



## SHARP POWER MEAN BOUNDS FOR THE SECOND NEUMAN MEAN

XIAO-HONG HE, YUE-YING YANG, AND WEI-MAO QIAN

*Received 20 November, 2016*

*Abstract.* In this paper, we prove that the double inequality  $M_\alpha(a, b) < N_{GQ}(a, b) < M_\beta(a, b)$  holds for all  $a, b > 0$  with  $a \neq b$  if and only if  $\alpha \leq 2\log 2 / (5\log 2 - 2\log \pi) = 1.1785\dots$  and  $\beta \geq 4/3$ , where  $N_{GQ}(a, b) = [G(a, b) + Q^2(a, b)/U(a, b)]/2$  is the second Neuman mean,  $G(a, b) = \sqrt{ab}$ ,  $Q(a, b) = \sqrt{(a^2 + b^2)/2}$  and  $U(a, b) = (a - b)/[\sqrt{2}\tan^{-1}((a - b)/\sqrt{2ab})]$  are the geometric, quadratic and Yang mean of  $a$  and  $b$ , respectively.

2010 *Mathematics Subject Classification:* 26E60

*Keywords:* Schwab-Borchardt mean, second Neuman mean, power mean, geometric mean, Yang mean

### 1. INTRODUCTION

For  $p \in \mathbb{R}$  and  $a, b > 0$  with  $a \neq b$ , the  $p$ th power mean  $M_p(a, b)$ [14] of  $a$  and  $b$  is defined by

$$M_p(a, b) = \begin{cases} \left(\frac{a^p + b^p}{2}\right)^{1/p} & \text{if } p \neq 0 \\ \sqrt{ab} & \text{if } p = 0. \end{cases}$$

It is well known that the power mean  $M_p(a, b)$  is continuous and strictly increasing with respect to  $p \in \mathbb{R}$  for fixed  $a, b > 0$  with  $a \neq b$ . Many bivariate means are the special cases of the power mean, for example,  $M_0(a, b) = G(a, b) = \sqrt{ab}$ ,  $M_1(a, b) = A(a, b) = (a + b)/2$  and  $M_2(a, b) = Q(a, b) = \sqrt{(a^2 + b^2)/2}$  are respectively the arithmetic, geometric and quadratic means. Many properties for the power mean can be found in the literature[2–5, 11, 22, 24, 26, 31, 36].

The Schwab-Borchardt mean  $SB(a, b)$ [16, 17] defined by

$$SB(a, b) = \begin{cases} \frac{\sqrt{b^2 - a^2}}{\cos^{-1}(a/b)} & \text{if } a < b \\ \frac{\sqrt{a^2 - b^2}}{\cosh^{-1}(a/b)} & \text{if } a > b, \end{cases}$$

where  $\cos^{-1}(x)$  and  $\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$  are the inverse cosine and inverse hyperbolic cosine functions, respectively. It is well-known that  $SB(a, b)$  is strictly

increasing in both  $a$  and  $b$ , nonsymmetric and homogeneous of degree 1 with respect to  $a$  and  $b$ . Many symmetric bivariate means are special cases of the Schwab-Borchardt mean, for example, the first Seiffert mean  $P(a, b)$ , second Seiffert mean  $T(a, b)$ , Neuman-Sándor mean  $M(a, b)$ , logarithmic mean  $L(a, b)$  and Yang mean  $U(a, b)$ [29] are respectively defined by

$$P(a, b) = \frac{a-b}{2 \sin^{-1} [(a-b)/(a+b)]} = SB[G(a, b), A(a, b)],$$

$$T(a, b) = \frac{a-b}{2 \tan^{-1} [(a-b)/(a+b)]} = SB[A(a, b), Q(a, b)],$$

$$M(a, b) = \frac{a-b}{2 \sinh^{-1} [(a-b)/(a+b)]} = SB[Q(a, b), A(a, b)],$$

$$L(a, b) = \frac{a-b}{2 \tanh^{-1} [(a-b)/(a+b)]} = SB[A(a, b), G(a, b)],$$

and

$$U(a, b) = \frac{a-b}{\sqrt{2} \tan^{-1} [(a-b)/\sqrt{2ab}]} = SB[G(a, b), Q(a, b)].$$

In 2014, Neuman [15] found a new bivariate means derived from the Schwab-Borchardt mean

$$N(a, b) = \frac{1}{2} \left[ a + \frac{b^2}{SB(a, b)} \right].$$

We call  $N(a, b)$  is the second Neuman mean[19]. Let  $a > b$ ,  $v = (a-b)/(a+b) \in (0, 1)$ , then Neuman [15] gave explicit formulas

$$N_{AG}(a, b) = \frac{1}{2} A(a, b) \left[ 1 + (1-v^2) \frac{\tanh^{-1}(v)}{v} \right], N_{GA}(a, b) = \frac{1}{2} A(a, b) \left[ \sqrt{1-v^2} + \frac{\sin^{-1}(v)}{v} \right],$$

$$N_{QA}(a, b) = \frac{1}{2} A(a, b) \left[ \sqrt{1+v^2} + \frac{\sinh^{-1}(v)}{v} \right], N_{AQ}(a, b) = \frac{1}{2} A(a, b) \left[ 1 + (1+v^2) \frac{\tan^{-1}(v)}{v} \right].$$

and proved that the inequalities

$$G(a, b) < N_{AG}(a, b) < N_{GA}(a, b) < A(a, b) < N_{QA}(a, b) < N_{AQ}(a, b) < Q(a, b)$$

for  $a, b > 0$  with  $a \neq b$ .

Very recently, Shen et. al. [21] found a new mean  $N_{GQ}(a, b)$  derived from the Schwab- Borchardt mean. Let  $a > b$ ,  $u = (a-b)/\sqrt{2ab} \in (0, +\infty)$ , then explicit formulas for  $N_{GQ}(a, b)$  be in the following:

$$N_{GQ}(a, b) = \frac{1}{2} G(a, b) \left[ 1 + (1+u^2) \frac{\tan^{-1}(u)}{u} \right].$$

Recently, the bounds involving the power and the Schwab-Borchardt means has been the subject of intensive research. In particular, many remarkable inequalities for the power mean, Schwab-Borchardt mean and their related means can be found in the literature [1, 6–10, 12, 13, 18–21, 23, 25, 27–30, 32–35].

Radó[20] (see also [13, 18, 23]) proved that the double inequalities

$$M_p(a, b) < L(a, b) < M_q(a, b), M_\lambda(a, b) < I(a, b) < M_\mu(a, b)$$

hold for all  $a, b > 0$  with  $a \neq b$  if and only if  $p \leq 0$ ,  $q \geq 1/3$ ,  $\lambda \leq 2/3$  and  $\mu \geq \log 2$ , where  $I(a, b) = (a^a/b^b)^{1/(a-b)}/e$  is the indetric mean of  $a$  and  $b$ .

In [7–10, 12, 28], the authors proved that  $p_1 = \log 2/\log \pi$ ,  $q_1 = 2/3$ ,  $p_2 = \log 2/(\log \pi - \log 2)$ ,  $q_2 = 5/3$ ,  $p_3 = \log 2/\log[2\log(1 + \sqrt{2})]$  and  $q_3 = 4/3$  are the best possible parameters such that the double inequalities

$$M_{p_1}(a, b) < P(a, b) < M_{q_1}(a, b),$$

$$M_{p_2}(a, b) < T(a, b) < M_{q_2}(a, b),$$

$$M_{p_3}(a, b) < M(a, b) < M_{q_3}(a, b)$$

hold for all  $a, b > 0$  with  $a \neq b$ .

Chu [6] and Yang [30] proved that the double inequalities

$$M_{\lambda_1}(a, b) < X(a, b) < M_{\mu_1}(a, b), M_{\lambda_2}(a, b) < U(a, b) < M_{\mu_2}(a, b)$$

hold for all  $a, b > 0$  with  $a \neq b$  if and only if  $\lambda_1 \leq 1/3$ ,  $\mu_1 \geq \log 2/(1 + \log 2)$ ,  $\lambda_2 \leq 2\log 2/(2\log \pi - \log 2)$  and  $\mu_2 \geq 4/3$ , where  $X(a, b) = Ae^{G/P-1}$  is the Sándor mean of  $a$  and  $b$ .

In [21], the authors proved the double inequalities

$$\alpha_1 Q(a, b) + (1 - \alpha_1)G(a, b) < N_{GQ}(a, b) < \beta_1 Q(a, b) + (1 - \beta_1)G(a, b),$$

$$\frac{\alpha_2}{G(a, b)} + \frac{1 - \alpha_2}{Q(a, b)} < \frac{1}{N_{GQ}(a, b)} < \frac{\beta_2}{G(a, b)} + \frac{1 - \beta_2}{Q(a, b)},$$

$$\alpha_3 Q(a, b) + (1 - \alpha_3)U(a, b) < N_{GQ}(a, b) < \beta_3 Q(a, b) + (1 - \beta_3)U(a, b)$$

hold for all  $a, b > 0$  with  $a \neq b$  if and only if  $\alpha_1 \leq 2/3$ ,  $\beta_1 \geq \pi/4$ ,  $\alpha_2 \leq 0$ ,  $\beta_2 \geq 1/3$ ,  $\alpha_3 \leq 0$  and  $\beta_3 \geq (\pi^2 - 8)/[4(\pi - 2)] = 0.4094\dots$

The main purpose of this paper is to present the best possible parameter  $\alpha$  and  $\beta$  such that the double inequalities  $M_\alpha(a, b) < N_{GQ}(a, b) < M_\beta(a, b)$  hold for all  $a, b > 0$  with  $a \neq b$ .

## 2. MAIN RESULT

In order to prove our main result we need a lemma, which we present in this section.

**Lemma 1.** *Let  $p \in \mathbb{R}$ , and*

$$f(x) = x^{2p+2} + x^{2p+1} + 5x^{2p} + x^{2p-1} + (2p-3)x^{p+3} - 4x^{p+2} + 4x^p - (2p-3)x^{p-1} - x^3 - 5x^2 - x - 1 \quad (2.1)$$

*Then the following statements are true:*

- (1) *If  $p = 4/3$ , then  $f(x) > 0$  for all  $x \in (1, +\infty)$ ;*

(2) If  $p = 2 \log 2 / (5 \log 2 - 2 \log \pi) = 1.1785 \dots$ , then there exists  $\lambda \in (1, +\infty)$  such that  $f(x) < 0$  for  $x \in (1, \lambda)$  and  $f(x) > 0$  for  $x \in (\lambda, +\infty)$ .

*Proof.* For part (1), if  $p = 4/3$ , then (2.1) becomes

$$\begin{aligned} f(x) &= \frac{1}{3}(x^{2/3} - 1)^3(3x^{8/3} - x^{7/3} + 9x^2 + 6x^{4/3} + 9x^{2/3} - x^{1/3} + 3) \\ &= \frac{1}{3}(x^{2/3} - 1)^3[2x^{8/3} + x^{7/3}(x^{1/3} - 1) + 9x^2 + 6x^{4/3} \\ &\quad + 8x^{2/3} + x^{1/3}(x^{1/3} - 1) + 3] \\ &> \frac{1}{3}(x^{2/3} - 1)^3(2x^{8/3} + 9x^2 + 6x^{4/3} + 8x^{2/3} + 3) \quad (2.2) \end{aligned}$$

for  $x \in (1, +\infty)$ .

Therefore, part (1) follows from (2.2).

For part (2), let  $p = 2 \log 2 / (5 \log 2 - 2 \log \pi) = 1.1785 \dots$ ,  $f_1(x) = f'(x)$ ,  $f_2(x) = f_1'(x)$ ,  $f_3(x) = f_2'(x)$ ,  $f_4(x) = x^{5-p} f_3'(x)$ . Then elaborated computations lead to

$$\lim_{x \rightarrow 1} f(x) = 0, \quad \lim_{x \rightarrow +\infty} f(x) = +\infty, \quad (2.3)$$

$$\begin{aligned} f_1(x) &= 2(p+1)x^{2p+1} + (2p+1)x^{2p} + 10px^{2p-1} + (2p-1)x^{2p-2} \\ &\quad + (p+3)(2p-3)x^{p+2} - 4(p+2)x^{p+1} + 4px^{p-1} \\ &\quad - (p-1)(2p-3)x^{p-2} - 3x^2 - 10x - 1 \\ \lim_{x \rightarrow 1} f_1(x) &= 24(p - \frac{4}{3}) < 0, \quad \lim_{x \rightarrow +\infty} f_1(x) = +\infty, \quad (2.4) \end{aligned}$$

$$\begin{aligned} f_2(x) &= 2(p+1)(2p+1)x^{2p} + 2p(2p+1)x^{2p-1} + 10p(2p-1)x^{2p-2} \\ &\quad + 2(p-1)(2p-1)x^{2p-3} + (p+2)(p+3)(2p-3)x^{p+1} \\ &\quad - 4(p+1)(p+2)x^p + 4p(p-1)x^{p-2} \\ &\quad - (p-1)(p-2)(2p-3)x^{p-3} - 6x - 10 \\ \lim_{x \rightarrow 1} f_2(x) &= 24(2p+1)(p - \frac{4}{3}) < 0, \quad \lim_{x \rightarrow +\infty} f_2(x) = +\infty, \quad (2.5) \end{aligned}$$

$$\begin{aligned} f_3(x) &= 4p(p+1)(2p+1)x^{2p-1} + 2p(4p^2-1)x^{2p-2} + 20p(p-1)(2p-1)x^{2p-3} \\ &\quad + 2(p-1)(2p-1)(2p-3)x^{2p-4} + (p+1)(p+2)(p+3)(2p-3)x^p \\ &\quad - 4p(p+1)(p+2)x^{p-1} + 4p(p-1)(p-2)x^{p-3} \\ &\quad - (p-1)(p-2)(p-3)(2p-3)x^{p-4} - 6 \\ \lim_{x \rightarrow 1} f_3(x) &= 4(22p^3 - 33p^2 + 17p - 12) < 0, \quad \lim_{x \rightarrow +\infty} f_3(x) = +\infty, \quad (2.6) \end{aligned}$$

$$\begin{aligned}
f_4(x) &= 4p(p+1)(4p^2-1)x^{p+3} + 4p(p-1)(4p^2-1)x^{p+2} \\
&+ 20p(p-1)(2p-1)(2p-3)x^{p+1} + 4(p-1)(p-2)(2p-1)(2p-3)x^p \\
&+ p(p+1)(p+2)(p+3)(2p-3)x^4 - 4p(p^2-1)(p+2)x^3 \\
&+ 4p(p-1)(p-2)(p-3)x - (p-1)(p-2)(p-3)(p-4)(2p-3) \\
&= a_0x^{p+3} + a_2x^{p+2} + a_4x^{p+1} + a_5x^p + a_1x^4 + a_3x^3 + a_6x + a_7. \quad (2.7)
\end{aligned}$$

Note that

$$p+3 > 4 > p+2 > 3 > p+1 > p > 1 > 0, \quad (2.8)$$

$$a_0 > 0, a_1 < 0, a_2 > 0, a_3 < 0, a_4 < 0, a_5 > 0, a_6 > 0, a_7 < 0, \quad (2.9)$$

$$23p^2 - 43p + 12 = -6.7311 \dots < 0, 2p^3 - 37p^2 + 89p - 48 = 8.7726 \dots > 0, \quad (2.10)$$

$$2p^3 + 119p^2 - 125p + 70 = 91.2430 \dots > 0, \quad (2.11)$$

$$a_0 + a_1 = p(p^2 - 1)(2p^2 + 25p + 22) > 0, \quad (2.12)$$

$$a_2 + a_3 + a_4 = 4p(p-1)(23p^2 - 43p + 12) \quad (2.13)$$

$$a_5 + a_6 + a_7 = (p-1)(2-p)(2p^3 - 37p^2 + 89p - 48), \quad (2.14)$$

$$\sum_{i=0}^4 a_i = p(p-1)(2p^3 + 119p^2 - 125p + 70), \quad (2.15)$$

It follows from (2.7)-(2.15) that

$$\begin{aligned}
f_4(x) &> (a_0 + a_1)x^4 + (a_2 + a_3 + a_4)x^3 + (a_5 + a_6 + a_7)x \\
&> \sum_{i=0}^4 a_i x^4 + (a_5 + a_6 + a_7)x > 0 \quad (2.16)
\end{aligned}$$

for  $x \in (1, +\infty)$ .

From (2.16) we clearly see that  $f_3(x)$  is strictly increasing on  $(1, +\infty)$ . Then (2.6) leads to the conclusion that there exists  $\lambda_1 > 1$  such that  $f_2(x)$  is strictly decreasing on  $(1, \lambda_1]$  and strictly increasing on  $[\lambda_1, +\infty)$ .

It follows from (2.5) and the piecewise monotonicity of  $f_2(x)$ , we conclude that there exists  $\lambda_2 \in (1, +\infty)$  such that  $f_1(x)$  is strictly decreasing on  $(1, \lambda_2]$  and strictly increasing on  $[\lambda_2, +\infty)$ .

From (2.4) and the piecewise monotonicity of  $f_1(x)$  that there exists  $\lambda_3 \in (1, +\infty)$  such that  $f(x)$  is strictly decreasing on  $(1, \lambda_3]$  and strictly increasing on  $[\lambda_3, +\infty)$ .

Therefore, part (2) follows from (2.3) and the piecewise monotonicity of  $f(x)$ .  $\square$

**Theorem 1.** *The double inequality*

$$M_\alpha(a, b) < N_{GQ}(a, b) < M_\beta(a, b),$$

holds for all  $a, b > 0$  with  $a \neq b$  if and only if  $\alpha \leq 2 \log 2 / (5 \log 2 - 2 \log \pi) = 1.1785 \dots$  and  $\beta \geq 4/3$ .

*Proof.* Since  $N_{GQ}(a, b)$  and  $M_p(a, b)$  are symmetric and homogenous of degree 1, we assume that  $a > b > 0$ . Let  $x = a/b \in (1, +\infty)$ ,  $p \in \mathbb{R}_+$ . Then we have

$$\begin{aligned} & \log [N_{GQ}(a, b)] - \log [M_p(a, b)] \\ &= \log [2\sqrt{x}(x-1) + \sqrt{2}(x^2+1)\tan^{-1}\left(\frac{x-1}{\sqrt{2x}}\right)] - \log [4(x-1)] - \frac{1}{p} \log \left(\frac{x^p+1}{2}\right). \end{aligned} \quad (2.17)$$

Let

$$\begin{aligned} F(x) &= \log [2\sqrt{x}(x-1) + \sqrt{2}(x^2+1)\tan^{-1}\left(\frac{x-1}{\sqrt{2x}}\right)] \\ &\quad - \log [4(x-1)] - \frac{1}{p} \log \left(\frac{x^p+1}{2}\right) \end{aligned} \quad (2.18)$$

Then simple computations lead to

$$\lim_{x \rightarrow 1^+} F(x) = 0, \quad (2.19)$$

$$\lim_{x \rightarrow +\infty} F(x) = \frac{1}{p} \log 2 + \log \pi - 5 \log \sqrt{2}, \quad (2.20)$$

$$F'(x) = \frac{x^{p+1} + 2x^p - x^{p-1} - x^2 + 2x + 1}{(x-1)(x^p+1)[2\sqrt{x}(x-1) + \sqrt{2}(x^2+1)\tan^{-1}\left(\frac{x-1}{\sqrt{2x}}\right)]} F_1(x), \quad (2.21)$$

where

$$\begin{aligned} F_1(x) &= \frac{2\sqrt{x}(x-1)(x^{p-1}+1)}{x^{p+1} + 2x^p - x^{p-1} - x^2 + 2x + 1} - \sqrt{2}\tan^{-1}\left(\frac{x-1}{\sqrt{2x}}\right), \\ \lim_{x \rightarrow 1} F_1(x) &= 0, \end{aligned} \quad (2.22)$$

$$\lim_{x \rightarrow +\infty} F_1(x) = -\frac{\sqrt{2}}{2}\pi < 0, \quad (2.23)$$

$$F'_1(x) = -\frac{2(x-1)}{\sqrt{x}(x^2+1)(x^{p+1} + 2x^p - x^{p-1} - x^2 + 2x + 1)^2} f(x), \quad (2.24)$$

where  $f(x)$  is defined by (2.1).

We divide the proof into four cases.

Case 1.  $p = 2 \log 2 / (5 \log 2 - 2 \log \pi)$  Then it follows from Lemma 1(2) and (2.24) that there exists  $\lambda \in (1, +\infty)$  such that  $F_1(x)$  is strictly increasing on  $(1, \lambda]$  and strictly decreasing on  $[\lambda, +\infty)$ .

Equations (2.21) and (2.22)-(2.23) together with the piecewise monotonicity of  $F_1(x)$  lead to the conclusion that there exists  $\lambda_0 \in (1, +\infty)$  such that  $F(x)$  is strictly increasing on  $(1, \lambda_0]$  and strictly decreasing on  $[\lambda_0, +\infty)$ .

Note that (2.20) becomes

$$\lim_{x \rightarrow +\infty} F(x) = 0, \quad (2.25)$$

Therefore,

$$N_{GQ}(a, b) > M_{2 \log 2 / (5 \log 2 - 2 \log \pi)}(a, b)$$

for all  $a, b > 0$  with  $a \neq b$  follows from (2.17)-(2.19) and (2.25) together with the piecewise monotonicity of  $F(x)$ .

Case 2 .  $p = 4/3$  Then it follows from Lemma 1(1) and (2.24) that  $F_1(x)$  is strictly decreasing on  $(1, +\infty)$ .

Therefore,

$$N_{GQ}(a, b) < M_{4/3}(a, b)$$

for all  $a, b > 0$  with  $a \neq b$  follows from (2.17)-(2.19) and (2.21)-(2.22) together with the monotonicity of  $F_1(x)$ .

Case 3 .  $p > 2 \log 2 / (5 \log 2 - 2 \log \pi)$  Then (2.20) leads to

$$\lim_{x \rightarrow +\infty} F(x) < 0, \quad (2.26)$$

Equations (2.17)-(2.18) together with inequality (2.26) imply that there exists large enough  $M_0 > 1$  such that

$$N_{GQ}(a, b) < M_p(a, b)$$

for all  $a, b > 0$  with  $x \in (M_0, +\infty)$ .

Case 4 .  $p < 4/3$  Let  $x > 0, x \rightarrow 0$ , then making use the Taylor expansion we get

$$\begin{aligned} & N_{GQ}(1, 1+x) - M_p(1, 1+x) \\ &= \frac{2x\sqrt{x+1} + \sqrt{2}[(x+1)^2 + 1] \tan^{-1}\left(\frac{x}{\sqrt{2(x+1)}}\right)}{4x} - \left[\frac{1+(1+x)^p}{2}\right]^{1/p} \\ &= \frac{4-3p}{24}x^2 + o(x^2). \end{aligned} \quad (2.27)$$

Equation (2.27) implies that there exists small enough  $\delta_0 > 0$  such that

$$N_{GQ}(1, 1+x) > M_p(1, 1+x)$$

for all  $a, b > 0$  with  $x \in (0, \delta_0)$ .

Therefore, Theorem 1 follows easily from Cases 1-4 and the monotonicity of the function  $p \rightarrow M_p(a, b)$ .  $\square$

#### ACKNOWLEDGEMENT

The research was supported by the Natural Science Foundation of China under Grants 61374086, 11171307 and 11401191, the Natural Science Foundation of Zhejiang Province under Grant LY13A010004, and the Natural Science Foundation of Zhejiang Broadcast and TV University under Grant XKT-17Z04 and XKT-17G26.

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#### Authors' addresses

##### **Xiao-Hong He**

Office of Academic Affairs, Quzhou Broadcast and TV University, Quzhou, ZheJiang, China

E-mail address: mcqmm2006@163.com

##### **Yue-Ying Yang**

School of Mechanical and Electrical Engineering, Huzhou Vocational & Technical College, Huzhou, ZheJiang, China

E-mail address: yyy1008hz@163.com

##### **Wei-Mao Qian**

School of Distance Education, Huzhou Broadcast and TV University, Huzhou, ZheJiang, China

E-mail address: qwm661977@126.com