Remark 2.1 from [4] that if A is a subset of a partial metric space (X,p) and $x\in X$, then

$$x\in\bar{A}\Longleftrightarrow p(x,A)=p(x,x).$$

We show by an example that this property is not longer true in a dislocated metric space.

Example 2. Let $X=\{0,1\}$ and $\sigma:X\times X\to\mathbb{R}_0^+$ be defined by

$$\sigma(0,0)=2\quad\text{and}\quad\sigma(x,y)=1\,if\;(x,y)\neq(0,0).$$

Then, (X,σ) is a dislocated metric space. Note that σ is not a partial metric on X because $\sigma(0,0)\geq\sigma(1,0)$.
We have $0\in\overline{X}(=X)$, but $\sigma(0,X)=\min\{\sigma(0,0),\sigma(0,1))=1\neq\sigma(0,0)$

Let (X,σ) be a dislocated metric space. For $A,B\in CB^\sigma(X)$, define

$$H_\sigma(A,B)=\max\{\delta_\sigma(A,B),\delta_\sigma(B,A)\}.$$

Now, we shall study some properties of $H_\sigma:CB^\sigma(X)\times CB^\sigma(X)\to[0,\infty)$.

Proposition 1. *Let (X,σ) be a dislocated metric space. For all $A,B,C\in CB^\sigma(X)$, we have the following:*

$$\begin{aligned}(i)&:H_\sigma(A,A)=\delta_\sigma(A,A)=\sup\{\sigma(a,A):a\in A\};\\(ii)&:H_\sigma(A,B)=H_\sigma(B,A);\\(iii)&:H_\sigma(A,B)=0\text{ implies that }A=B;\\(iv)&:H_\sigma(A,B)\leq H_\sigma(A,C)+H_\sigma(C,B).\end{aligned}$$

Proof. (i) and (ii) are clear.
(iii) Suppose that $H_\sigma(A,B)=0$. Then,

$$\sup_{a\in A}\sigma(a,B)=0.$$

Mention that $\sup_{a\in A}\sigma(a,B)=0$, implies $\forall a\in A, \sigma(a,B)=0$. Then, by lemma 1, $a\in\overline{B}=B$. As a is arbitrary in A, we conclude that $A\subset B$.
Similarly, $H_\sigma(B,A)=0$ implies $B\subset A$.

(iv) Let $a\in A$, $b\in B$ and $c\in C$. As

$$\sigma(a,b)\leq\sigma(a,c)+\sigma(c,b),$$

so we have

$$\sigma(a,B)\leq\sigma(a,c)+\sigma(c,B)\leq\sigma(a,c)+\delta_\sigma(C,B)\leq\sigma(a,C)+\delta_\sigma(C,B),$$

since c is an arbitrary element of C. As a is an arbitrary element of A, it follows

$$\delta_\sigma(A,B)\leq\delta_\sigma(A,C)+\delta_\sigma(C,B)\leq H_\sigma(A,C)+H_\sigma(C,B).$$

Similarly, due to symmetry of H_σ, we have

$$\delta_\sigma(B,A)\leq H_\sigma(A,C)+H_\sigma(C,B).$$

Combining the two above inequalities, we get (iv).

□

Remark 2. The converse of assertion (iii) from Proposition 1 is not true in general as it is clear from the following example.

Example 3. Let $X=\{0,1\}$ be endowed with the dislocated metric $\sigma: X\times X\to[0,\infty)$ defined by

$$\sigma(1,1)=2 \quad \text{and } \sigma(0,0)=\sigma(0,1)=\sigma(1,0)=1.$$

Note that σ is not a partial metric since $\sigma(1,1)>\sigma(1,0)$. From (i) of Proposition 1, we have

$$\begin{aligned}H_\sigma(X,X)=\delta_\sigma(X,X)&=\sup\{\sigma(x,X),\ x\in\{0,1\}\}\\&=\max\{\sigma(0,\{0,1\}),\sigma(1,\{0,1\})\}=1\neq0.\end{aligned}$$

In view of Proposition 1, we call the mapping
$H_\sigma: CB^\sigma(X)\times CB^\sigma(X)\to[0,+\infty)$, a Hausdorff dislocated metric induced by σ.

Remark 3. It is easy to show that any Hausdorff metric is a Hausdorff dislocated metric . The converse is not true (see Example 3).

3. Fixed point of multi-valued contraction mappings

We start with the following simple useful lemma. One may find its analogous for the partial metric case in [5].

Lemma 2. *Let $A,B\in CB^\sigma(X)$ and $a\in A$. Then, for all $\varepsilon>0$, there exists a point $b\in B$ such that $\sigma(a,b)\leq H_\sigma(A,B)+\varepsilon$.*

The inequality from Lemma 2 also appears in Nadler's paper [18]. Now, we state and prove our main result.

Theorem 2. *Let (X,σ) be a complete dislocated metric space. If $T : X \to CB^\sigma(X)$ is a multi-valued mapping such that for all $x, y \in X$, we have*

$$H_\sigma(Tx,Ty) \le k\,M(x,y) \tag{3.1}$$

where $k \in [0,1)$ and

$$M(x,y) = \max\left\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\frac{1}{4}(\sigma(x,Ty)+\sigma(y,Tx))\right\}.$$

Then, T has a fixed point.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. Clearly, if $\sigma(x_0,x_1) = 0$, then $x_0 = x_1$ and x_0 is a fixed point of T. Assume $\sigma(x_0,x_1) > 0$. Since $Tx_0, Tx_1 \in CB^\sigma(X)$ and $x_1 \in Tx_0$, Lemma 2 implies the existence of a point $x_2 \in Tx_1$ such that

$$\sigma(x_2,x_1) \le H_\sigma(Tx_1,Tx_0) + \frac{1-k}{2}M(x_1,x_0). \tag{3.2}$$

If $\sigma(x_2,x_1) = 0$, then $x_2 = x_1$ and x_1 is a fixed point of T. Assuming $\sigma(x_2,x_1) > 0$, then, by Lemma 2, there is a point $x_3 \in Tx_2$ such that

$$\sigma(x_3,x_2) \le H_\sigma(Tx_2,Tx_1) + \frac{1-k}{2}M(x_2,x_1). \tag{3.3}$$

Continuing in this fashion, we complete a sequence $(x_n) \subset X$ such that $x_{n+1} \in Tx_n$ and $\sigma(x_n,x_{n+1}) > 0$ with

$$\sigma(x_{n+1},x_n) \le H_\sigma(Tx_n,Tx_{n-1}) + \frac{1-k}{2}M(x_n,x_{n-1}).$$

Then, we get

$$\begin{aligned}
&\sigma(x_{n+1},x_n)\\
&\le kM(x_n,x_{n-1}) + \frac{1-k}{2}M(x_n,x_{n-1})\\
&= \frac{1+k}{2}M(x_n,x_{n-1})\\
&\le \frac{1+k}{2}\max\{\sigma(x_n,x_{n-1}),\sigma(x_n,x_{n+1}),\frac{1}{4}[\sigma(x_n,x_n)+\sigma(x_{n-1},x_{n+1})]\}.
\end{aligned}$$

By a triangular inequality, we get

$$\begin{aligned}
\frac{1}{4}(\sigma(x_n,x_n)+\sigma(x_{n-1},x_{n+1})) &\le \frac{1}{4}(3\sigma(x_n,x_{n-1})+\sigma(x_{n+1},x_n))\\
&\le \max\{\sigma(x_n,x_{n-1}),\sigma(x_n,x_{n+1})\}.
\end{aligned}$$

Then

$$\sigma(x_n, x_{n+1}) \leq \frac{1+k}{2} \max\{\sigma(x_{n-1}, x_n), \sigma(x_n, x_{n+1})\}.$$

Now, if $\sigma(x_n, x_{n+1}) > \sigma(x_{n-1}, x_n)$, then we have

$$\sigma(x_n, x_{n+1}) \leq \frac{1+k}{2}\sigma(x_n, x_{n+1}) < \sigma(x_n, x_{n+1}),$$

which is a contradiction. So, for all $n \geq 1$, $\sigma(x_n, x_{n+1}) \leq \sigma(x_n, x_{n-1})$. Finally, we get

$$\sigma(x_n, x_{n+1}) \leq \frac{1+k}{2}\sigma(x_{n-1}, x_n), \ \forall n \geq 1.$$

Moreover, by induction, one finds

$$\sigma(x_n, x_{n+1}) \leq (\frac{1+k}{2})^n \sigma(x_0, x_1), \forall n \geq 1.$$

Since $k \in [0,1)$, we have $\sum_{n\geq 0}(\frac{1+k}{2})^n < \infty$. So, for all $p \geq 0$, we have

$$\begin{aligned}
\sigma(x_n, x_{n+p}) &\leq \sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+2}) + \ldots + \sigma(x_{n+p-1}, x_{n+p}) \\
&\leq \sum_{i=n}^{n+p-1} (\frac{1+k}{2})^i \sigma(x_0, x_1) \\
&\leq \sum_{i=n}^{\infty} (\frac{1+k}{2})^i \sigma(x_0, x_1) \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.4}$$

Thus, by symmetry of σ, we obtain

$$\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \tag{3.5}$$

This yields that the sequence $\{x_n\}$ is Cauchy. Since (X, σ) is complete, the sequence $\{x_n\}$ converges to a point $x^\star \in X$, i.e,

$$\lim_{n\to\infty} \sigma(x_n, x^\star) = \sigma(x^\star, x^\star) = \lim_{n,m\to\infty} \sigma(x_n, x_m) = 0. \tag{3.6}$$

We have $\sigma(x^\star, Tx^\star) \leq \sigma(x^\star, x_{n+1}) + \sigma(x_{n+1}, Tx^\star)$.
Since $x_{n+1} \in Tx_n$, it follows

$$\begin{aligned}
\sigma(x^\star, Tx^\star) &\leq \sigma(x^\star, x_{n+1}) + \delta_\sigma(Tx_n, Tx^\star) \\
&\leq \sigma(x^\star, x_{n+1}) + H_\sigma(Tx_n, Tx^\star) \\
&\leq \sigma(x^\star, x_{n+1}) + kM(x_n, x^\star),
\end{aligned}$$

where

$$M(x_n, x^\star)$$

$$= \max\left\{\sigma(x_n,x^\star),\sigma(x_n,Tx_n),\sigma(x^\star,Tx^\star),\frac{1}{4}\left(\sigma(x_n,Tx^\star)+\sigma(x^\star,Tx_n)\right)\right\}.$$

We have

$$\sigma(x_n,Tx_n) \le \sigma(x_n,x_{n+1}),$$
$$\sigma(x^\star,Tx_n) \le \sigma(x^\star,x_{n+1}).$$

When passing to limit, it should be mentioned that, by Lemma 1 and (3.6),

$$\sigma(x^\star,Tx_n) \to \sigma(x^\star,Tx^\star).$$

Again, by taking $n \to \infty$ and using (3.6), we obtain

$$\begin{aligned}\sigma(x^\star,Tx^\star) &\le k\max\left\{\sigma(x^\star,Tx^\star),\frac{1}{4}\sigma(x^\star,Tx^\star)\right\}\\ &= k\sigma(x^\star,Tx^\star).\end{aligned}$$

Since, $k \in [0,1)$, we have $\sigma(x^\star,Tx^\star)=0$. Finally, by lemma 1, we have $x^\star \in \overline{Tx^\star} = Tx^\star$. Then, $x^\star$ is a fixed point of T. □

As consequences of our main result, we may state the following immediate corollaries.

Corollary 1 (Hardy-Rogers type [10])**.** *Let (X,σ) be a complete dislocated metric space. If $T: X \to CB^\sigma(X)$ is a multi-valued mapping such that for all $x,y \in X$, we have*

$$H_\sigma(Tx,Ty) \le a\sigma(x,y)+b\sigma(x,Tx)+c\sigma(y,Ty)+d[\sigma(x,Ty)+\sigma(y,Tx)] \tag{3.7}$$

where $a,b,c,d \in [0,1)$ such that $a+b+c+4d<1$. Then, T has a fixed point.

Corollary 2 (Kannan type [14])**.** *Let (X,σ) be a complete dislocated metric space. If $T: X \to CB^\sigma(X)$ is a multi-valued mapping such that for all $x,y \in X$, we have*

$$H_\sigma(Tx,Ty) \le a\sigma(x,y)+b\sigma(x,Tx)+c\sigma(y,Ty) \tag{3.8}$$

where $a,b,c \in [0,1)$ such that $a+b+c<1$. Then, T has a fixed point.

Corollary 3. *Let (X,σ) be a complete dislocated metric space. If $T: X \to CB^\sigma(X)$ is a multi-valued mapping such that for all $x,y \in X$, we have*

$$H_\sigma(Tx,Ty) \le k\,\sigma(x,y) \tag{3.9}$$

where $k \in [0,1)$. Then, T has a fixed point.

Corollary 4 ([4])**.** *Let (X,σ) be a complete partial metric space. If $T: X \to CB^\sigma(X)$ is a multi-valued mapping such that for all $x,y \in X$, we have*

$$H_\sigma(Tx,Ty) \le k\,\sigma(x,y) \tag{3.10}$$

where $k \in [0,1)$. Then, T has a fixed point.

Corollary 5 ([18]). *Let (X,σ) be a complete metric space. If $T : X \to CB^{\sigma}(X)$ is a multi-valued mapping such that for all $x,y \in X$, we have*

$$H_{\sigma}(Tx,Ty) \le k\,\sigma(x,y) \tag{3.11}$$

where $k \in [0,1)$. Then, T has a fixed point.

Corollary 6. *Let (X,σ) be a complete dislocated metric space. If $T : X \to X$ is a single-valued mapping such that for all $x,y \in X$, we have*

$$\begin{aligned}&\sigma(Tx,Ty) &(3.12)\\ &\le k \max\left\{\sigma(x,y),\sigma(x,Tx),\sigma(y,Ty),\frac{1}{4}(\sigma(x,Ty)+\sigma(y,Tx))\right\}\end{aligned}$$

where $k \in [0,1)$. Then, T has a fixed point $x \in X$, that is, $Tx = x$.

4. Examples and an application

First, we give the following illustrative examples where the main result of Aydi et al. [4] is not applicable..

Example 4. Let $X = \{0,1,2\}$ and $\sigma : X \times X \to [0,\infty)$ defined by

$$\sigma(0,0) = \sigma(1,1) = 0, \qquad \sigma(2,2) = \frac{23}{48}$$

$$\sigma(0,1) = \sigma(1,0) = \frac{1}{3}, \qquad \sigma(0,2) = \sigma(2,0) = \frac{11}{24} \quad \text{and} \quad \sigma(1,2) = \sigma(2,1) = \frac{1}{2}.$$

Then, (X,σ) is a complete dislocated metric space. Note that $\sigma$is not a partial metric on X because $\sigma(2,2) \ge \sigma(2,0)$.
Define the map $T : X \to CB^{\sigma}(X)$ by

$$T0 = T1 = \{0\}, \qquad T2 = \{0,1\}$$

Note that it easy that Tx is bounded and is closed for all $x \in X$ in the dislocated metric space (X,σ).
We shall show that

$$H_{\sigma}(Tx,Ty) \le \frac{8}{11}M(x,y), \qquad \forall x,y \in X.$$

For this, we distinguish the following cases:
<u>case1</u> : $x,y \in \{0,1\}$. We have

$$H_{\sigma}(Tx,Ty) = \sigma(0,0) = 0 \le \frac{8}{11}\sigma(x,y) \le \frac{8}{11}M(x,y).$$

<u>case2</u> : $x \in \{0,1\}$, $y = 2$. We have

$$\begin{aligned}H_{\sigma}(Tx,Ty) = H_{\sigma}(\{0\},\{0,1\}) &= \max\{\sigma(0,\{0,1\}),\max\{\sigma(0,0),\sigma(0,1)\}\}\\ &= \max\{\min\{\sigma(0,0),\sigma(0,1)\},\frac{1}{3}\} = \frac{1}{3}\end{aligned}$$

$$\leq \frac{8}{11}\sigma(x,y) \leq \frac{8}{11}M(x,y).$$

case3 : $x = y = 2$. We have

$$H_\sigma(Tx,Ty) = H_\sigma(\{0,1\},\{0,1\}) = \max\{\sigma(0,\{0,1\}),\sigma(1,\{0,1\})\}$$
$$= \min\{\sigma(0,1),\sigma(1,1)\} = 0 \leq \frac{8}{11}\sigma(2,2) \leq \frac{8}{11}M(2,2).$$

Thus, all the required hypotheses of Theorem 2 are satisfied. Then, T has a fixed point. Here, $x = 0$ is the unique fixed point of T.

Example 5. Let $X = \{0,1,2\}$ and $\sigma : X \times X \to [0,\infty)$ defined by

$$\sigma(0,0) = 0,\ \sigma(1,1) = 3,\ \sigma(2,2) = 1$$
$$\sigma(0,1) = \sigma(1,0) = 7, \quad \sigma(0,2) = \sigma(2,0) = 3 \quad \text{and} quad \sigma(1,2) = \sigma(2,1) = 4.$$

Then, (X,σ) is a complete dislocated metric space. Note that σ is not a partial metric on X because $\sigma(0,1) \geq \sigma(2,0)+\sigma(2,1)-\sigma(2,2)$.
Define the map $T : X \to CB^\sigma(X)$ by

$$T0 = T2 = \{0\} \quad \text{and } T1 = \{0,2\}.$$

Note that Tx is bounded and is closed for all $x \in X$ in the dislocated metric space (X,σ).
We shall show that

$$H_\sigma(Tx,Ty) \leq \frac{3}{4}M(x,y), \qquad \forall x,y \in X.$$

For this, we consider the following cases:
case1 : $x,y \in \{0,2\}$. We have

$$H_\sigma(Tx,Ty) = \sigma(0,0) = 0 \leq \frac{3}{4}M(x,y).$$

case2 : $x \in \{0,2\}$, $y = 1$. We have

$$H_\sigma(Tx,Ty) = H_\sigma(\{0\},\{0,2\}) = \max\{\sigma(0,\{0,2\}),\max\{\sigma(0,0),\sigma(0,2)\}\}$$
$$= \max\{0,3\} = 3 \leq \frac{3}{4}\sigma(x,y) \leq \frac{3}{4}M(x,y).$$

case3 : $x = y = 1$. We have

$$H_\sigma(Tx,Ty) = H_\sigma(\{0,2\},\{0,2\}) = \max\{\sigma(0,\{0,2\}),\sigma(2,\{0,2\})\}$$
$$= \min\{\sigma(0,2),\sigma(2,2)\} = 1 \leq \frac{3}{4}\sigma(1,1) \leq \frac{3}{4}M(1,1).$$

Therefore, all the required hypotheses of Theorem 2 are satisfied. Here, $x = 0$ is the unique fixed point of T

Example 6. Let $X=[0,1]$ and $\sigma : X\times X\to[0,\infty)$ defined by

$$\sigma(x,y)=x+y, \quad \forall x,y\in X$$

Then, (X,σ) is a complete dislocated metric space. Note that σ is not a partial metric on X because $\sigma(x,x)>\sigma(x,y)$ for all $x>y$. σ is not also a metric on X since $\sigma(1,1)=2$.
Define the map $T : X\to CB^{\sigma}(X)$ by

$$Tx=\{0,\frac{x^2}{1+x}\}, \quad \forall x\in X$$

It is easy that Tx is bounded and is closed for all $x\in X$ in the dislocated metric space (X,σ).
We shall show that

$$H_\sigma(Tx,Ty)\le\frac{1}{2}M(x,y), \qquad \forall x,y\in X.$$

For this, we consider the following cases:
<u>case1</u> : $\quad x=y$. We have

$$\begin{aligned}H_\sigma(Tx,Ty)&=\max\{\sigma(0,Tx),\sigma(\frac{x^2}{1+x},Tx)\}\\&=\max\{\min\{\sigma(0,0),\sigma(0,\frac{x^2}{1+x})\},\ \min\{\sigma(0,\frac{x^2}{1+x}),\sigma(\frac{x^2}{1+x},\frac{x^2}{1+x})\}\}\\&=\max\{0,\frac{x^2}{1+x}\}=\frac{x^2}{1+x}\le x=\frac{1}{2}\sigma(x,x)\le\frac{1}{2}M(x,y).\end{aligned}$$

<u>case2</u> : $\quad x\ne y$. Since σ is symmetric, we suppose $x>y$. We have

$$\begin{aligned}&H_\sigma(Tx,Ty)\\&=H_\sigma(\{0,\frac{x^2}{1+x}\},\{0,\frac{y^2}{1+y}\})\\&=\sup\{\max\{\sigma(0,\{0,\frac{y^2}{1+y}\}),\sigma(\frac{x^2}{1+x},\{0,\frac{y^2}{1+y}\})\},\\&\max\{\sigma(0,\{0,\frac{x^2}{1+x}\}),\sigma(\frac{y^2}{1+y},\{0,\frac{x^2}{1+x}\})\}\}\\&=\max\{\sigma(\frac{x^2}{1+x},\{0,\frac{y^2}{1+y}\})\},\sigma(\frac{y^2}{1+y},\{0,\frac{x^2}{1+x}\})\}\}\\&=\max\{\frac{x^2}{1+x},\frac{y^2}{1+y}\}=\frac{x^2}{1+x}\le\frac{1}{2}x\le\frac{1}{2}(x+y)=\frac{1}{2}\sigma(x,y)\le\frac{1}{2}M(x,y).\end{aligned}$$

Thus, all the required hypotheses of Theorem 2 are satisfied. Here, $x=0$ is the unique fixed point of T.

Now, we provide an application on the research of a solution of an integral equation. For instance, using Corollary 6, we will prove the existence of a solution of the following integral equation.

$$x(t) = \int_a^b K(t, x(s))\,ds, \tag{4.1}$$

where $K : [a,b] \times \mathbb{R} \to [0,\infty)$ is a continuous nonnegative function.

Throughout this part, let $X = C([a,b],[0,\infty))$ be the set of real nonnegative continuous functions defined on $[a,b]$. Take the dislocated metric $\sigma : X \times X \to [0,\infty)$ defined by

$$\sigma(x,y) = \|x\|_\infty + \|y\|_\infty = \max_{s\in[a,b]} x(s) + \max_{s\in[a,b]} y(s) \quad \text{for all } x,y \in X.$$

Mention that σ is not partial metric on X. But, it is easy that (X,d) is a complete dislocated metric space.

Now, take the operator $T : X \to X$ defined by

$$Tx(t) = \int_a^b K(t, x(s))\,ds. \tag{4.2}$$

Mention that (4.1) has a solution if and only if the operator T has a fixed point.

The main result is

Theorem 3. *Assume that there exists $\lambda \in (0,1)$, such that for every $s \in [a,b]$ and $u \in X$, we have*

$$K(s,u(s)) \le \frac{\lambda}{b-a} u(s).$$

Then, T has a fixed point in X.

Proof. For all $x \in X$

$$\begin{aligned} |T(x)(t)| &\le \int_a^b |K(t,s,x(s))|\,ds \\ &\le \frac{\lambda}{b-a}\int_a^b x(s)\,ds \le \lambda \|x\|_\infty. \end{aligned}$$

It follows that for all $x, y \in X$

$$\sigma(Tx,Ty) \le \lambda\sigma(x,y) \le \lambda M(x,y). \tag{4.3}$$

Therefore, all the hypotheses of Corollary 6 are satisfied. Consequently, T has a fixed point, that is, (4.1) has a solution $x \in X$.

□

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