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EXISTENCE OF SOLUTIONS FOR FRACTIONAL INTERVAL–VALUED DIFFERENTIAL EQUATIONS BY THE METHOD OF UPPER AND LOWER SOLUTIONS

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Abstract. In this work we firstly study some important properties of fractional calculus for interval-valued functions and introduce the concepts of upper and lower solutions for interval-valued Caputo fractional differential equations. Then, we prove an existence result for interval-valued Caputo fractional differential equations by use of the method of upper and lower solutions. Finally several examples will be presented to illustrate our abstract results.

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1. Introduction

Let $R(I)$ be the set of nonempty bounded, closed interval of $\mathbb{R}$. The main purpose of this paper is to introduce and investigate the following interval-valued differential equation with fractional derivative operator of Caputo type ((IVCFDE), for short):

$$(IVCFDE)\begin{cases} {}^{C}\mathfrak{D}_t^{\alpha} x(t) = f(t, x(t)),\ t \in J := [0, T], \\ x(0) = x_0, \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$, ${}^{C}\mathfrak{D}_t^{\alpha}$ stands the Caputo fractional derivative for interval-valued functions (see Definition 14 below), $x : J \to R(I)$, $f \in C(J \times R(I); R(I))$ and $x_0 \in R(I)$, which will be be specified in Section 4. Interval analysis was introduced as an attempt to handle interval (nonstatistical, non-probabilistic) uncertainty which appears in many mathematical or computer models of some deterministic real-world phenomena. The development has experienced a long historical time. The first celebrated monograph which mainly dealt with interval analysis was written by Moore [34]. With this monograph, it began a new era of applications to error analysis for digital computers and engineering problems. For more details, please, see [3, 12, 15, 25, 33, 35–37] and the references therein.

Based on interval analysis, the concepts of integral and Hukuhara derivative (H-derivative, for short) for set functions were introduced by Aumann [5] and Hukuhara

[10], respectively. Recently, the interval valued differential equations and fuzzy differential equations have began to be intensively studied (see, e.g., [6, 13, 16, 26–31, 38]). Plotnikova in [38] has studied a class of linear differential equations with π-derivative and linear differential inclusion; Abbas and Lupulescu [1] have investigated and obtained some results about set functional differential equations. According to the concept of H-derivative, some new notions of derivatives for interval-valued functions have been introduced, such as G-derivative, gH-derivative and π-derivative. It is known that, if the set of switching points is finite, then the notions of G-differentiability, gH-differentiability and π-differentiability will be coincided with interval-valued functions (for more details, ones can refer to [6, 42]). In accordance with this statement, we just consider the concept of gH-differentiability in this paper.

On the other hand, fractional differential equations (FDEs, for short) have become more and more popular and importance due to their numerous applications in many fields of science and engineering. Which involved in physics, population dynamics, chemical technology, biotechnology, aerodynamics, electrodynamics of complex mediums, polymer rheology, control of dynamical systems, etc. (see, e.g., [2, 8, 9, 14, 18–24, 47, 48]).

Although fractional calculus and interval-valued differential equations both are playing important roles in our real life. Up to now, fractional calculus for interval valued functions have just been studied by Lupulescu in [28] and the interval-valued Caputo fractional differential equations have not yet been addressed. Based on this motivation, the purpose of this paper is to develop properties of interval-valued Caputo fractional derivative operator and theory of existence of solutions for interval-valued Caputo fractional differential equation (1.1). Also, we believe that the presented results will be useful for development about the theory of interval-valued fractional differential equations and the relevant applications, such as, digital computers.

The rest of this paper is organized as follows. In Section 2, some preliminaries are provided. In Section 3, we shall introduce the concepts of fractional calculus for interval-valued functions and present some important properties of partial order relationship '$\preceq_{LU}$'. Furthermore, in Section 4, we prove the existence and uniqueness results for interval-valued fractional integro-differential equations (IVFIE, for short) by employing the method of upper and lower solutions and then obtain the existence of solutions for (1.1). Finally, several examples will be illustrated.

2. Preliminaries

Let $R(I)$ be the space of nonempty bounded, closed interval of $\mathbb{R}$. For any element A of $R(I)$, we denote $A=[a^L,a^U]$, here a^L and a^U stand for the lower and upper bounds of A, respectively. Let $A=[a^L,a^U]$, $B=[b^L,b^U]\in R(I)$ and $\lambda\in\mathbb{R}$. The usual interval operations, i.e., well-known as Minkowski addition and scalar multiplication, are defined as follows

$$A+B=[a^L,a^U]+[b^L,b^U]=[a^L+b^L,a^U+b^U],$$

and

$$\lambda A=\lambda[a^L,a^U]=\begin{cases}[\lambda a^L,\lambda a^U] & \text{if } \lambda>0,\\ \{0\} & \text{if } \lambda=0,\\ [\lambda a^U,\lambda a^L] & \text{if } \lambda<0,\end{cases}$$

respectively. Particularly, when $\lambda=-1$, which has

$$-A:=(-1)\cdot A=(-1)\cdot[a^L,a^U]=[-a^U,-a^L].$$

So, the Minkowski difference can be written as

$$A-B=A+(-1)\cdot B=[a^L-b^U,a^U-b^L].$$

In fact, based on Minkowski addition and scalar multiplication, $R(I)$ merely constructs a quasi-linear space (more detail ones can refer to Markov [33] and Schneider [40]). For $A=[a^L,a^U]\in R(I)$, we define

$$\|A\|:=\max\{|a^L|,|a^U|\}.$$

Then, it is easily to get that $\|\cdot\|$ is a norm of quasi-linear space $R(I)$, and therefore $(R(I),+,\cdot,\|\cdot\|)$ is a normed quasilinear space. At the same time, the Hausdorff–Pompeiu distance $\mathscr{H}:R(I)\times R(I)\to\mathbb{R}_+:=[0,\infty)$ will be defined by

$$\mathscr{H}(A,B)=\max\{|a^L-b^L|,|a^U-b^U|\},\ \text{for any } A=[a^L,a^U],B=[b^L,b^U]\in R(I).$$

By the concept of $\mathscr{H}$, we readily obtain its properties as follows

$$\mathscr{H}(\lambda A,\lambda B)=|\lambda|\mathscr{H}(A,B)\ \text{for all } \lambda\in\mathbb{R},$$
$$\mathscr{H}(A+C,B+C)=\mathscr{H}(A,B),$$
$$\mathscr{H}(A+B,C+D)\le\mathscr{H}(A,C)+\mathscr{H}(B,D).$$

Indeed, it is obviously that the Hausdorff metric $\mathscr{H}$ is associated with the norm $\|\cdot\|$ by $\|A\|=\mathscr{H}(A,\{0\})$. Moreover, it follows from Li, Ogura and Kreinovich in [17], which knows that $(R(I),\mathscr{H})$ is a complete, locally compact and separable metric space.

For any $A\in R(I)$, it can be seen $A+(-A)\neq 0$ in general, thus the opposite of A is not the inverse of A with respect to the Minkowski addition unless $A=\{a\}$ is a singleton. To partially overcome this situation, the Hukuhara difference (or H-difference see c.f. [10]) has been introduced as a set C, for which is $A\ominus B=C$ if and only if $A=B+C$. The most important property of $\ominus$ is that $A\ominus A=\{0\},\ \forall A\in R(I)$ and $(A+B)\ominus B=A,\ \forall A,B\in R(I)$. Despite, the H-difference is unique, but it does not always exist, for example $A=[1,2],B=[0,4]$, it is impossible

$C=[1,-2]$. In order to solve this problem, the generalized Hukuhara difference is proposed in recently.

Definition 1 ([33, 41])**.** The generalized Hukuhara difference (or gH-difference) of two intervals $A=[a^L,a^U],\ B=[b^L,b^U]\in R(I)$ is defined as follows

$$\begin{aligned} A\ominus_{gH} B &= [a^L,a^U]\ominus_{gH}[b^L,b^U] \\ &= [\min\{a^L-b^L,a^U-b^U\},\max\{a^L-b^L,a^U-b^U\}]. \end{aligned} \tag{2.1}$$

Remark 1. For any $A=[a^L,a^U]\in R(I)$, the width of A can be denoted by $w(A)=a^U-a^L$ and the midpoint of A by $m(A)=\frac{a^U+a^L}{2}$. Then, (2.1) is equivalent to

$$A\ominus_{gH} B=\begin{cases} [a^L-b^L,a^U-b^U], & \text{if } w(A)\geq w(B) \\ [a^U-b^U,a^L-b^L], & \text{if } w(A)<w(B). \end{cases}$$

Hence, for any $A,B,C\in R(I)$, it has

$$A\ominus_{gH} B=C \Leftrightarrow \begin{cases} A\ominus B=C, & \text{if } w(A)\geq w(B) \\ B\ominus A=(-C) & \text{if } w(A)<w(B). \end{cases}$$

$$\Leftrightarrow \begin{cases} A=B+C, & \text{if } w(A)\geq w(B) \\ B=A+(-C) & \text{if } w(A)<w(B). \end{cases}$$

and $\mathcal{H}(A,B)=\|A\ominus_{gH} B\|$.

Now, we shall recall some properties of gH-difference.

Proposition 1 ([32, 41])**.** *Let $A,B,C,D\in R(I)$, $\alpha,\beta\in\mathbb{R}$ and denote*

$$W_1=\Big(w(A)-w(C)\Big)\Big(w(B)-w(D)\Big),\quad W_2=\Big(w(A)-w(B)\Big)\Big(w(C)-w(D)\Big).$$

Then the following results hold

(i) $(-A)\ominus_{gH} B=(-B)\ominus_{gH} A$

(ii) $(A+B)\ominus_{gH}(C+D)=\begin{cases} (A\ominus_{gH} C)+(B\ominus_{gH} D), & \textit{if } W_1\geq 0, \\ (A\ominus_{gH} C)\ominus_{gH}(-(B\ominus_{gH} D)), & \textit{if } W_1<0, \end{cases}$

(iii) $(A\ominus_{gH} B)+(C\ominus_{gH} D)$

$$=\begin{cases} (A\ominus_{gH}(-C))\ominus_{gH}(B\ominus_{gH}(-D)), & \textit{if } W_1\geq 0,\ W_2<0, \\ (A\ominus_{gH}(-C))+((-B)\ominus_{gH} D), & \textit{if } W_1<0,\ W_2<0, \\ (A+C)\ominus_{gH}(B+D), & \textit{if } W_2\geq 0, \end{cases}$$

(iv) $(A\ominus_{gH} B)\ominus_{gH}(C\ominus_{gH} D)$

$$=\begin{cases} (A\ominus_{gH} C)\ominus_{gH}(B\ominus_{gH} D), & \textit{if } W_2\geq 0,\ W_1\geq 0, \\ (A\ominus_{gH} C)+(-(B\ominus_{gH} D)), & \textit{if } W_2\geq 0,\ W_1<0, \\ (A+(-C))\ominus_{gH}(B+(-D)), & \textit{if } W_2<0, \end{cases}$$

The function $f:[a,b]\to R(I)$, which defined on $[a,b]$, is named to be an interval-valued function, i.e., $f(t)$ is a closed interval in $\mathbb{R}$ for each $t\in[a,b]$ and can also be written as $f(t)=[f^L(t),f^U(t)]$, where f^L and f^U are two real valued functions defined on $[a,b]$ which satisfied $f^L(t)\le f^U(t)$ for every $t\in[a,b]$. In addition, an interval-valued function f is called to be w-monotone, if real valued function $w(f(t))=f^U(t)-f^L(t)$ is monotone.

Definition 2 ([32])**.** Let $f:[a,b]\to R(I)$ be an interval-valued function. Then $\lim_{t\to t_0} f(t)$ exists, if and only if $\lim_{t\to t_0} f^L(t)$ and $\lim_{t\to t_0} f^U(t)$ exist as finite numbers, i.e.,

$$\lim_{t\to t_0} f(t)=\left[\lim_{t\to t_0} f^L(t),\lim_{t\to t_0} f^U(t)\right].$$

Thus f is continuous if and only if f^L and f^U are both continuous.

Definition 3 ([41])**.** Let $f,g:[a,b]\to R(I)$ be two interval-valued functions, then we define the interval-valued function $f\ominus_{gH} g:[a,b]\to R(I)$ by

$$(f\ominus_{gH} g)(t):=f(t)\ominus_{gH} g(t),\ \text{for all } t\in[a,b].$$

Remark 2. By above definitions, we know that if there exist $\lim_{t\to t_0} f(t)=A$, $\lim_{t\to t_0} g(t)=B$, then $\lim_{t\to t_0}(f\ominus_{gH} g)(t)$ exists, and

$$\lim_{t\to t_0}(f\ominus_{gH} g)(t)=A\ominus_{gH} B.$$

Besides, it is well-known that if f and g are two continuous interval-valued functions then $f+g$, $\lambda f,\forall\lambda\in\mathbb{R}$ and the single-valued function $w(f(t))$ are continuous also (more details we can refer to [32]).

Let $C([a,b];R(I))$ be the set of continuous interval-valued functions from $[a,b]$ into $R(I)$. Then $C([a,b];R(I))$ is a complete metric space with respect to the metric

$$\mathcal{H}_{\mathcal{C}}(f,g):=\|f\ominus_{gH} g\|_{\mathcal{C}}=\sup_{a\le t\le b}\mathcal{H}(f(t),g(t)),$$

where $f,g\in C([a,b];R(I))$.

Definition 4 ([32, 41])**.** Let $f:[a,b]\to R(I)$ be an interval-valued function and $t_0\in[a,b]$. We define $f'(t_0)\in R(I)$ (provided it exists) as

$$f'(t_0)=\lim_{h\to 0}\frac{f(t_0+h)\ominus_{gH} f(t_0)}{h}.$$

The $f'(t_0)$ is called the generalized Hukuhara derivative (gH-derivative, for short) of f at $t_0\in[a,b]$. Also, we define the left gH-derivative $f'_-(t_0)\in R(I)$ (provided it exists) as

$$f'_-(t_0)=\lim_{h\to 0^-}\frac{f(t_0+h)\ominus_{gH} f(t_0)}{h},$$

and the right gH-derivative $f'_+(t_0) \in R(I)$ (provided it exists) as

$$f'_+(t_0) = \lim_{h \to 0^+} \frac{f(t_0+h) \ominus_{gH} f(t_0)}{h}.$$

Saying that f is generalized Hukuhara differentiable (gH-differentiable, for short) on $[a,b]$ if $f'(t) \in R(I)$ exists at each point $t \in [a,b]$. At the end points of $[a,b]$ only one sided gH-derivatives would be considered. The interval-valued function $f' : [a,b] \to R(I)$ is then called the gH-derivative of f on $[a,b]$.

Next, let us recall some important properties about gH-differential of interval-valued functions.

Theorem 1 ([32,41]). *Let $f : [a,b] \to R(I)$ be such that $f(t) = [f^L(t), f^U(t)]$, $t \in [a,b]$. If the real valued functions f^L and f^U are both differential at $t \in [a,b]$, then f is gH-differential at $t \in [a,b]$ and*

$$f'(t) = \left[\min\left\{ \frac{d}{dt} f^L(t), \frac{d}{dt} f^U(t) \right\}, \max\left\{ \frac{d}{dt} f^L(t), \frac{d}{dt} f^U(t) \right\} \right], \ \forall t \in [a,b].$$

Theorem 2 ([32,41]). *Let $f : [a,b] \to R(I)$ be gH-differentiable on $[a,b]$. Then for all $C \in R(I)$ and for all $\lambda \in \mathbb{R}$, the interval-valued functions $f + C$, $f \ominus_{gH} C$ and λf are gH-differentiable on $[a,b]$, where $(f+C)' = f'$, $(f \ominus_{gH} C)' = f'$, $(\lambda f)' = \lambda f'$.*

Definition 5 ([28]). Let $f : [a,b] \to R(I)$ be an interval-valued function. f is called a simple interval-valued function, if it is constant on each sets of $I_k \subset [a,b]$, $k = 1,2,\cdots,m$, which produce a finite system of pair wise disjoint, Lebesgue measurable sets covering $[a,b]$.

Definition 6 ([28]). Let $f : [a,b] \to R(I)$ be an interval-valued function. f is said to be measurable if there exists a sequence $f_n : [a,b] \to R(I)$ of simple interval-valued functions which satisfies

$$\lim_{n\to\infty} \mathcal{H}(f_n(t), f(t)) = 0, \text{ for a.e. } t \in [a,b].$$

Definition 7 ([28]). Let $f : [a,b] \to R(I)$ be an interval-valued function such that $f(t) = [f^L(t), f^U(t)]$. Also, both f^L and f^U are measurable and Lebesgue integrable on $[a,b]$. Then, we defined the Lebesgue integral of f by $\int_a^b f(s)ds$, i.e.

$$\int_a^b f(s)ds := \left[\int_a^b f^L(s)ds, \int_a^b f^U(s)ds \right].$$

Remark 3. $\lim_{n\to\infty} \mathcal{H}(f_n(t), f(t)) = 0$ if and only if $\lim_{n\to\infty} |f_n^L(t) - f^L(t)| = 0$ and $\lim_{n\to\infty} |f_n^U(t) - f^U(t)| = 0$. Then it can be easily get that an interval-valued function f is measurable on $[a,b]$ if and only if f^L and f^U are measurable on $[a,b]$. Moreover, f is Lebesgue integrable on $[a,b]$ if and only if f is measurable and real function $\|f(t)\|$ is Lebesgue integrable on $[a,b]$ (those results have been obtained by [5, 28]).

Definition 8 ([28])**.** Let $f:[a,b]\to R(I)$ be an interval-valued function. f is said to be absolutely continuous, if for each $\epsilon>0$, there exists $\delta>0$ such that for each family $\{(s_k,t_k)\subset[a,b];\ k=1,2,\ldots,n\}$ of disjoint open intervals with $\sum_{k=1}^{n}(t_k-s_k)<\delta$, we have

$$\sum_{k=1}^{n}\mathcal{H}(f(s_k),f(t_k))<\epsilon.$$

Let $AC([a,b];R(I))$ denote the set of all absolutely continuous interval-valued functions from $[a,b]$ into $R(I)$. The following theorem gives us a necessary and sufficient condition for absolutely continuous interval-valued functions.

Proposition 2 ([4, 28, 32, 41])**.** *An interval-valued function $f:[a,b]\to R(I)$ is absolutely continuous if and only if f^L and f^U are both absolutely continuous.*

It follows well-known result (see Corollary 7.23 of [43]) that if $\phi:[a,b]\to\mathbb{R}$ is an absolutely continuous function, then $\frac{d}{dt}\phi(t)$ exists a.e. $t\in[a,b]$. Therefore, we have the following theorem.

Theorem 3 ([5, 28])**.** *If $f\in AC([a,b];R(I))$, then f is gH-difference a.e. $t\in[a,b]$ and $f'\in L^1([a,b];R(I))$.*

3. Fractional calculus for interval-valued functions and partial order relationship

In this section, we will discuss the properties of fractional calculus for interval-valued functions. J will be denoted as $J=:[0,T]$.

Definition 9 ([14, 39])**.** Let $\phi\in L^1(J;\mathbb{R})$. The Riemann-Liouville fractional integral of order $\alpha>0$ for ϕ is defined by (provided it exists)

$$I_t^{\alpha}\phi(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\phi(s)ds,\ \ t\in J,$$

where Γ is the well-known Gamma function.

In this paper, we denote $I_t^0\phi(t)=\phi(t)$ apparently.

Definition 10 ([14, 39])**.** Let $\phi\in L^1(J,\mathbb{R})$. The Riemann-Liouville derivative of order $\alpha\in(0,1]$ for ϕ is defined (provided exists)

$${}^{L}D_t^{\alpha}\phi(t)=\frac{d}{dt}I_t^{1-\alpha}\phi(t),\ \text{a.e. } t\in J,$$

thus is,

$${}^{L}D_t^{\alpha}\phi(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-s)^{-\alpha}\phi(s)ds,\ \text{a.e. } t\in J.$$

Definition 11 ([14, 39]). Let $\phi \in L^1(J,\mathbb{R})$ be such that ${}^L D_t^\alpha \phi(t)$ exists for a.e. $t \in J$. The Caputo fractional derivative of order $\alpha \in (0,1]$ for ϕ is defined as (provided exists)

$$ {}^C D_t^\alpha \phi(t) = {}^L D_t^\alpha [\phi(t) - \phi(0)], \text{ a.e. } t \in J, $$

thus is,

$$ {}^C D_t^\alpha \phi(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} [\phi(s) - \phi(0)] ds, \text{ a.e. } t \in J. $$

Remark 4. In particular, when $\alpha = 1$ and $\phi \in AC(J;\mathbb{R})$, then it has

$$ {}^L D_t^1 \phi(t) = \frac{d}{dt} \phi(t), \quad \text{a.e. } t \in J. $$

In the sequel, we denote $L^p(J;R(I))$ the set of all interval-valued functions $f : J \to R(I)$ such that the real function $t \mapsto \|f(t)\|$ belongs to $L^P(J;\mathbb{R})$. In fact, we readily get from [17] that $L^p(J;R(I))$ is a complete metric space with respect to the metric $\mathcal{H}_p$ defined by $\mathcal{H}_p(f,g) := \|f \ominus_{gH} g\|_p$ for any $f,g \in L^p(J;R(I))$, where

$$ \|f\|_p := \begin{cases} \left(\int_0^T \|f(t)\|^p dt \right)^{\frac{1}{p}}, & \text{if } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{t \in J} \|f(t)\|, & \text{if } p = \infty. \end{cases} $$

Now, the definition of Riemann-Liouville fractional integral for the interval-valued functions will be given.

Definition 12 ([28]). Let $f \in L^1(J;R(I))$ be an interval-valued function such that $f(t) = [f^L(t), f^U(t)]$ and $\alpha > 0$. The interval-valued Riemann-Liouville fractional integral of order α for f is defined by (provided it exists)

$$ \mathfrak{I}_t^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \ t \in J, $$

that is,

$$ \mathfrak{I}_t^\alpha f(t) := [I_t^\alpha f^L(t), I_t^\alpha f^U(t)], \ t \in J, $$

Remark 5. In fact, operator $\mathfrak{I}_t^\alpha$ has good properties, such as (more details ones can refer to c.f. [28]) if $f,g \in L^p(J;R(I))$ and $\alpha, \beta > 0$

(i) $w(\mathfrak{I}_t^\alpha f(t)) = I_t^\alpha w(f(t))$ (see [28, Reamrk 5]);
(ii) $\mathfrak{I}_t^\alpha(\mathfrak{I}_t^\beta f(t)) = \mathfrak{I}_t^{\alpha+\beta} f(t)$ (see [28, Reamrk 5]);
(iii) $\mathfrak{I}_t^\alpha c f(t)) = c\mathfrak{I}_t^\alpha f(t)$ for every $c \in [0,\infty)$ (see [28, Reamrk 5]);
(iv) $\mathfrak{I}_t^\alpha(f+g)(t)) = \mathfrak{I}_t^\alpha f(t) + \mathfrak{I}_t^\alpha g(t)$ (see [28, Reamrk 5]);
(v) $\mathfrak{I}_t^\alpha(f \ominus_{gH} g)(t)) \supseteq \mathfrak{I}_t^\alpha f(t) \ominus_{gH} \mathfrak{I}_t^\alpha g(t)$ (see [28, Theorem 1]);
(vi) $\mathfrak{I}_t^\alpha(f \ominus_{gH} g)(t)) = \mathfrak{I}_t^\alpha f(t) \ominus_{gH} \mathfrak{I}_t^\alpha g(t)$, if $w(f(t)) - w(g(t))$ has a constant sign on J (see [28, Theorem 1]).

Definition 13 ([28]). Let $f \in L^1(J;R(I))$ be an interval-valued function such that $f(t) = [f^L(t), f^U(t)]$ and $0 < \alpha \leq 1$. The interval-valued Riemann-Liouville fractional derivative (or Riemann-Liouville gH-fractional derivative) of order α is defined by (provided it exists)

$$^L\mathfrak{D}_t^\alpha f(t)(t) := (\mathfrak{I}_t^{1-\alpha} f(t))', \text{ a.e. } t \in J,$$

thus is,

$$^L\mathfrak{D}_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\left(\int_0^t (t-s)^{-\alpha} f(s)ds\right)', \text{ a.e. } t \in J.$$

Remark 6. In this paper, we denote $f_{1-\alpha}(t) := \mathfrak{I}_t^{1-\alpha} f(t)$

$$f_{1-\alpha}(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} f(s)ds, \text{ for all } t \in J.$$

Hence, it knows from [28] that if $f \in C(J;R(I))$, then $f_{1-\alpha}$ is absolutely continuous and

(i) if either f is w-increasing on J or f is w-decreasing and $f_{1-\alpha}$ is w-increasing on J then

$$^L\mathfrak{D}_t^\alpha f(t) = [^L D_t^\alpha f^L(t), {}^L D_t^\alpha f^U(t)], \text{ for a.e. } t \in J; \tag{3.1}$$

(ii) if $f_{1-\alpha}$ is w-decreasing on J then

$$^L\mathfrak{D}_t^\alpha f(t) = [^L D_t^\alpha f^U(t), {}^L D_t^\alpha f^L(t)], \text{ for a.e. } t \in J. \tag{3.2}$$

Definition 14 ([28]). Let $f \in L^1(J;R(I))$ be an interval-valued function such that $f(t) = [f^L(t), f^U(t)]$ and the Riemann-Liouville gH-fractional derivative $^L\mathfrak{D}_t^\alpha f(t)$, $\alpha \in (0,1]$, exists a.e. $t \in J$. The interval-valued Caputo fractional derivative (or Caputo gH-fractional derivative) of order $\alpha \in (0,1]$ is defined by (provided it exists)

$$^C\mathfrak{D}_t^\alpha f(t) := {}^L\mathfrak{D}_t^\alpha[f(t) \ominus_{gH} f(0)], \text{ a.e. } t \in J.$$

Next, the following partial ordering $\preceq_{LU}$ will be introduced (see, Inuiguchi and Kume in [11]; Wu in [44,45]).

Definition 15 ([11,44,45]). Let $A = [a^L, a^U]$, $B = [b^L, b^U]$ be two closed intervals and $f, g \in C(J;R(I))$ be two interval-valued functions.

(i) We say $A \preceq_{LU} B$ if and only if

$$a^L \leq b^L \text{ and } a^U \leq b^U.$$

Also It can be written as $A \prec_{LU} B$ if and only if

$$A \preceq_{LU} B \text{ and } A \neq B,$$

thus is,

$$a^L < b^L,\ a^U < b^U;$$

$$\text{or }, a^L \le b^L,\ a^U < b^U;$$
$$\text{or }, a^L < b^L,\ a^U \le b^U.$$

(ii) We say that $f \preceq_{LU} g$ if and only if

$$f(t) \preceq_{LU} g(t),\ \forall t \in J.$$

In the sequel, $A \preceq_{LU} B$ also means $B \succeq_{LU} A$. In fact, by the definition of $\preceq_{LU}$, the following lemmas can be obtain

Lemma 1. *Let $A, B, C, D \in R(I)$ and $t \in \mathbb{R}$. Then, it has*

(i) *$A = B$ if and only if $A \preceq_{LU} B$ and $B \preceq_{LU} A$;*
(ii) *if $A \preceq_{LU} B$, then $A + C \preceq_{LU} B + C$;*
(iii) *if $A \preceq_{LU} B$ and $C \preceq_{LU} D$, then $A + C \preceq_{LU} B + D$;*
(iv) *if $A \preceq_{LU} B$ and $t \ge 0$, then $tA \preceq_{LU} tB$.*

Lemma 2. *Let $f, g \in C(J, R(I))$ be two interval-valued functions with $f \preceq_{LU} g$, i.e., $f(t) \preceq_{LU} g(t)$ for each $t \in J$. Then, for each $\alpha \in (0,1]$ we get*

$$\Im_t^\alpha f(t) \preceq_{LU} \Im_t^\alpha g(t), \text{ for any } t \in J. \tag{3.3}$$

Proof. By the definition of $\Im_t^\alpha$, it has

$$\Im_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds = \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} f^L(s)ds, \int_0^t (t-s)^{\alpha-1} f^U(s)ds\right],$$
$$\Im_t^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)ds = \frac{1}{\Gamma(\alpha)}\left[\int_0^t (t-s)^{\alpha-1} g^L(s)ds, \int_0^t (t-s)^{\alpha-1} g^U(s)ds\right].$$

Because of $f^L(t) \le g^L(t)$, $f^U(t) \le g^U(t)$ for all $t \in J$ and $(t-s)^{\alpha-1} > 0$, which can easily has $(t-s)^{\alpha-1} f^L(t) \le (t-s)^{\alpha-1} g^L(t)$ and $(t-s)^{\alpha-1} f^U(t) \le (t-s)^{\alpha-1} g^U(t)$, hence

$$\int_0^t (t-s)^{\alpha-1} f^L(s)ds \le \int_0^t (t-s)^{\alpha-1} g^L(s)ds,$$
$$\int_0^t (t-s)^{\alpha-1} f^U(s)ds \le \int_0^t (t-s)^{\alpha-1} g^U(s)ds.$$

thus is $\int_0^t (t-s)^{\alpha-1} f(s)ds \preceq_{LU} \int_0^t (t-s)^{\alpha-1} g(s)ds$. Besides, it is immediate from item (iv) of Lemma 1 that for each $\alpha \in (0,1]$ it concludes $\Im_t^\alpha f(t) \preceq_{LU} \Im_t^\alpha g(t),\ \forall t \in J$. □

Lemma 3. *Let $f, g \in C(J; R(I))$ be two interval-valued functions with $f \preceq_{LU} g$ and $A \in R(I)$. Then, for every $\alpha \in (0,1]$ the following results hold:*

$$A \ominus (-1) \cdot \Im_t^\alpha f(t) \preceq_{LU} A \ominus (-1) \cdot \Im_t^\alpha g(t),\ \forall t \in J,$$

providing $A \ominus (-1) \cdot \Im_t^\alpha f(t)$ and $A \ominus (-1) \cdot \Im_t^\alpha g(t)$ are well-defined for each $t \in J$.

Proof. For each $\alpha \in (0,1]$, which has

$$\mathfrak{I}_t^\alpha f(t) = [I_t^\alpha f^L(t), I_t^\alpha f^U(t)],$$
$$\mathfrak{I}_t^\alpha g(t) = [I_t^\alpha g^L(t), I_t^\alpha g^U(t)].$$

By the assumptions and Lemma 2, which deduce

$$\mathfrak{I}_t^\alpha f(t) \preceq_{LU} \mathfrak{I}_t^\alpha g(t),$$

i.e., for every $\alpha \in (0,1]$,

$$I_t^\alpha f^L(t) \le I_t^\alpha g^L(t) \text{ and } I_t^\alpha f^U(t) \le I_t^\alpha g^U(t), \ \forall t \in J. \tag{3.4}$$

By the definition of Hukuhara difference, we obtain

$$A \ominus (-1)\cdot \mathfrak{I}_t^\alpha f(t) = [a^L + I_t^\alpha f^U(t), a^U + I_t^\alpha f^L(t)],$$
$$A \ominus (-1)\cdot \mathfrak{I}_t^\alpha g(t) = [a^L + I_t^\alpha g^U(t), a^U + I_t^\alpha g^L(t)],$$

It follows from (3.4) that

$$a^L + I_t^\alpha f^U(t) \le a^L + I_t^\alpha g^U(t) \text{ and}$$
$$a^U + I_t^\alpha f^L(t) \le a^U + I_t^\alpha g^L(t).$$

Which means that for each $\alpha \in (0,1]$ it has $A \ominus (-1)\cdot \mathfrak{I}_t^\alpha f(t) \preceq_{LU} A \ominus (-1)\cdot \mathfrak{I}_t^\alpha g(t)$, $\forall t \in J$. □

Lemma 4. *Let $\{f_n\} \subset C(J;R(I))$ and $g \in C(J;R(I))$. Assuming that $\{f_n\}$ satisfies*

$$f_n \preceq_{LU} g, \ \forall n \in \mathbb{N},$$

and $f_n(t)$ converges to $f(t)$ in $R(I)$ for each $t \in J$, then which has $f \preceq_{LU} g$.

Proof. For each $n \in \mathbb{N}$, because of $f_n \preceq_{LU} g$ then for all $t \in J$ we have

$$f_n(t) \preceq_{LU} g(t),$$

thus is

$$f_n^L(t) \le g^L(t) \text{ and } f_n^U(t) \le g^U(t).$$

Due to for all $t \in J$, $f_n(t) \to f(t)$ in $R(I)$ as $n \to \infty$, this implies

$$\lim_{n\to\infty} \mathscr{H}(f_n(t), f(t)) = \lim_{n\to\infty} \max\{|f_n^L(t) - f^L(t)|, |f_n^U(t) - f^U(t)|\} = 0.$$

Thereby, it can easily get for each $t \in J$, $f_n^L(t) \to f^L(t)$ and $f_n^U(t) \to f^U(t)$ in $\mathbb{R}$ as $n \to \infty$. Hence, for any $t \in J$ we obtain

$$f^L(t) = \lim_{n\to\infty} f_n^L(t) \le g^L(t),$$
$$f^U(t) = \lim_{n\to\infty} f_n^U(t) \le g^U(t).$$

That means $f \preceq_{LU} g$. □

Now the concept of monotone sequences will be introduced.

Definition 16. Let $\{f_n\}$ be an interval-valued functions sequence. The sequence $\{f_n\}$ is said to be

(i) nondecreasing sequence iff,

$$f_n \preceq_{LU} f_m, \text{ for any } m \geq n;$$

(ii) nonincreasing sequence iff,

$$f_m \preceq_{LU} f_n, \text{ for any } m \geq n;$$

(iii) monotone sequence it is nondecreasing sequence or nonincreasing sequence.

Lemma 5. *Let $\{f_n\} \subset C(J;R(I))$ be a monotone sequence. If it has a subsequence which convergent to $f \in C(J;R(I))$, then $\{f_n\}$ converges to f in $C(J;R(I))$ and*

$$f_n \preceq_{LU} f \text{ if } \{f_n\} \text{ is nondecreasing},$$
$$f \preceq_{LU} f_n \text{ if } \{f_n\} \text{ is nonincreasing}.$$

Proof. According to the hypothesises, assuming that there is a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ convergent to f. Then, for any $\epsilon > 0$ there exists a $k_0 \in \mathbb{N}$ such that

$$\|f_n - f\|_{\mathcal{C}} = \sup_{t\in J} \mathcal{H}(f_{n_k}(t), f(t)) < \epsilon, \ \forall k \geq k_0. \tag{3.5}$$

Because $\{f_n\}$ is monotone. Hence, real valued function sequences $\{f_n^L(t)\}$ and $\{f_n^U(t)\}$ have the same monotonicity in $\mathbb{R}$. However, using the condition of (3.5), we know that for each $t \in J$

$$f_{n_k}^L(t) \to f^L(t) \text{ in } \mathbb{R},$$
$$f_{n_k}^U(t) \to f^U(t) \text{ in } \mathbb{R}.$$

Which means that $f^L(t)$ and $f^U(t)$ are supremum of $\{f_n^L(t)\}$ and $\{f_n^U(t)\}$, respectively, provided $\{f_n\}$ is nondecreasing; however if $\{f_n\}$ is nonincreasing then $f^L(t)$ and $f^U(t)$ are both infremum of $\{f_n^L(t)\}$ and $\{f_n^U(t)\}$ (depending on the nondecreasing or nonincreasing character of the sequence, see [43]). Besides, for any $n \in \mathbb{N}$ we can find $k^* \in \mathbb{N}$ such that $n_{k^*} > n$, $f_{n_{k^*}}^L(t) \in \{f_{n_k}^L(t)\}$ and $f_{n_{k^*}}^U(t) \in \{f_{n_k}^U(t)\}$ satisfied

$$f_n^L(t) \leq f_{n_{k^*}}^L \leq f^L(t) \text{ and } f_n^U(t) \leq f_{n_{k^*}}^U \leq f^U(t) \text{ if } \{f_n\} \text{ is nondecreasing},$$
$$f_n^L(t) \geq f_{n_{k^*}}^L \geq f^L(t) \text{ and } f_n^U(t) \geq f_{n_{k^*}}^U \geq f^U(t) \text{ if } \{f_n\} \text{ is nonincreasing}.$$

That implies for each $n \in \mathbb{N}$

$$f_n \preceq_{LU} f \text{ if } \{f_n\} \text{ is nondecreasing},$$
$$f \preceq_{LU} f_n \text{ if } \{f_n\} \text{ is nonincreasing}.$$

On the other hand, for each $m \in \mathbb{N}$ such that $m > n_{k_0}$ (see (3.5)) we have

$$f_{n_{k_0}} \preceq_{LU} f_m \preceq_{LU} f \text{ if } \{f_n\} \text{ is nondecreasing,}$$
$$f \preceq_{LU} f_m \preceq_{LU} f_{n_{k_0}} \text{ if } \{f_n\} \text{ is nonincreasing.}$$

So, it follows from (3.5) that

$$\|f_m - f\|_{\mathcal{C}} = \sup_{t\in J} \mathcal{H}(f_m(t), f(t)) \leq \sup_{t\in J} \mathcal{H}(f_{n_{k_0}}(t), f(t)) < \epsilon.$$

This concludes that $\{f_n\}$ converges to f in $C(J;R(I))$. □

4. Main results

In this section, we shall investigate the existence of solutions for (IVCFDE) (1.1). To do so, the existence of solutions for interval-valued fractional integral equation (IVFIE, for short) will be considered firstly

$$x(t) \ominus_{gH} x_0 = \mathfrak{I}_t^{\alpha} f(t, x(t)),\ t \in J, \tag{4.1}$$

by method of upper and lower solutions.

Remark 7. Indeed, if $x \in C(J;R(I))$ is such that $w(x(t)) \geq w(x(0))$ for all $t \in J$, then (4.1) becomes the type I as

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))ds,\ t \in J. \tag{4.2}$$

However, when $x \in C(J;R(I))$ satisfies $w(x(t)) \leq w(x(0))$ for all $t \in J$, then (4.1) can be written type II by

$$x(t) = x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))ds,\ t \in J. \tag{4.3}$$

In particularly, if $x \in C(J;R(I))$ is w-increasing then (4.1) can be written as (4.2). On the other hand, (4.1) also can be written as (4.3), when $x \in C(J;R(I))$ is w-decreasing.

Now the definition of upper and lower solutions for problem (4.1) will be given, which plays an essential role in this paper.

Definition 17. Let $(\underline{x}_1, \overline{x}_1) \in C(J;R(I)) \times C(J;R(I))$ and $(\underline{x}_2, \overline{x}_2) \in C(J;R(I)) \times C(J;R(I))$ be two pair given functions. It says that

(i) $\underline{x}_1$ and $\overline{x}_1$ are upper and lower solutions of the type I for problem (4.1), respectively, if

$$\overline{x}_1(t) \succeq_{LU} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{x}_1(s))ds,\ t \in J,$$

and

$$\underline{x}_1(t) \preceq_{LU} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \underline{x}_1(s)) ds, \ t \in J,$$

(ii) $\underline{x}_2$ and $\overline{x}_2$ are upper and lower solutions of the type II for problem (4.1), respectively, if

$$\overline{x}_2(t) \succeq_{LU} x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{x}_2(s)) ds, \ t \in J,$$

and

$$\underline{x}_2(t) \preceq_{LU} x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \underline{x}_2(s)) ds, \ t \in J,$$

providing $x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \overline{x}_2(s)) ds$ and $x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \underline{x}_2(s)) ds$ are well-defined for each $t \in J$.

Definition 18. Let $(\underline{x}_1, \overline{x}_1)$ and $(\underline{x}_2, \overline{x}_2)$ be a pair of upper and lower solutions of the type I and type II for problem (4.1), respectively. The admissible sets U_1 and U_2 of solutions for problem (4.1) governed by $(\underline{x}_1, \overline{x}_1)$ and $(\underline{x}_2, \overline{x}_2)$ respectively, which are defined as

$$U_1 := \left\{ x \in C(J; R(I)) : \ \underline{x}_1 \preceq_{LU} x \preceq_{LU} \overline{x}_1 \text{ and } x \text{ is a type I solution of (4.1)} \right\},$$

$$U_2 := \left\{ x \in C(J; R(I)) : \ \underline{x}_2 \preceq_{LU} x \preceq_{LU} \overline{x}_2 \text{ and } x \text{ is a type II solution of (4.1)} \right\}.$$

Based on the partial ordering $\preceq_{LU}$, we now consider the concepts of maximal and minimal solutions in admissible sets U_1 and U_2.

Definition 19. Let admissible sets U_1 and U_2 be governed by $(\underline{x}_1, \overline{x}_1)$ and $(\underline{x}_2, \overline{x}_2)$, respectively. It says that

(i) $x : J \to R(I)$ is a solution of (4.1) depended on $(\underline{x}_1, \overline{x}_1)$ or $(\underline{x}_2, \overline{x}_2)$, if $x \in U_1 \cup U_2$;

(ii) $x_M \in U_1$ (resp. U_2) is a maximal solution of problem (4.1) in U_1 (resp. U_2), if

$$x \preceq_{LU} x_M, \quad \text{for each } x \in U_1 \text{ (resp. } U_2);$$

(iii) $x_L \in U_1$ (resp. U_2) is a minimal solution of problem (4.1) in U_1 (resp. U_2), if

$$x_L \preceq_{LU} x, \quad \text{for each } x \in U_1 \text{ (resp. } U_2).$$

Then, we illustrate the existence of solutions for (4.1) delimited by a pair of upper and lower solutions.

Theorem 4. *Let $f \in C(J \times R(I); R(I))$ be given. Assuming that one of the following conditions holds:*

(A_1) *There exists a pair of upper and lower solutions $(\underline{x}_1, \overline{x}_1)$ of type I for problem (4.1) such that*

$$\underline{x}_1 \preceq_{LU} \overline{x}_1,$$

and $f(t,\cdot) : [\min_{t\in J} \underline{x}_1^L(t), \max_{t\in J} \overline{x}_1^U(t)] \to R(I)$ is nondecreasing for all $t \in J$, i.e. for any $A, B \in R(I)$ with $A \preceq_{LU} B$ then $f(t,A) \preceq_{LU} f(t,B)$ for all $t \in J$.

(A_2) *There exists a pair of upper and lower solutions $(\underline{x}_2, \overline{x}_2)$ of type II for problem (4.1) such that*

$$\underline{x}_2 \preceq_{LU} \overline{x}_2,$$

which is also the sequences satisfied

$$\begin{cases} y_0 = \underline{x}_2, \\ y_{n+1} = x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_n(s))ds, \ t \in J, \ \ n = 0,1,2,\cdots, \end{cases}$$

and

$$\begin{cases} z_0 = \overline{x}_2, \\ z_{n+1} = x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z_n(s))ds, \ t \in J, \ n = 0,1,2,\cdots, \end{cases}$$

They are well-defined, for all $t \in J$, and $f(t,\cdot) : [\min_{t\in J} \underline{x}_2^L(t), \max_{t\in J} \overline{x}_2^U(t)] \to R(I)$ is nondecreasing for all $t \in J$.

Then there exist maximal and minimal solutions $x_M, x_L \in U_1$ in the case (A_1) (or $x_M, x_L \in U_2$ in the case (A_2)) satisfied for (4.1).

Proof. For one thing, we assume that (A_1) satisfied, then construct the sequence $\{y_n\}_{n=0}^{\infty}$ as follows

$$y_0 = \underline{x}_1,$$

$$y_{n+1}(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_n(s))ds, \ \forall t \in J \text{ and } n = 0,1,2,\cdots.$$

For another, constructing another sequence $\{z_n\}_{n=0}^{\infty}$ also as

$$z_0 = \overline{x}_1,$$

$$z_{n+1}(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z_n(s))ds, \ \forall t \in J \text{ and } n = 1,2,\cdots.$$

Then, we have the next claims.

Claim 1. *The sequences* $\{y_n\}_{n=0}^{\infty}$ *and* $\{z_n\}_{n=0}^{\infty}$ *are both satisfied the following relationship*

$$\underline{x}_1 = y_0 \preceq_{LU} y_1 \preceq_{LU} \cdots \preceq_{LU} y_n \preceq_{LU} \cdots \preceq_{LU} z_n \preceq_{LU} \cdots \preceq_{LU} z_1 \preceq_{LU} z_0 = \overline{x}_1. \tag{4.4}$$

To do so, we start with proving that $\{y_n\}_{n=0}^{\infty}$ is nondecreasing sequence and satisfies

$$y_n \preceq_{LU} z_0,\ n = 0, 1, \cdots.$$

By (A_1), it has $\underline{x}_1 = y_0 \preceq_{LU} z_0 = \overline{x}_1$. However, for $n = 0$, since $y_0 = \underline{x}_1$ is a lower solution of type I for problem (4.1) then

$$y_1(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_0(s))ds \succeq_{LU} y_0,\ \forall t \in J.$$

On the other hand, because $f(t,\cdot) : [\min_{t\in J} \underline{x}_1^L(t), \max_{t\in J} \overline{x}_1^U(t)] \to R(I)$ is nondecreasing for each $t \in J$, then we readily get

$$f(s, y_0(s)) \preceq_{LU} f(s, z_0(s)),\ \forall s \in [0,t],\ t \in J.$$

It is of course immediate from Lemma 2 and (ii) of Lemma 1 that

$$\begin{aligned} &y_1(t) \\ &= x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_0(s))ds \\ &\preceq_{LU} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z_0(s))ds \\ &\preceq_{LU} z_0(t),\ \forall t \in J. \end{aligned}$$

That implies $y_0 \preceq_{LU} y_1 \preceq_{LU} z_0$. Now, applying the inductive method

$$y_{n-1} \preceq_{LU} y_n \preceq_{LU} z_0. \tag{4.5}$$

Hence, obtain that for each $t \in J$

$$y_n(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_{n-1}(s))ds,$$

$$y_{n+1}(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_n(s))ds.$$

Due to the nondecreasing property of f and $y_{n-1} \preceq_{LU} y_n$, then we apply Lemma 1 and Lemma 2 again to obtain by (4.5)

$$y_n \preceq_{LU} y_{n+1} \preceq_{LU} z_0.$$

By the same arguments, one implies that $y_0 \preceq_{LU} z_{n+1} \preceq_{LU} z_n \preceq_{LU} z_0$, for each $n = 0, 1, 2, \cdots$.

To finish the proof of this claim, we need to show $y_n \preceq_{LU} z_n$, for all $n \in \mathbb{N}$. The induction can also be used to prove this conclusion. When $n = 0$, it is obvious $\underline{x}_1 = y_0 \preceq_{LU} z_0 = \overline{x}_1$ thanks to (A_1). So, assuming that

$$y_n \preceq_{LU} z_n. \tag{4.6}$$

Then, by the definitions of y_{n+1} and z_{n+1}, It has

$$y_{n+1}(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y_n(s))ds,$$
$$z_{n+1}(t) = x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, z_n(s))ds.$$

Similarly, we obtain

$$y_{n+1} \preceq_{LU} z_{n+1},$$

because f is nondecreasing with respect to the second variable and (4.6). Consequently, the proof of claim finished.

Claim 2. *$\{y_n\}$ and $\{z_n\}$ are relatively compact in $C(J;R(I))$.*

By Claim 1 and the continuity of f, we know that $\{y_n(t)\}$ and $\{z_n(t)\}$ are both bounded in $R(I)$ for each $t \in J$. Then, it deduces that there exists a constant $M_f > 0$ such that for each $n \in \mathbb{N}$ and $t \in J$

$$\mathcal{H}(f(t, y_n(t)), \{0\}) \le M_f, \quad \mathcal{H}(f(t, z_n(t)), \{0\}) \le M_f.$$

For each $n \in \mathbb{N}$ and $\forall t \in J$, we have the following estimate

$$\begin{aligned}
\mathcal{H}(y_{n+1}(t), x_0) &= \mathcal{H}\left(x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y_n(s))ds, x_0\right) \\
&= \mathcal{H}\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y_n(s))ds, \{0\}\right) \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \mathcal{H}(f(s, y_n(s)), \{0\})ds \\
&\le \frac{M_f}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} ds \le \frac{M_f T^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

which means that $\{y_n\}$ is uniformly bounded. In addition, for any $t_1, t_2 \in J$, without loss of generality we assume $t_1 \le t_2$, then calculate

$$\begin{aligned}
&\mathcal{H}(y_{n+1}(t_1), y_{n+1}(t_2)) \\
&= \mathcal{H}\Bigg(x_0 + \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_1-s)^{\alpha-1} f(s, y_n(s))ds, x_0 \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{t_2} (t_2-s)^{\alpha-1} f(s, y_n(s))ds\Bigg)
\end{aligned} \tag{4.7}$$

$$
\begin{aligned}
&= \mathcal{H}\left(\frac{1}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,y_n(s))ds, \frac{1}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}f(s,y_n(s))ds\right)\\
&= \frac{1}{\Gamma(\alpha)}\mathcal{H}\left(\int_0^{t_1}(t_1-s)^{\alpha-1}f(s,y_n(s))ds, \int_0^{t_2}(t_2-s)^{\alpha-1}f(s,y_n(s))ds\right)\\
&= \frac{1}{\Gamma(\alpha)}\max\left\{\left|\int_0^{t_2}(t_2-s)^{\alpha-1}f^L(s,y_n(s))ds-\int_0^{t_1}(t_1-s)^{\alpha-1}f^L(s,y_n(s))ds\right|,\right.\\
&\quad\left.\left|\int_0^{t_2}(t_2-s)^{\alpha-1}f^U(s,y_n(s))ds-\int_0^{t_1}(t_1-s)^{\alpha-1}f^U(s,y_n(s))ds\right|\right\}\\
&\le \frac{M_f}{\Gamma(\alpha)}\left(\int_0^{t_1}[(t_1-s)^{\alpha-1}-(t_2-s)^{\alpha-1}]ds+\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}ds\right)\\
&= \frac{M_f}{\Gamma(\alpha+1)}\left(t_1^\alpha-t_2^\alpha+2(t_2-t_1)^\alpha\right)\to 0, \text{ as } |t_1-t_2|\to 0.
\end{aligned}
$$

This means that $\{y_n\}$ is equicontinuous in $C(J;R(I))$. Therefore, utilizing Arzela-Ascoli Theorem (see [46]) it concludes that $\{y_n\}$ is relatively compact in $C(J;R(I))$. Similarly, we can also get that $\{z_n\}$ is relatively compact in $C(J;R(I))$.

By virtue of Claim 1 and Claim 2, it knows that $\{y_n\}$ and $\{z_n\}$ are monotone and relatively compact. Thereby, applying Lemma 5 that $\{y_n\}$ and $\{z_n\}$ are two Cauchy sequences, i.e., there exists continuous functions x_L and x_M such that $\{y_n\}$ and $\{z_n\}$ converge uniformly to x_L and x_M in $C(J;R(I))$, respectively. Hence, we readily obtain for each $t\in J$

$$
\begin{aligned}
x_L(t) &= x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x_L(s))ds,\\
x_M(t) &= x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x_M(s))ds.
\end{aligned}
$$

Also, according to Lemma 4 and Lemma 5, it has $\underline{x}_1 \preceq_{LU} x_L \preceq_{LU} x_M \preceq_{LU} \overline{x}_1$, i.e., $x_L, x_M \in U_1$. Furthermore, we get that x_L and x_M are minimal and maximal solutions in U_1, respectively. Assuming that $x\in U_1$ then, it has

$$\underline{x}_1 \preceq_{LU} x \preceq_{LU} \overline{x}_1.$$

Since f is nondecreasing, then using the similar methods, we obtain $y_1 \preceq_{LU} x \preceq_{LU} z_1$. So, that inducts

$$\underline{x}_1 \preceq_{LU} y_n \preceq_{LU} x \preceq_{LU} z_n \preceq_{LU} \overline{x}_1, \quad \text{for each } n\in\mathbb{N}. \tag{4.8}$$

Taking limits as $n\to\infty$ into (4.8), one yields

$$\underline{x}_1 \preceq_{LU} x_L \preceq_{LU} x \preceq_{LU} x_M \preceq_{LU} \overline{x}_1.$$

That finishes the proof, when (A_1) is satisfied.

However, if (A_2) holds. Then, it can also be written as $y_0 = \underline{x}_2$, $z_0 = \overline{x}_2$ and then construct sequences $\{y_n\}$ and $\{z_n\}$ as follows

$$y_{n+1}(t) = x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_n(s)) ds,$$
$$z_{n+1}(t) = x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, z_n(s)) ds.$$

By the same arguments of the previous part, we assert that

$$\underline{x}_2 = y_0 \preceq_{LU} y_1 \preceq_{LU} \cdots \preceq_{LU} y_n \preceq_{LU} \cdots \preceq_{LU} z_n \preceq_{LU} \cdots \preceq_{LU} z_1 \preceq_{LU} z_0 = \overline{x}_2.$$

Following the same trend above in addition to Lemma 3 to prove the theorem. □

Theorem 5. *Let $f : J \times R(I) \to R(I)$ be a continuous interval-valued function. Then a w-monotone interval-valued function $x \in C(J; R(I))$ is a solution of (IVFIE) (4.1) then it also satisfies (IVCFDE) (1.1).*

Proof. Let $x \in C(J; R(I))$ be a w-monotone solution of (IVFIE) (4.1), then it can be written as

$$x(t) \ominus_{gH} x_0 = \mathfrak{I}_t^\alpha f(t, x(t)), \ \forall t \in J.$$

By the continuity of f, we easily get that $f^L(t, x(t))$ and $f^U(t, x(t))$ are both continuous. Then, $t \mapsto I_t^\alpha f^L(t, x(t))$ and $t \mapsto I_t^\alpha f^U(t, x(t))$ are absolutely continuous on J, i.e., $\mathfrak{I}_t^\alpha f(t)$ is absolutely continuous by Proposition 2. Hence, it has $x(0) \ominus_{gH} x_0 = 0$, thus is $x(0) = x_0$. Since x is w-monotone on J then it follows from [7, Lemma 1] that $x(t) \ominus_{gH} x_0$ is w-increasing on J. Therefore $t \mapsto \mathfrak{I}_t^\alpha f(t, x(t))$ is also w-increasing and

$$\mathfrak{I}_t^\alpha f(t, x(t)) = [I_t^\alpha f^L(t, x(t)), I_t^\alpha f^U(t, x(t))], \ \forall t \in J.$$

Hence,

$$\begin{aligned}
{}^C\mathfrak{D}_t^\alpha \mathfrak{I}_t^\alpha f(t, x(t)) &= {}^L\mathfrak{D}_t^\alpha [\mathfrak{I}_t^\alpha f(t, x(t)) \ominus_{gH} (\mathfrak{I}_t^\alpha f(t, x(t))|_{t=0})] \\
&= {}^L\mathfrak{D}_t^\alpha [\mathfrak{I}_t^\alpha f(t, x(t)) \ominus_{gH} \{0\}] \\
&= {}^L\mathfrak{D}_t^\alpha \mathfrak{I}_t^\alpha f(t, x(t)) \\
&= {}^L\mathfrak{D}_t^\alpha [I_t^\alpha f^L(t, x(t)), I_t^\alpha f^U(t, x(t))] \\
&= \frac{d}{dt} \mathfrak{I}_t^{1-\alpha} [I_t^\alpha f^L(t, x(t)), I_t^\alpha f^U(t, x(t))] \\
&= \frac{d}{dt} [I_t^{1-\alpha} I_t^\alpha f^L(t, x(t)), I_t^{1-\alpha} I_t^\alpha f^U(t, x(t))] \\
&= \frac{d}{dt} \left[\int_0^t f^L(s, x(s)) ds, \int_0^t f^U(s, x(s)) ds \right] \\
&= [f^L(t, x(t)), f^U(t, x(t))]
\end{aligned}$$

$$= f(t,x(t)) \text{ for a.e. } t \in J. \tag{4.9}$$

On the other hand, applying Theorem 2 it directly obtains that

$$\begin{aligned}
&{}^{C}\mathfrak{D}_t^{\alpha}[x(t) \ominus_{gH} x_0] \\
&= {}^{L}\mathfrak{D}_t^{\alpha}[(x(t) \ominus_{gH} x_0) \ominus_{gH} (x(t) \ominus_{gH} x_0|_{t=0})] \\
&= {}^{L}\mathfrak{D}_t^{\alpha}[(x(t) \ominus_{gH} x_0) \ominus_{gH} \{0\}] \\
&= {}^{L}\mathfrak{D}_t^{\alpha}[x(t) \ominus_{gH} x_0] \\
&= {}^{L}\mathfrak{D}_t^{\alpha}[x(t) \ominus_{gH} x(0)] \\
&= {}^{C}\mathfrak{D}_t^{\alpha} x(t), \text{ for a.e. } t \in J.
\end{aligned} \tag{4.10}$$

Combining (4.9) with (4.10) it concludes

$$\begin{cases} {}^{C}\mathfrak{D}_t^{\alpha} x(t) = f(t,x(t)), \text{ for a.e. } t \in J, \\ x(0) = x_0. \end{cases}$$

This means that $x \in C(J;R(I))$ is a solution of (IVCFDE) (1.1). □

Now, we denote U as the solution set of (IVCFDE) (1.1).

Theorem 6. *Assuming that the same hypothesises as in Theorem 4 hold. If, in addition, $t \mapsto \mathfrak{I}_t^{\alpha} f(t,x(t))$ is w-increasing, for each $\underline{x}_1 \preceq_{LU} x \preceq_{LU} \overline{x}_1$ (or $\underline{x}_2 \preceq_{LU} x \preceq_{LU} \overline{x}_2$). Then, it has $U_1 \subseteq U$ (or $U_2 \subseteq U$), thus is (IVCFDE) (1.1) has at least one w-monotone solution.*

Proof. By the assumptions, if (A_1) is satisfied, then we know that $U_1 \neq \varnothing$. For any $x \in U_1$, thus is

$$x(t) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,x(s))ds, \ \forall t \in J.$$

Hence, it calculate

$$w(x(t)) = w(x_0) + w(\mathfrak{I}_t^{\alpha} f(t,x(t))).$$

Therefore, x is w-increasing, because $t \mapsto \mathfrak{I}_t^{\alpha} f(t,x(t))$ is w-increasing, for each $\underline{x}_1 \preceq_{LU} x \preceq_{LU} \overline{x}_1$. Consequently, applying Theorem 5 we conclude that x is also a solution of (IVCFDE) (1.1). This means $U_1 \subseteq U$.

Similarly, when (A_2) holds, we also get that for each $x \in U_2$, x is w-decreasing. So, applying Theorem 5 again, it implies $U_2 \subseteq U$. □

5. Numerical examples

In this section, we mainly use foregoing analysis to present some simple examples to solve the interval-valued Caputo fractional differential equation.

Example 1. Consider the following interval-valued Caputo fractional differential equation

$$\begin{cases} {}^C\mathfrak{D}_t^{\frac{1}{2}}x(t) = [\frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}} - t^2 + x^L(t), \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}} - t - 1 + x^U(t)],\ t \in [0, \frac{1}{2}], \\ x(0) = [0,1]. \end{cases} \tag{5.1}$$

Proof. Denote $f(t,x(t)) = [\frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}} - t^2 + x^L(t), \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}} - t - 1 + x^U(t)],\ t \in [0, \frac{1}{2}]$. In this probelm, we just discuss

$$x(t) = x_0 + \mathfrak{I}_t^{\frac{1}{2}}[\frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}} - t^2 + x^L(t), \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}} - t - 1 + x^U(t)],\ t \in [0, \frac{1}{2}]. \tag{5.2}$$

Let $\underline{x}_1 = [0,1]$, $\overline{x}_1 = [t^2 + t^3, 1 + t + t^2]$ and

$$B_1 = \left[\min_{t\in[0,0.5]} \underline{x}_1^L(t), \max_{t\in[0,0.5]} \underline{x}_1^U(t)\right] = \left[0, \frac{7}{4}\right].$$

By the form of f, we readily get f is continuous and nondecreasing in B_1 with respect to the second variable. Also, $\underline{x}_1 \preceq_{LU} \overline{x}_1$, this means that all conditions of (A_1) in Theorem 4 are satisfied. So, it constructs the sequences $\{y_n\}$ and $\{z_n\}$ as follows

$$y_0 = \underline{x}_1,$$
$$y_{n+1} = x_0 + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}} f(s, y_n(s))ds,$$

and

$$z_0 = \overline{x}_1,$$
$$z_{n+1} = x_0 + \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}} f(s, z_n(s))ds.$$

By Theorem 4, it knows that $y_n \to x_L$ in $C([0,0.5]; R(I))$ and $z_n \to x_M$ in $C([0,0.5]; R(I))$. In fact, may calculate that

$$x_L(t) = x_M(t) = x(t) = [t^2, 1+t],\ t \in [0, 0.5].$$

However, $w(x(t)) = 1 + t - t^2$ is w-increasing on $[0, 0.5]$ then by Theorem 5, we conclude that x is a solution of (5.1). Moreover, by applying the numerical scheme described in the previous processes, we obtain Fig. 1 and Table 1. □

Example 2. Consider the following interval-valued Caputo fractional differential equation

$$\begin{cases} {}^C\mathfrak{D}_t^{\frac{1}{2}}x(t) = [\frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}} - t + x^L(t), \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}} - t^2 - 1 + x^U(t)], \\ x(0) = [0,1]. \end{cases} \tag{5.3}$$

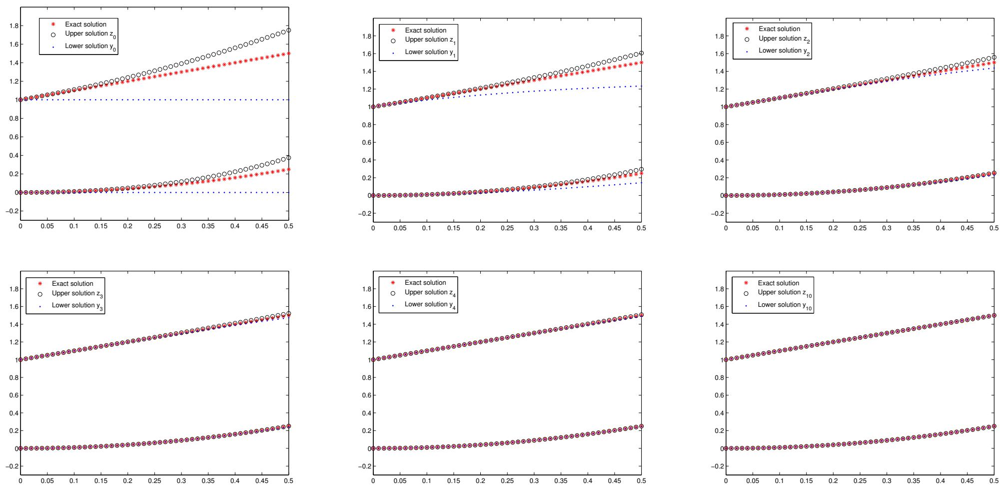

FIGURE 1. A plot of y_k and z_k, $k = 0, 1, 2, 3, 4, 10$, for Example 1

TABLE 1. Error Analysis

Iterations / Absolute Error	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$\sup_{t\in[0,0.5]} \lvert y_n^L(t) - x^L(t)\rvert$	2.60×10^{-3}	5.90×10^{-6}	5.38×10^{-9}	2.52×10^{-12}
$\sup_{t\in[0,0.5]} \lvert y_n^U(t) - x^U(t)\rvert$	1.04×10^{-2}	3.84×10^{-5}	4.84×10^{-8}	2.90×10^{-11}
$\sup_{t\in[0,0.5]} \lvert z_n^L(t) - x^L(t)\rvert$	3.91×10^{-4}	5.90×10^{-7}	4.04×10^{-10}	1.51×10^{-13}
$\sup_{t\in[0,0.5]} \lvert z_n^U(t) - x^U(t)\rvert$	7.36×10^{-3}	1.67×10^{-5}	1.52×10^{-8}	7.14×10^{-12}

Proof. Let $f(t,x(t)) := [\frac{8}{3\sqrt{\pi}}t^{\frac{3}{2}} - t + x^L(t), \frac{2}{\sqrt{\pi}}t^{\frac{1}{2}} - t^2 - 1 + x^U(t)]$, $t \in [0, \frac{1}{2}]$, $t \in [0, 0.5]$. For problem (5.3), we just discuss

$$x(t) = x_0 \ominus \frac{(-1)}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}} f(s, x(s))ds,\ t \in [0, 0.5]. \tag{5.4}$$

Consider the upper and lower solutions of (5.4) as

$$\underline{x}_2(t) = [0, 1],$$
$$\overline{x}_2(t) = [t + t^2, 1 + t^2 + t^3].$$

We may verify that all condition of (A_2) in Theorem 4 also hold. Consequently, it constructs the sequences $\{y_n\}$ and $\{z_n\}$ as follows

$$y_{n+1}(t) = x_0 \ominus (-1)\cdot\frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-s)^{-\frac{1}{2}} f(s, y_n(s))ds,$$

$$z_{n+1}(t) = x_0 \ominus (-1) \cdot \frac{1}{\Gamma(\frac{1}{2})} \int_0^t (t-s)^{-\frac{1}{2}} f(s, z_n(s)) ds.$$

It directly calculate that $y_n \to [t, t^2+1]$ and $z_n \to [t, t^2+1]$. So, $x(t) = [t, t^2+1]$ is a solution of (5.4). Besides, $w(x(t)) = 1 + t^2 - t$ is w-decreasing, then applying Theorem 5 again to obtain that x is also a solution of (5.3). In addition, we also obtain Fig. 2 and Table 2. □

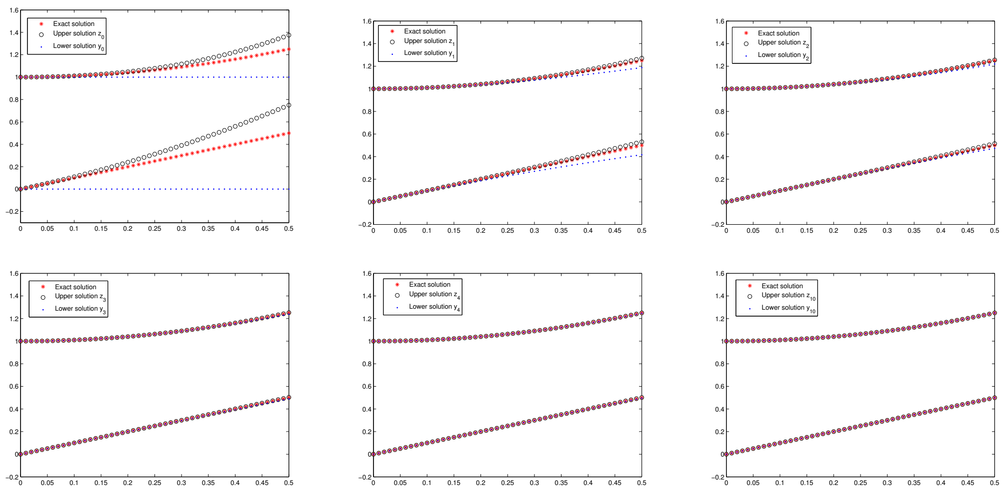

FIGURE 2. A plot of y_k and z_k, $k = 0, 1, 2, 3, 4, 10$, for Example 2

TABLE 2. Error Analysis

Absolute Error \ Iterations	$n = 5$	$n = 10$	$n = 15$	$n = 20$
$\sup_{t\in[0,0.5]} \lvert y_n^L(t) - x^L(t)\rvert$	1.04×10^{-2}	3.84×10^{-5}	2.15×10^{-8}	2.90×10^{-11}
$\sup_{t\in[0,0.5]} \lvert y_n^U(t) - x^U(t)\rvert$	1.04×10^{-2}	2.36×10^{-5}	4.82×10^{-8}	1.01×10^{-11}
$\sup_{t\in[0,0.5]} \lvert z_n^L(t) - x^L(t)\rvert$	3.13×10^{-3}	1.18×10^{-5}	3.23×10^{-9}	5.05×10^{-12}
$\sup_{t\in[0,0.5]} \lvert z_n^U(t) - x^U(t)\rvert$	2.60×10^{-3}	2.36×10^{-6}	5.38×10^{-9}	6.06×10^{-13}

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REFERENCES

[1] U. Abbas and V. Lupulescu, "Set functional functional differential equations," *Communications on Applied Nonlinear Analysis*, vol. 18, pp. 97–110, 2011.

[2] B. Ahmad and S. Ntouyas, "Initial value problems for functional and neutral functional Hadamard type fractional differential inclusions," *Miskolc Mathematical Notes*, vol. 17, pp. 15–27, 2016, doi: 0.18514/MMN.2016.1632.

[3] G. Alefeld and G. Mayer, "Interval analysis: theory and applications," *J. Comput. Appl. Math.*, vol. 121, pp. 421–464, 2000, doi: 10.1016/S0377-0427(00)00342-3.

[4] Z. Arstein, "On the calculus of closed set-valued functions," *Indiana Univ. Math. J.*, vol. 24, pp. 433–441, 1974, doi: http://www.jstor.org/stable/24890804.

[5] R. Aumann, "Integrals of set-valued functions," *J. Math. Anal. Appl.*, vol. 12, pp. 1–12, 1965, doi: 10.1016/0022-247X(65)90049-1.

[6] Y. Chalco-Cano, H. Román-Flores, and M. Jiménez-Gamero, "Generalized derivative and π-derivative for set-valued functions," *Inf. Sci.*, vol. 181, pp. 2177–2188, 2011, doi: 10.1016/j.ins.2011.01.023.

[7] Y. Chalco-Cano, A. Rufán-Lizana, H. Román-Flores, and M. Jiménez-Gamero, "Calculus for interval-valued functions using generalized Hukuhara derivative and applications," *Fuzzy Sets Syst.*, vol. 219, pp. 119–141, 2013, doi: 10.1016/j.fss.2012.12.004.

[8] Y. Huang, Z. Liu, and R. Wang, "Quasilinearization for higher order impulsive fractional differential equations," *Appl. Comput. Math.*, vol. 15, pp. 159–171, 2016.

[9] Y. Huang, J. Lv, and Z. Liu, "Existence results for Riemann-Liouville fractional evolution inclusions," *Miskolc Mathematical Notes*, vol. 17, pp. 305–325, 2016, doi: 10.18514/MMN.2016.1360.

[10] M. Hukuhara, "Intégration des applications mesurables dont la valeur est un compact convex," *Funkcial. Ekvac.*, vol. 10, pp. 205–229, 1967.

[11] M. Inuiguchi and Y. Kume, "Goal programming problems with interval coefficients and target intervals," *European Journal of Operational Research*, vol. 52, pp. 345–360, 1991, doi: 10.1016/0377-2217(91)90169-V.

[12] B. Kearfott and V. Kreinovich, *Applications of Interval Computations*. Dordrecht: Kluwer Academic Publishers, 1996.

[13] A. Khastan and J. Nieto, "A boundary value problem for second order fuzzy differential equations," *Nonlinear Anal.*, vol. 72, pp. 3583–3593, 2010, doi: 10.1016/j.na.2009.12.038.

[14] A. Kilbas, H. Srivastava, and J. Trujillo, *Theory an Applications of Fractional Differential Equations*. Amsterdam: Elsevier Science B.V., 2006.

[15] L. Kolev, *Interval Methods for Circuit Analysis*. Singapore: World Scientific, 1993.

[16] V. Lakshmikantham, T. G. Bhaskar, and J. V. Devi, *Theory of Set Differential Equations in Metric Spaces*. Cambridge: Cambridge Scientific Publishers, 2002.

[17] S. Li, Y. Ogura, and V. Kreinovich, *Limit Theorems and Applications of Set-Valued and Fuzzy Set-Valued Random Variables*. Dordrecht: Kluwer Academic Publishers, 2002.

[18] X. Liu and Z. Liu, "On the 'bang-bang' principle for a class of fractional semilinear evolution inclusions," *Proceedings of the Royal Society of Edinburgh*, vol. 144, pp. 333–349, 2014, doi: 10.1017/S030821051200128X.

[19] X. Liu, Z. Liu, and X. Fu, "Relaxation in nonconvex optimal control problems described by fractional differential equations," *J. Math. Anal. Appl.*, vol. 409, pp. 446–458, 2014, doi: 10.1016/j.jmaa.2013.07.032.

[20] Z. Liu and X. Li, "Existence and uniqueness of solutions for the nonlinear impulsive fractional differential equations," *Commun Nonlinear Sci. Numer. Simulat.*, vol. 18, pp. 1362–1373, 2013, doi: 10.1016/j.cnsns.2012.10.010.

[21] Z. Liu and X. Li, "Approximate controllability of fractional evolution systems with Riemann-Liouville fractional derivatives," *SIAM J. Control Optim.*, vol. 53, pp. 1920–1933, 2015, doi: 10.1137/120903853.

[22] Z. Liu and J. Sun, "Nonlinear boundary value problems of fractional differential systems," *Computers and Mathematics with Applications*, vol. 64, pp. 463–475, 2012, doi: j.camwa.2011.12.020.

[23] Z. Liu, J. Sun, and I. Szanto, "Monotone Iterative Technique for Riemann-Liouville Fractional Integro-Differential Equations with Advanced Arguments," *Results. Math.*, vol. 63, pp. 1277–1287, 2013, doi: 10.1007/s00025-012-0268-4.

[24] Z. Liu, S. Zeng, and Y. Bai, "Maximum Principles for multi-term space-time variable-order fractional diffusion equations and their applications," *Fract. Calc. Appl. Anal.*, vol. 19, pp. 188–211, 2016, doi: 10.1515/fca-2016-0011.

[25] J. Llobet, *Application of Modal Interval Analysis to the simulation of the behavior of dynamical systems with uncertain parameters*. PhD. Thesis: University of Girona, 1999.

[26] V. Lupulescu, "On a class of fuzzy functional differential equations," *Fuzzy Sets Syst.*, vol. 160, pp. 1547–1562, 2009, doi: 10.1016/j.fss.2008.07.005.

[27] V. Lupulescu, "Hukuhara differentiability of interval-valued functions and interval differential equations on times scales," *Inf. Sci.*, vol. 248, pp. 50–67, 2013, doi: 10.1016/j.ins.2013.06.004.

[28] V. Lupulescu, "Fractional calculus for interval-valued functions," *Fuzzy Sets Syst.*, vol. 265, pp. 63–85, 2015, doi: 10.1016/j.fss.2014.04.005.

[29] M. Malinowski, "Random fuzzy differential equations under generalized Lipschitz condition," *Nonlinear Anal.*, vol. 13, pp. 860–881, 2012, doi: j.nonrwa.2011.08.022.

[30] M. Malinowski, "On a new set-valued stochastic integral with respect to semimartingales and its applications," *J. Math. Anal. Appl.*, vol. 408, pp. 669–680, 2013, doi: 10.1016/j.jmaa.2013.06.054.

[31] M. Malinowski, "Some properties of strong solutions to stochastic fuzzy differential equations," *Inf. Sci.*, vol. 252, pp. 62–80, 2013, doi: 10.1016/j.ins.2013.02.053.

[32] S. Markov, "Calculus for interval functions of a real variables," *Computing*, vol. 22, pp. 325–337, 1979, doi: 10.1007/BF02265313.

[33] S. Markov, "On the algebraic properties of convex bodies and some applications," *J. Convex Anal.*, vol. 7, pp. 129–166, 2000, doi: http://eudml.org/doc/121073.

[34] R. Moore, *Interval Analysis*. New York: Prentice-Hall, Englewood Cliffs, 1966.

[35] R. Moore, *Methods and applications of Interval Analysis*. Philadelphia: SIAM Stuides in Applied Mathematics, SIAM, 1966.

[36] A. Neumaier, *Interval Methods for Systems of Equations*. Cambridge: Cambridge University Press, 1990.

[37] H. Nguyen, V. Kreinovich, and Q. Zuo, "Interval-valued degrees of belief: applications of interval computations to expert systems and intelligent control," *Int. J. Uncertain. Fuzziness Knowl.-Based Syst.*, vol. 5, pp. 317–358, 1997, doi: 10.1142/S0218488597000257.

[38] N. Plotnikova, "Systems of linear differential equations with π-derivative and linear differential inclusions," *Sb. Math.*, vol. 196, pp. 1677–1691, 2005, doi: 10.1070/SM2005v196n11ABEH003726.

[39] I. Podlubny, *Fractional Differential Equation*. San Diego: Academic Press, 1989.

[40] R. Scgbeuderm, *Convex Bodies: The Brunn-Minkowski Theory*. Cambridge: Cambridge University Press, 1993.

[41] L. Stefanini, "A generalization of Hukuhara difference and division for interval and fuzzy arithemetic," *Fuzzy Sets Syst.*, vol. 161, pp. 1564–1584, 2010, doi: 10.1016/j.fss.2009.06.009.

[42] L. Stefanini and B. Bede, "Generalized Hukuhara differentiability of interval-valued functions and interval differential equations," *Nonlinear Anal.*, vol. 71, pp. 1311–1328, 2009, doi: 10.1016/j.na.2008.12.005.

[43] R. Wheeden and A. Zygmund, *Measure and Integral: An Introduction to Real Analysis*. New York: Marcel Dekker, Inc., 1977.

[44] H. Wu, "The Duality theory in fuzzy linear programming problems based on the concept of necessity," *Fuzzy Sets Syst.*, vol. 139, pp. 363–367, 2003, doi: 10.1023/A:1022852314914.

[45] H. Wu, "The Karush-Kuhn-Tucker optimality conditions in an optimization problem with interval-valued objective function," *European Journal of Operational Research*, vol. 176, pp. 46–59, 2007, doi: 10.1016/j.ejor.2005.09.007.

[46] E. Zeidler, *Nonlinear Functional Analysisc and its Applications, vol. II/B, Nonlinear Monotone Operators*. New York: Springer, 1990.

[47] S. Zeng and S. Migórski, "A Class of time-fractional hemivariational inequalities with application to frictional contact problem," *Commun Nonlinear Sci. Numer. Simulat.*, vol. 56, pp. 34–48, 2018, doi: 10.1016/j.cnsns.2017.07.016.

[48] J. Zhao and R. Wang, "Mixed monotone iterative technique for fractional impulsive evolution equations," *Miskolc Mathematical Note*, vol. 17, pp. 683–696, 2016, doi: 10.18514/MMN.2016.1380.

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