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SOME REMARKS ON REGULARIZED MULTIVALUED NONCONVEX EQUILIBRIUM PROBLEMS

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Abstract. This work focuses on the class of multivalued regularized equilibrium problems in the context of uniformly prox-regular sets introduced and studied in [Noor, M.A.: Multivalued regularized equilibrium problems. J. Global Optim. 35, 483–492 (2006)]. The algorithms and results presented in the paper cited above are investigated and analyzed and some fatal errors in them are detected. Meanwhile, the correct version of the corresponding algorithm and results is given.

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1. INTRODUCTION

It is well-known that the equilibrium problem (EP) is an unified model of several problems including variational inequality problems, optimization problems, problems of Nash equilibria, saddle point problems, fixed point problems and complementarity problems, etc, see, for example, [1, 10, 12, 19] and the references therein. Because of the wide applications to optimization, economics, finance, physics, image reconstruction, network, ecology, transportation and engineering sciences, EP have been extended and generalized in different directions, see, for example, [1, 2, 4, 6, 11, 23] and the references therein. An important and useful generalization of EP is the generalized multivalued equilibrium problem [23] involving a nonlinear bifunction. It has been shown that a wide class of unrelated odd order and nonsymmetric free, moving, obstacle and equilibrium problems can be studied via the multivalued equilibrium problems.

It is worth to mention that most of the results regarding the existence and iterative approximation of solutions to variational inequality problems and equilibrium problems have been investigated and considered so far to the case where the underlying set is a convex set. This is because all the techniques are based on the properties of the projection operator over convex sets, which may not hold in general, when the sets are nonconvex. In recent years, the concept of convex set has been generalized in many directions, which has potential and important applications in various fields. It is well known that the uniformly prox-regular sets are nonconvex and include the convex sets as special cases, for more details, see, for example, [8, 9, 17, 18, 25].

In the recent past, several authors considered and studied different classes of variational inequalities and equilibrium problems in the setting of uniformly prox-regular sets, see, for example, [3, 5, 7, 14, 21, 24] and the references therein.

Recently, Noor [23] considered and studied a class of equilibrium problems known as the multivalued regularized equilibrium problems in the context of uniformly prox-regular sets. He used the auxiliary principle technique [20] to suggest some iterative methods for solving the multivalued regularized equilibrium problems. He also studied the convergence analysis of the proposed methods under some certain conditions.

The main objective of this paper is to investigate and analyze the algorithms and results presented in [23]. Some fatal errors in the algorithms and main results of [23] are detected. Meanwile, the correct version of the algorithms and convergence results corresponding to the algorithms and results given in [23] is presented.

2. Notations and Preliminaries

Throughout the paper, unless otherwise specified, we use the following notations, terminology and assumptions. Let $\mathcal{H}$ be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$, respectively. Let K be a nonempty closed subset of $\mathcal{H}$. We denote by $d_K(.)$ or $d(.,K)$ the usual distance function from a point to a set K, that is, $d_K(u) = \inf\limits_{v \in K} \|u - v\|$.

Definition 1. Let $u \in \mathcal{H}$ be a point not lying in K. A point $v \in K$ is called *a closest point or a projection of u onto K* if $d_K(u) = \|u - v\|$. The set of all such closest points is denoted by $P_K(u)$, that is,

$$P_K(u) := \{v \in K : d_K(u) = \|u - v\|\}.$$

Definition 2. The *proximal normal cone* of K at a point $u \in K$ is given by

$$N_K^P(u) := \{\xi \in \mathcal{H} : \exists \alpha > 0 \text{ such that } u \in P_K(u + \alpha\xi)\}.$$

The following lemma gives a characterization of the proximal normal cone.

Lemma 1 ([17, Proposition 1.1.5])**.** *Let K be a nonempty closed subset of $\mathcal{H}$. Then $\xi \in N_K^P(u)$ if and only if there exists a constant $\alpha = \alpha(\xi, u) > 0$ such that $\langle \xi, v - u \rangle \leq \alpha \|v - u\|^2$ for all $v \in K$.*

Definition 3 ([16])**.** Let $f : \mathcal{H} \to \mathbb{R}$ be locally Lipschitz near a point x. The *Clarke's directional derivative* of f at x in the direction v, denoted by $f^\circ(x; v)$, is defined by

$$f^\circ(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t},$$

where y is a vector in $\mathcal{H}$ and t is a positive scalar.

The *tangent cone* to K at a point $x \in K$, denoted by $T_K(x)$, is defined by

$$T_K(x) := \left\{v \in \mathcal{H} : d_K^\circ(x;v) = 0\right\}.$$

The *normal cone* to K at $x \in K$, denoted by $N_K(x)$, is defined by

$$N_K(x) := \{\xi \in \mathcal{H} : \langle \xi, v\rangle \leq 0 \text{ for all } v \in T_K(x)\}.$$

The *Clarke normal cone*, denoted by $N_K^C(x)$, is defined by

$$N_K^C(x) = \overline{co}[N_K^P(x)],$$

where $\overline{co}[S]$ denotes the closure of the convex hull of S.

Clearly, $N_K^P(x) \subseteq N_K^C(x)$. Note that $N_K^C(x)$ is a closed and convex cone, whereas $N_K^P(x)$ is convex, but may not be closed. For further details on this topic, we refer to [16, 17, 25] and the references therein.

In 1995, Clarke et al. [18] introduced a nonconvex set, called *proximally smooth set*. Subsequently, it has been investigated by Poliquin et al. [25] but under the name of uniformly prox-regular set. Such kind of sets are used in many nonconvex applications in optimization, economic models, dynamical systems, differential inclusions, etc. For further details and applications, we refer to [13] and the references therein. This class of nonconvex sets seems particularly well suited to overcome the difficulties which arise due to the nonconvexity assumption.

Definition 4 ([18])**.** For a given $r \in (0, +\infty]$, a subset K of $\mathcal{H}$ is said to be *normalized uniformly prox-regular* (or *uniformly r-prox-regular*) if for all $\bar{x} \in K$ and all $\mathbf{0} \neq \xi \in N_K^P(\bar{x})$,

$$\left\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \right\rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2, \quad \text{for all } x \in K.$$

The class of normalized uniformly prox-regular sets includes the class of convex sets, p-convex sets [15], $C^{1,1}$ submanifolds (possibly with boundary) of $\mathcal{H}$, the images under a $C^{1,1}$ diffeomorphism of convex sets and many other nonconvex sets [18].

Lemma 2 ([18])**.** *A closed set $K \subseteq \mathcal{H}$ is convex if and only if it is uniformly r-prox-regular for every $r > 0$.*

If $r = +\infty$, then in view of Definition 4 and Lemma 2, the uniform r-prox-regularity of K is equivalent to the convexity of K.

The union of two disjoint intervals $[a,b]$ and $[c,d]$ is uniformly r-prox-regular with $r = \frac{c-b}{2}$ [14, 17, 25]. The finite union of disjoint intervals is also uniformly r-prox-regular and r depends on the distances between the intervals.

3. Regularized Multivalued Nonconvex Equilibrium Problems

Let $C(\mathscr{H})$ denote the family of all the nonempty compact subsets of $\mathscr{H}$. Suppose that $g : \mathscr{H} \to \mathscr{H}$ is a single-valued operator and $T : \mathscr{H} \to C(\mathscr{H})$ is a multivalued operator. For a given bifunction $F(.,.) : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$, Noor [23] considered the problem of finding $u \in \mathscr{H} : g(u) \in K$ and $\nu \in T(u)$ such that

$$F(\nu, g(v)) + \gamma \|g(v) - g(u)\|^2 \geq 0, \quad \forall v \in \mathscr{H} : g(v) \in K, \tag{3.1}$$

where $\gamma = \frac{1}{2r}$. He called it *multivalued regularized equilibrium problem*.

It should be pointed out that there is a small mistake in the context of problem (2.1) in [23]. In fact, the bifunction $F : \mathscr{H} \times \mathscr{H} \to \mathscr{H}$ in problem (2.1) in [23] should be replaced by $F : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$.

Lemma 3 ([22])**.** *Let X be a complete metric space, and $T : X \to C(X)$ be a multivalued mapping. Then for any $x, y \in X$, $u \in T(x)$, there exists $v \in T(y)$ such that*

$$d(u, v) \leq M(T(x), T(y)), \tag{3.2}$$

where $M(.,.)$ is the Hausdorff metric on $C(X)$ defined by

$$M(A, B) = \max\left\{\sup_{x \in A} \inf_{y \in B} \|x - y\|, \sup_{y \in B} \inf_{x \in A} \|x - y\|\right\}, \quad \forall A, B \in C(X).$$

Noor [23] considered the following auxiliary multivalued regularized equilibrium problem: For $u \in \mathscr{H}$ with $g(u) \in K$ and $\nu \in T(u)$, find $w \in \mathscr{H}$ with $g(w) \in K$ and $\xi \in T(w)$ such that, for all $g(v) \in K$, the following relation holds

$$F(\xi, g(v)) + \langle g(w) - g(u), g(v) - g(w) \rangle + \gamma \|g(v) - g(w)\|^2 \geq 0. \tag{3.3}$$

He claimed that if $w = u$, then w is a solution of the multivalued regularized equilibrium problem (3.1). Based on this fact and by utilizing Lemma 3, he suggested the following predictor-corrector algorithm for solving problem (3.3).

Algorithm 1 ([23, Algorithm 3.1]). For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned} \rho F(\eta_{n+1}, g(v)) &+ \langle g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1}) \rangle \\ &+ \gamma \|g(u_{n+1}) - g(u_n)\|^2 \geq 0, \quad \forall g(v) \in K, \end{aligned} \tag{3.4}$$

$$\eta_n \in T(u_n) : \|\eta_{n+1} - \eta_n\| \leq M(T(u_{n+1}), T(u_n)), \tag{3.5}$$

where $\rho > 0$ is a constant and $n = 0, 1, 2, \ldots$.

In order to study the convergence analysis of Algorithm 1, Noor [23] used the following definition.

Definition 5 ([23, Definition 2.4])**.** The bifunction $F(.,.)$ is said to be:

(a) *partially relaxed strongly g-monotone* if there exists a constant $\alpha > 0$ such that, for all $u_1, u_2, z \in \mathcal{H}$, $w_1 \in T(u_1), w_2 \in T(u_2)$ the following relation holds

$$F(w_1, g(u_2)) + F(w_2, g(z)) \leq \alpha \|g(z) - g(u_1)\|^2,$$

(b) *g-monotone* if

$$F(w_1, g(u_2)) + F(w_2, g(u_1)) \leq 0,$$

$$\forall u_1, u_2, z \in \mathcal{H},\ w_1 \in T(u_1), w_2 \in T(u_2);$$

(c) *g-pseudomonotone* if for all $u_1, u_2, z \in \mathcal{H}$, $w_1 \in T(u_1)$, $w_2 \in T(u_2)$,

$$F(w_1, g(u_2)) + \gamma \|g(u_2) - g(u_1)\|^2 \geq 0$$

$$\Rightarrow\ -F(w_2, g(u_1)) + \gamma \|g(u_2) - g(u_1)\|^2 \geq 0.$$

The following theorem played an important role in establishing the strong convergence of the iterative sequences generated by Algorithm 1.

Theorem 1 ([23, Theorem 3.1])**.** *Let $u \in \mathcal{H}$ be a solution of* (3.1) *and u_{n+1} be the approximate solution obtained from Algorithm* 1*. If $F(.,.)$ is g-pseudomonotone, then*

$$(1-\gamma)\|g(u_{n+1}) - g(u)\|^2 \leq \|g(u_n) - g(u)\|^2 - (1-\gamma)\|g(u_{n+1}) - g(u_n)\|^2. \quad (3.6)$$

By a careful reading of the proof of Theorem 1 (that is, [23, Theorem 3.1]), we found that there are fatal errors in it. In fact, by assuming $u \in \mathcal{H} : g(u) \in K$ and $\nu \in T(u)$ as a solution of the problem (3.1) and taking $v = u_{n+1}$ in (3.1), Noor [23] deduced the following inequality:

$$F(\nu, g(u_{n+1})) + \gamma \|g(u_{n+1}) - g(u)\|^2 \geq 0. \quad (3.7)$$

By (3.7) and with the help of the concept of g-pseudomonotonicity of the bifunction F presented in part (c) of Definition 5, he obtained the inequality (3.6) in [23] as follows:

$$-F(\eta_{n+1}, g(u)) + \gamma \|g(u_{n+1}) - g(u)\|^2 \geq 0. \quad (3.8)$$

By setting $v = u$ in (3.4), Noor [23] derived the following inequality:

$$\begin{aligned} \rho F(\eta_{n+1}, g(u)) + &\langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1}) \rangle \\ &+ \gamma \|g(u) - g(u_{n+1})\|^2 \geq 0, \end{aligned} \quad (3.9)$$

and then relying on (3.8), he deduced the inequality (3.7) in [23] as follows:

$$\begin{aligned} &\langle g(u_{n+1}) - g(u_n), g(u) - g(u_{n+1}) \rangle \\ &\geq -\rho F(\eta_{n+1}, g(u)) - \gamma \|g(u) - g(u_{n+1})\|^2 \\ &\geq -\gamma \|g(u) - g(u_{n+1})\|^2 - \gamma \|g(u_{n+1}) - g(u)\|^2. \end{aligned} \quad (3.10)$$

Letting $u = g(u)-g(u_{n+1})$ and $v = g(u_{n+1})-g(u_n)$ and by utilizing the well-known property of the inner product, he obtained the inequality (3.8) in [23] as follows:

$$2\langle g(u_{n+1})-g(u_n), g(u)-g(u_{n+1})\rangle = \|g(u)-g(u_n)\|^2 - \|g(u_{n+1})-g(u_n)\|^2 - \|g(u)-g(u_{n+1})\|^2. \tag{3.11}$$

Combining (3.10) and (3.11), Noor [23] deduced the required inequality (3.6). We now show that under the conditions mentioned in Theorem 1, what yields is not the inequality (3.6). In fact, taking $v = u$ in (3.4), what we obtain is the inequality

$$\begin{aligned}\rho F(\eta_{n+1}, g(u)) + \langle g(u_{n+1})-g(u_n), g(u)-g(u_{n+1})\rangle \\ +\gamma\|g(u_{n+1})-g(u_n)\|^2 \geq 0,\end{aligned} \tag{3.12}$$

not the inequality (3.9). Furthermore, by virtue of (3.8) and (3.12), we obtain the inequality

$$\begin{aligned}&\langle g(u_{n+1})-g(u_n), g(u)-g(u_{n+1})\rangle \\ &\geq -\rho F(\eta_{n+1}, g(u)) - \gamma\|g(u_{n+1})-g(u_n)\|^2 \\ &\geq -\rho\gamma\|g(u)-g(u_{n+1})\|^2 - \gamma\|g(u_{n+1})-g(u_n)\|^2,\end{aligned} \tag{3.13}$$

not the inequality (3.10). At long last, by (3.11) and (3.13), what we get is the inequality

$$(1-2\rho\gamma)\|g(u_{n+1})-g(u)\|^2 \leq \|g(u)-g(u_n)\|^2 - (1-2\gamma)\|g(u_{n+1})-g(u_n)\|^2,$$

not the inequality (3.6).

Noor [23] claimed that the sequence generated by Algorithm 1 is strongly convergent to a solution of the problem (3.1).

Theorem 2 ([23, Theorem 3.2]). *Let $\mathcal{H}$ be a finite dimensional space and let $g: \mathcal{H} \to \mathcal{H}$ be injective. Let $T: \mathcal{H} \to C(\mathcal{H})$ be M-Lipschitz continuous operator. If $\gamma \leq 1$, then the sequence $\{u_n\}$ given by Algorithm 1 converges to a solution u of* (3.1).

Now we analyze the proof of Theorem 2 (that is, [23, Theorem 3.2]).

Theorem 1 plays a crucial role in the proof of Theorem 2. But, as we have pointed out the assertion of Theorem 1 is not true necessarily. Beside this fact, we also found that, by a careful reading, there are some fatal errors in the proof of Theorem 2.

In the first place, Noor [23] asserted that the inequality (3.6) implies the boundedness of the sequence $\{g(u_n)\}$ and then he deduced the boundedness of the sequence $\{u_n\}$ under the injectivity assumption of the operator g.

In fact, relying on (3.6), Noor [23] claimed that

$$\|g(u_{n+1})-g(u)\| \leq \|g(u_n)-g(u)\|, \tag{3.14}$$

that is, the sequence $\{\|g(u_n)-g(u)\|\}$ is nonincreasing. Taking into account of the facts that the sequence $\{\|g(u_n)-g(u)\}$ is nonincreasing and the operator g is injective, he deduced the boundedness of the sequences $\{g(u_n)\}$ and $\{u_n\}$. However, using the inequality (3.6), we obtain the inequality

$$\|g(u_{n+1})-g(u)\| \leq \frac{1}{\sqrt{1-\gamma}}\|g(u_n)-g(u)\|, \tag{3.15}$$

not the inequality (3.14). It goes without saying that the inequality (3.15) does not guarantee that the sequence $\{\|g(u_n)-g(u)\}$ is nonincreasing. Therefore, in light of the conditions mentioned in Theorem 2, the sequence $\{g(u_n)\}$ is not necessarily bounded and so the sequence $\{u_n\}$ is not also necessarily bounded.

In the second place, the author claimed that by applying the inequality (3.6), one can deduce the following inequality:

$$\sum_{n=0}^{\infty}(1-\gamma)\|g(u_{n+1})-g(u_n)\|^2 \leq \|g(u_0)-g(u)\|^2, \tag{3.16}$$

which implies that

$$\lim_{n\to\infty}\|g(u_{n+1})-g(u_n)\| = 0. \tag{3.17}$$

However, by using the inequality (3.6), what one can get is the following inequality:

$$\begin{aligned}&\sum_{n=0}^{\infty}(1-\gamma)\|g(u_{n+1})-g(u)\|^2\\ &\leq \|g(u_0)-g(u)\|^2+\sum_{n=0}^{\infty}\gamma\|g(u_{n+1})-g(u)\|^2,\end{aligned} \tag{3.18}$$

not the inequality (3.16). Obviously, the inequality (3.18) does not imply the relation (3.17). It is worth mentioning that the boundedness of the sequence $\{u_n\}$ and the relation (3.17) are main tools to establish the statement of Theorem 2.

Thirdly, on page 489, at the end of the proof of Theorem 2 (that is, [23, Theorem 3.2]), the author derived the following inequality by using M-Lipschitz continuity with constant δ of T:

$$\|\nu_n-\nu\| \leq M(T(u_n),T(u)) \leq \delta\|u_n-u\|.$$

Unfortunately, there is a fatal error in the above inequality. In fact, in view of the preceding inequality, the author used the fact that $\nu \in T(u)$ before proving it. Even, without considering this fact, the following example illustrates that for any given $x, y \in \mathcal{H}$, $u \in T(x)$ and $v \in T(y)$, the inequality (3.2) need not be hold.

Example 1. Let X be the space of all bounded real sequences, that is,

$$X = l^{\infty} = \Big\{x = (x_n)_{n\in\mathbb{N}} : x_n \in \mathbb{R}, \sup_{n\in\mathbb{N}}|x_n| < \infty\Big\}.$$

Then X equipped with the ∞-norm (or sup-norm) defined by

$$\|x\|_\infty = \|(x_n)_{n\in\mathbb{N}}\|_\infty = \sup_{n\in\mathbb{N}} |x_n|, \qquad \forall x = (x_n)_{n\in\mathbb{N}} \in l^\infty,$$

is a Banach space, and the set $\{e_1, e_2, \ldots, e_n, \ldots\}$, where for each $n \in \mathbb{N}$, $e_n = (0,0,\ldots,1,0,\ldots)$, 1 at the n^{th} coordinate and all other coordinates are zero, is a Schauder basis of l^∞. Let the multivalued mapping $T : X \to CB(X)$ be defined by

$$T(x) = \begin{cases} \{(\frac{\gamma}{n})_{n\in\mathbb{N}}, e_n : n = 1,2,\ldots\}, & x \neq e_l, \\ \{e_n\}_{n\in\mathbb{N}}, & x = e_l, \end{cases}$$

where $\gamma \in [-1,0)$ is an arbitrary real constant and $l \in \mathbb{N}$ is arbitrary but fixed. Take $x \neq e_l$, an arbitrary element belonging to X, $y = e_l$ and $u = (\frac{\gamma}{n})_{n\in\mathbb{N}}$. If $a = (\frac{\gamma}{n})_{n\in\mathbb{N}}$, then considering the fact that $\gamma \in [-1,0)$, for any $k \in \mathbb{N}$, we have

$$\begin{aligned} d(a, e_k) &= \left\| (\frac{\gamma}{n})_{n\in\mathbb{N}} - e_k \right\|_\infty = \sup_{i\in\mathbb{N}} \left| \frac{\gamma}{i} - e_{k,i} \right| \\ &= \sup\left\{ \left|\frac{\gamma}{i}\right|, \left|\frac{\gamma}{k} - 1\right| : i \in \mathbb{N}, i \neq k \right\} \\ &= \left| \frac{\gamma}{k} - 1 \right| = 1 - \frac{\gamma}{k}, \end{aligned}$$

where

$$e_{k,i} = \delta_{k,i} = \begin{cases} 1 & i = k, \\ 0 & otherwise, \end{cases}$$

and so the fact that $\gamma \in [-1,0)$ implies that

$$d(a, T(y)) = \inf_{b\in T(y)} d(a,b) = \inf\left\{ 1 - \frac{\gamma}{k} : k \in \mathbb{N} \right\} = 1.$$

If $a = e_n$, for some $n \in \mathbb{N}$, then for each $k \in \mathbb{N}$, $k \neq n$, we have

$$d(a, e_k) = \|e_n - e_k\|_\infty = \sup_{i\in\mathbb{N}} \left| e_{n,i} - e_{k,i} \right| = 1$$

and

$$d(a, e_n) = \|e_n - e_n\|_\infty = 0.$$

Therefore,

$$d(a, T(y)) = \inf_{b\in T(y)} d(a,b) = 0,$$

consequently,

$$\sup_{a\in T(x)} d(a, T(y)) = 1.$$

If $b = e_k$, for some $k \in \mathbb{N}$, then for $a = (\frac{\gamma}{n})_{n\in\mathbb{N}}$, we obtain

$$d(e_k, a) = \left\| e_k - (\frac{\gamma}{n})_{n\in\mathbb{N}} \right\|_\infty = 1 - \frac{\gamma}{k}$$

and

$$d(e_k, e_n) = \|e_k - e_n\|_\infty = \sup_{i\in\mathbb{N}} \left| e_{k,i} - e_{n,i} \right| = 1$$

for each $n \neq k$. For the case when $n = k$, we have

$$d(e_k, e_k) = \|e_k - e_k\|_\infty = 0.$$

Accordingly,

$$d(T(x), b) = \inf_{a\in T(x)} d(a, b) = 0,$$

and so

$$\sup_{b\in T(y)} d(T(x), b) = 0.$$

Hence,

$$M(T(x), T(y)) = \max\left\{ \sup_{a\in T(x)} d(a, T(y)), \sup_{b\in T(y)} d(T(x), b) \right\} = 1.$$

Taking into account of the fact that for each $k \in \mathbb{N}$,

$$\left\| (\frac{\gamma}{n})_{n\in\mathbb{N}} - e_k \right\|_\infty = 1 - \frac{\gamma}{k} > 1,$$

because $\gamma \in [-1, 0)$, it follows that for any $v \in T(y)$,

$$d(u, v) = \|u - v\|_\infty > M(T(x), T(y)).$$

In view of the above mentioned arguments, the statements of Theorems 1 and 2 are not valid in general.

In order to overcome these difficulties, we present the correct versions of the problems (3.1) and (3.3) and Algorithm 1.

Let $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a nonlinear bifunction and let $T : K \to C(\mathcal{H})$ be a multivalued operator. Replacing the operator $g : \mathcal{H} \to \mathcal{H}$ in the problem (3.1) by a surjective operator $g : K \to K$, we consider the problem of finding $u \in K$ and $\nu \in T(u)$ such that

$$F(\nu, g(v)) + \gamma \|g(v) - g(u)\|^2 \geq 0, \qquad \forall v \in K, \tag{3.19}$$

where $\gamma = \frac{1}{2r}$, and is called the *regularized multivalued nonconvex equilibrium problem* (in short, RMNEP). In the sequel, we denote by RMNEP(T, F, g, K) the set of solutions of RMNEP (3.19).

For given $u \in K$ and $\nu \in T(u)$, we consider the following auxiliary regularized multivalued nonconvex equilibrium problem of finding $w \in K$ and $\xi \in T(w)$ such that

$$\rho F(\xi, g(v)) + \langle w - u, v - w\rangle + \rho\gamma \|g(v) - g(w)\|^2 \geq 0, \qquad \forall v \in K, \tag{3.20}$$

where $\rho > 0$ is a constant. We observe that if $w = u$, then clearly w is a solution of RMNEP (3.19). This observation allows us to suggest the following iterative method for solving RMNEP (3.19).

Algorithm 2. For given $u_0 \in K$ and $\xi_0 \in T(u_0)$, compute the iterative sequences $\{u_n\}$ and $\{\xi_n\}$ by the iterative schemes

$$\rho F(\xi_n, g(v)) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + \rho\gamma \|g(v) - g(u_{n+1})\|^2 \geq 0, \ \forall v \in K, \tag{3.21}$$

$$\xi_n \in T(u_n) : \|\xi_{n+1} - \xi_n\| \leq M(T(u_{n+1}), T(u_n)), \tag{3.22}$$

where $\rho > 0$ is a constant and $n = 0, 1, 2, \ldots$.

To establish the strong convergence of the sequence generated by Algorithm 2, we need the following definitions.

Definition 6. A multivalued operator $T : \mathcal{H} \to C(\mathcal{H})$ is said to be *M-Lipschitz continuous* with constant δ if there exists a constant $\delta > 0$ such that

$$M(T(u), T(v)) \leq \delta \|u - v\|, \qquad \forall u, v \in \mathcal{H},$$

where $M(.,.)$ is the Hausdorff metric on $C(\mathcal{H})$.

Definition 7. Let $T : K \to C(\mathcal{H})$ be a multivalued operator and $g : K \to K$ be a nonlinear operator. For a given positive real constant γ, the bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is said to be g-pseudomonotone with respect to T with constant γ, iff

$$F(\xi_1, g(u_2)) + \gamma \|g(u_2) - g(u_1)\|^2 \geq 0$$

implies that

$$F(\xi_2, g(u_1)) + \gamma \|g(u_2) - g(u_1)\|^2 \leq 0, \qquad \forall u_1, u_2 \in K, \xi_1 \in T(u_1), \xi_2 \in T(u_2).$$

The next proposition plays a key role in establishing the strong convergence of the iterative sequences generated by Algorithm 2.

Proposition 1. *Let T, F, g and γ be the same as in* RMNEP (3.19) *and let $u \in K$, $\nu \in T(u)$ be the solution of* RMNEP (3.19). *Suppose further that $\{u_n\}$ and $\{\xi_n\}$ are the sequences generated by Algorithm 2. If F is g-pseudomonotone with respect to T with constant γ, then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2, \qquad \forall n \geq 0. \tag{3.23}$$

Proof. Since $u \in K$ and $\nu \in T(u)$ are the solution of RMNEP (3.19), we have

$$\rho F(\nu, g(v)) + \rho\gamma \|g(v) - g(u)\|^2 \geq 0, \quad \forall v \in K, \tag{3.24}$$

where the real constant $\rho > 0$ is the same as in Algorithm 2.

Taking $v = u_n$ in (3.24), we obtain

$$\rho F(\nu, g(u_n)) + \rho\gamma \|g(u_n) - g(u)\|^2 \geq 0. \tag{3.25}$$

Thanks to the fact that F is g-pseudomonotone with respect to T with constant γ, from the inequality (3.25), it follows that

$$\rho F(\xi_n, g(u)) + \rho\gamma\|g(u_n) - g(u)\|^2 \leq 0. \tag{3.26}$$

Letting $v = u$ in (3.21), we get

$$\rho F(\xi_n, g(u)) + \langle u_{n+1} - u_n, u - u_{n+1}\rangle + \rho\gamma\|g(u) - g(u_n)\|^2 \geq 0. \tag{3.27}$$

By combining (3.26) and (3.27), we conclude that

$$\langle u_{n+1} - u_n, u - u_{n+1}\rangle \geq -\rho F(\xi_n, g(u)) - \rho\gamma\|g(u) - g(u_n)\|^2 \geq 0. \tag{3.28}$$

On the other hand, by assuming $x = u_{n+1} - u_n$ and $y = u - u_{n+1}$, and by utilizing the well known property of the inner product, we obtain

$$2\langle u_{n+1} - u_n, u - u_{n+1}\rangle = \|u - u_n\|^2 - \|u - u_{n+1}\|^2 - \|u_{n+1} - u_n\|^2. \tag{3.29}$$

Making use of (3.28) and (3.29), we derive the required inequality (3.23). □

We now prove the strong convergence of the sequences generated by Algorithm 2 to a solution of RMNEP (3.19).

Theorem 3. *Let $\mathcal{H}$ be a finite dimensional real Hilbert space and let $g : K \to K$ be a continuous surjective operator. Suppose that the operator $T : K \to C(\mathcal{H})$ is M-Lipschitz continuous with constant δ and the bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is continuous in the first argument. Furthermore, let all the conditions of Proposition 1 hold and* RMNEP$(T, F, g, K) \neq \varnothing$. *Then, the iterative sequences $\{u_n\}$ and $\{\xi_n\}$ generated by Algorithm 2 converge strongly to $\hat{u} \in K$ and $\hat{\nu} \in T(\hat{u})$, respectively, and $(\hat{u}, \hat{\nu})$ is a solution of* RMNEP (3.19).

Proof. Let $u \in K$ and $\nu \in T(u)$ be the solution of RMNEP (3.19). From the inequality (3.23) it follows that the sequence $\{\|u - u_n\|\}$ is nonincreasing and so the sequence $\{u_n\}$ is bounded. Moreover, in virtue of the inequality (3.23), we yield

$$\sum_{n=0}^{\infty} \|u_n - u_{n+1}\|^2 \leq \|u - u_0\|^2,$$

which guarantees $\|u_n - u_{n+1}\| \to 0$, as $n \to \infty$. Let $\hat{u}$ be a cluster point of the sequence $\{u_n\}$. With the help of the boundedness of the sequence $\{u_n\}$, we deduce that there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \to \hat{u}$, as $i \to \infty$. Utilizing the inequality (3.22) and considering the fact that the operator T is M-Lipschitz continuous with constant δ, we obtain

$$\|\xi_{n_i+1} - \xi_{n_i}\| \leq M(T(u_{n_i+1}), T(u_{n_i})) \leq \delta\|u_{n_i+1} - u_{n_i}\|. \tag{3.30}$$

The inequality (3.30) implies that $\|\xi_{n_i+1} - \xi_{n_i}\| \to 0$, as $i \to \infty$. Therefore, $\{\xi_{n_i}\}$ is a Cauchy sequence in $\mathcal{H}$. Hence, $\xi_{n_i} \to \hat{\nu}$, as $i \to \infty$, for some $\hat{\nu} \in \mathcal{H}$. By (3.21), we get, for all $v \in K$, that

$$\rho F(\xi_{n_i}, g(v)) + \langle u_{n_i+1} - u_{n_i}, v - u_{n_i+1}\rangle + \rho\gamma\|g(v) - g(u_{n_i})\|^2 \geq 0. \tag{3.31}$$

Taking into consideration the facts that F is continuous in the first argument and g is continuous, by taking the limit in the relation (3.31) as $i \to \infty$, it follows that

$$F(\hat{\nu}, g(v)) + \gamma \|g(v) - g(\hat{u})\|^2 \geq 0, \quad \forall v \in K. \tag{3.32}$$

In the meanwhile, from M-Lipschitz continuity with constant δ of T, we deduce that

$$\begin{aligned} d(\hat{\nu}, T(\hat{u})) &= \inf\left\{\|\hat{\nu} - q\| : q \in T(\hat{u})\right\} \\ &\leq \|\hat{\nu} - \xi_{n_i}\| + d(\xi_{n_i}, T(\hat{u})) \\ &\leq \|\hat{\nu} - \xi_{n_i}\| + M(T(u_{n_i}), T(\hat{u})) \\ &\leq \|\hat{\nu} - \xi_{n_i}\| + \delta\|u_{n_i} - \hat{u}\|. \end{aligned}$$

We observe that the right-hand side of the above inequality tends to zero as $i \to \infty$. Thanks to the fact that $T(\hat{u}) \in CB(\mathcal{H})$, we conclude that $\hat{\nu} \in T(\hat{u})$. Then, the inequality (3.32) implies that $\hat{u} \in K$ and $\hat{\nu} \in T(\hat{u})$ are the solution of RMNEP (3.19). Thus, relying on Proposition 1, we have

$$\|u_{n+1} - \hat{u}\| \leq \|u_n - \hat{u}\|, \quad \forall n \geq 0.$$

The preceding inequality guarantees that $u_n \to \hat{u}$, as $n \to \infty$. Consequently, the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$. By (3.22) and M-Lipschitz continuity with constant δ of T, we conclude that for all $n \geq 0$,

$$\|\xi_{n+1} - \xi_n\| \leq M(T(u_{n+1}), T(u_n)) \leq \delta\|u_{n+1} - u_n\|. \tag{3.33}$$

The inequality (3.33) implies that $\{\xi_n\}$ is also a Cauchy sequence in $\mathcal{H}$. Considering the fact that $\hat{\nu}$ is a cluster point of the sequence $\{\xi_n\}$, we conclude that $\xi_n \to \hat{\nu}$, as $n \to \infty$, that is, the sequence $\{\xi_n\}$ has also exactly one cluster point $\hat{\nu}$. This gives us the desired result. □

It is well known that to implement the proximal point methods, one has to calculate the approximate solution implicity, which is in itself a difficult problem. To overcome this drawback, Noor [23] considered the following auxiliary uniformly regularized equilibrium problem: For a given $u \in \mathcal{H} : g(u) \in K$, $\nu \in T(u)$, find $w \in \mathcal{H} : g(w) \in K$ such that

$$\begin{aligned} \rho F(\nu, g(v)) + &\langle g(w) - g(u), g(v) - g(w)\rangle \\ &+ \gamma\|g(v) - g(w)\|^2 \geq 0, \quad \forall g(v) \in K, \end{aligned} \tag{3.34}$$

where $\rho > 0$ is a constant.

Noor [23] claimed that if $w = u$, then clearly w is a solution of the problem (3.1). Based on this fact, he proposed the following iterative method for solving the problem (3.1).

Algorithm 3 ([23, Algorithm 3.2]). For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative scheme

$$\begin{aligned}\rho F(\nu_n, g(v)) + \langle g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle \\ + \gamma\|g(v) - g(u_{n+1})\|^2 \geq 0,\ \forall g(v) \in K,\end{aligned} \tag{3.35}$$

$$\nu_n \in T(u_n) : \|\nu_{n+1} - \nu_n\| \leq M(T(u_{n+1}), T(u_n)), \quad n = 0, 1, 2, \ldots. \tag{3.36}$$

By an easy checking, we found that unlike the claim of the author in [23], if $w = u$ then w is not necessarily a solution of the problem (3.1). In fact, if $w = u$, then the auxiliary problem (3.34) reduces to the following regularized multivalued nonconvex equilibrium problem:

Find $u \in \mathcal{H} : g(u) \in K$ such that

$$\rho F(\nu, g(v)) + \gamma\|g(v) - g(u)\|^2 \geq 0, \quad \forall v \in \mathcal{H} : g(v) \in K. \tag{3.37}$$

However, the following example shows that every solution of the problem (3.37) is not necessary a solution of the problem (3.1).

Example 2. Let $\mathcal{H} = \mathbb{R}$ and $K = [\alpha, \beta] \cup [\delta, \sigma]$ be the union of two disjoint intervals $[\alpha, \beta]$ and $[\delta, \sigma]$ where $0 < \alpha < \beta < \delta < \sigma$. Then, K is a uniformly r-prox-regular set with $r = \frac{\delta-\beta}{2}$ and so we have $\gamma = \frac{1}{2r} = \frac{1}{\delta-\beta}$. Let us define the operators $T : \mathcal{H} \to C(\mathcal{H})$ and $g : \mathcal{H} \to \mathcal{H}$ by

$$T(x) = \left\{\lambda_i(a_i^{m_i x} + x^{n_i}) : i = 1, 2, \ldots, l\right\}, \quad g(x) = \mu \sqrt[p]{x^q}, \quad \forall x \in \mathcal{H}$$

and the bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by

$$F(x, y) = \theta x(y - \beta) \qquad \forall x, y \in \mathcal{H},$$

where $p, q \in \mathbb{N}\backslash\{1\}$ and $n_i, l \in \mathbb{N}$ are arbitrary but fixed natural numbers, $\mu, \theta > 0$, $m_i \in \mathbb{R}$, $a_i > 1$ and

$$\lambda_i < \frac{\beta - \sigma}{\theta(\delta - \beta)\left(a_i^{m_i \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_i}}\right)}, \text{(where } i = 1, 2, \ldots, l)$$

are arbitrary real constants. Take

$$u = \sqrt[q]{(\frac{\beta}{\mu})^p} \text{ and } \nu = \lambda_j\left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}}\right),$$

where $j \in \{1, 2, \ldots, l\}$ is arbitrary but fixed. Meanwhile, let

$$\rho = \min\left\{-\frac{1}{\theta\lambda_i\left(a_i^{m_i \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_i}}\right)} : i = 1, 2, \ldots, l\right\}.$$

Then, for all $v \in \mathcal{H}$, we have

$$
\begin{aligned}
&\rho F(v, g(v)) + \gamma \|g(v) - g(u)\|^2 \\
&= \rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right) (\mu \sqrt[p]{v^q} - \beta) + \frac{1}{\delta - \beta} (\mu \sqrt[p]{v^q} - \beta)^2 \\
&= (\mu \sqrt[p]{v^q} - \beta) \left[\rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right) + \frac{1}{\delta - \beta} (\mu \sqrt[p]{v^q} - \beta) \right].
\end{aligned}
\tag{3.38}
$$

If $v \in \left[\sqrt[q]{(\frac{\alpha}{\mu})^p}, \sqrt[q]{(\frac{\beta}{\mu})^p} \right]$, then $\alpha - \beta \leq \mu \sqrt[p]{v^q} - \beta \leq 0$ and

$$
\begin{aligned}
&\rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right) + \frac{\alpha - \beta}{\delta - \beta} \\
&\leq \rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right) + \frac{1}{\delta - \beta} (\mu \sqrt[p]{v^q} - \beta) \\
&\leq \rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right).
\end{aligned}
\tag{3.39}
$$

For the case when $v \in \left[\sqrt[q]{(\frac{\delta}{\mu})^p}, \sqrt[q]{(\frac{\sigma}{\mu})^p} \right]$, we have $\mu \sqrt[p]{v^q} - \beta \in [\delta - \beta, \sigma - \beta]$ and

$$
\begin{aligned}
&\rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right) + 1 \\
&\leq \rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right) + \frac{1}{\delta - \beta} (\mu \sqrt[p]{v^q} - \beta) \\
&\leq \rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right) + \frac{\sigma - \beta}{\delta - \beta}.
\end{aligned}
\tag{3.40}
$$

By (3.39) and (3.40) and taking into consideration the fact that

$$
\lambda_i < 0 < \rho \leq - \frac{1}{\theta\lambda_i \left(a_i^{m_i \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_i}} \right)}, \text{ for each } i \in \{1, 2, \ldots, l\},
$$

it follows that

$$
(\mu \sqrt[p]{v^q} - \beta) \left[\rho\theta\lambda_j \left(a_j^{m_j \sqrt[q]{(\frac{\beta}{\mu})^p}} + \sqrt[q]{(\frac{\beta}{\mu})^{pn_j}} \right) + \tfrac{1}{\delta - \beta} (\mu \sqrt[p]{v^q} - \beta) \right] \geq 0,
$$

$$
\forall v \in \left[\sqrt[q]{(\tfrac{\alpha}{\mu})^p}, \sqrt[q]{(\tfrac{\beta}{\mu})^p} \right] \cup \left[\sqrt[q]{(\tfrac{\delta}{\mu})^p}, \sqrt[q]{(\tfrac{\sigma}{\mu})^p} \right],
$$

which leads to

$$(\mu\sqrt[p]{v^q}-\beta)\left[\rho\theta\lambda_j\left(a_j^{m_j\sqrt[q]{(\frac{\beta}{\mu})^p}}+\sqrt[q]{(\frac{\beta}{\mu})^{pn_j}}\right)+\frac{1}{\delta-\beta}(\mu\sqrt[p]{v^q}-\beta)\right]\geq 0,$$
$$\forall v\in\mathscr{H}:\mu\sqrt[p]{v^q}\in[\alpha,\beta]\cup[\delta,\sigma]. \tag{3.41}$$

Now, in virtue of (3.38) and (3.41), we deduce that

$$\rho F(v,g(v))+\gamma\|g(v)-g(u)\|^2\geq 0,\qquad \forall v\in\mathscr{H}:g(v)\in K.$$

On the other hand, for all $v\in\mathscr{H}$, we have

$$\begin{aligned}
&F(v,g(v))+\gamma\|g(v)-g(u)\|^2\\
&=\theta\lambda_j\left(a_j^{m_j\sqrt[q]{(\frac{\beta}{\mu})^p}}+\sqrt[q]{(\frac{\beta}{\mu})^{pn_j}}\right)(\mu\sqrt[p]{v^q}-\beta)+\frac{1}{\delta-\beta}(\mu\sqrt[p]{v^q}-\beta)^2\\
&=(\mu\sqrt[p]{v^q}-\beta)\left[\theta\lambda_j\left(a_j^{m_j\sqrt[q]{(\frac{\beta}{\mu})^p}}+\sqrt[q]{(\frac{\beta}{\mu})^{pn_j}}\right)+\frac{1}{\delta-\beta}(\mu\sqrt[p]{v^q}-\beta)\right].
\end{aligned}$$

The fact that $\lambda_i<\frac{\beta-\sigma}{\theta(\delta-\beta)\left(a_i^{m_i\sqrt[q]{(\frac{\beta}{\mu})^p}}+\sqrt[q]{(\frac{\beta}{\mu})^{pn_i}}\right)}$, for each $i\in\{1,2,\ldots,l\}$, implies that

$$\theta\lambda_j\left(a_j^{m_j\sqrt[q]{(\frac{\beta}{\mu})^p}}+\sqrt[q]{(\frac{\beta}{\mu})^{pn_j}}\right)+\frac{1}{\delta-\beta}(\mu\sqrt[p]{v^q}-\beta)<0,$$
$$\forall v\in\left[\sqrt[q]{(\frac{\delta}{\mu})^p},\sqrt[q]{(\frac{\sigma}{\mu})^p}\right].$$

Since $\mu\sqrt[p]{v^q}-\beta\in[\delta-\beta,\sigma-\beta]$, for all $v\in\left[\sqrt[q]{(\frac{\delta}{\mu})^p},\sqrt[q]{(\frac{\sigma}{\mu})^p}\right]$, it follows that

$$(\mu\sqrt[p]{v^q}-\beta)\left[\theta\lambda_j\left(a_j^{m_j\sqrt[q]{(\frac{\beta}{\mu})^p}}+\sqrt[q]{(\frac{\beta}{\mu})^{pn_j}}\right)+\frac{1}{\delta-\beta}(\mu\sqrt[p]{v^q}-\beta)\right]<0,$$
$$\forall v\in\left[\sqrt[q]{(\frac{\delta}{\mu})^p},\sqrt[q]{(\frac{\sigma}{\mu})^p}\right],$$

that is,

$$(\mu\sqrt[p]{v^q}-\beta)\left[\theta\lambda_j\left(a_j^{m_j\sqrt[q]{(\frac{\beta}{\mu})^p}}+\sqrt[q]{(\frac{\beta}{\mu})^{pn_j}}\right)+\frac{1}{\delta-\beta}(\mu\sqrt[p]{v^q}-\beta)\right]<0,$$
$$\forall v\in\mathscr{H}:g(v)\in[\delta,\sigma].$$

Hence, the inequality

$$F(v,g(v))+\gamma\|g(v)-g(u)\|^2\geq 0$$

cannot hold for all $v\in\mathcal{H}$ with $g(v)\in K$. Relying on this fact, we conclude that every solution of the problem (3.37) is not a solution of the problem (3.1) necessarily. Accordingly, for given $u\in\mathcal{H}:g(u)\in K$ and $\nu\in T(u)$, if $w=u$ is a solution of the auxiliary regularized equilibrium problem (3.34), then w need not be a solution of the problem (3.1).

Noor [23] claimed that using essentially the technique of Theorem 2, one can study the convergence analysis of Algorithm 3. For this end, he first asserted that the following statement holds.

Theorem 4 ([23, Theorem 3.3]). *Let the bifunction $F(.,.)$ be partially relaxed strongly g-monotone with constant $\alpha>0$. If u_{n+1} is the approximate solution obtained from Algorithm 3 and $u\in\mathcal{H}$ is a solution of (3.1), then*

$$\begin{aligned}&(1-\gamma)\|g(u)-g(u_{n+1})\|^2\\&\leq\|g(u)-g(u_n)\|^2-(1-2\rho\alpha-\gamma)\|g(u_n)-g(u_{n+1})\|^2.\end{aligned}\tag{3.42}$$

Now we analyze Algorithm 3 and the proof of Theorem 4 (that is, [23, Theorem 3.3]).

Algorithm 3 is constructed based on the fact that if $w=u$, where $u\in\mathcal{H}$ with $g(u)\in K$ and $\nu\in T(u)$ are given elements satisfying the problem (3.1), then w is a solution of the problem (3.1). Whereas, it was shown that for given $u\in\mathcal{H}:g(u)\in K$ and $\nu\in T(u)$, if $w=u$ then the problem (3.20) does not reduce to the problem (3.1), that is, w is not a solution of the problem (3.1). Hence, by considering the problem (3.34) for a given $u\in\mathcal{H}$ with $g(u)\in K$ and $\nu\in T(u)$, and in light of the above mentioned arguments, it should be pointed out that unlike the claim in [23], Algorithm 3 cannot be used for solving the problem (3.1). In order to overcome these difficulties, we need to replace $\gamma\|g(v)-g(w)\|^2$ and $\gamma\|g(v)-g(u_{n+1})\|^2$ by $\rho\gamma\|g(v)-g(w)\|^2$ and $\rho\gamma\|g(v)-g(u_{n+1})\|^2$ in (3.34) and (3.35), respectively. In the meanwhile, in view of the proof of Theorem 4 (that is, Theorem 3.3 in [23]), η_n in (3.36) must be replaced by ν_n.

By a careful reading the proof of Theorem 4 (that is, [23, Theorem 3.3]), we discovered that under the assumptions mentioned in Theorem 4, the relation (3.42) does not hold necessarily. By assuming $u\in\mathcal{H}$ with $g(u)\in K$ and $\nu\in T(u)$ as the solution of the problem (3.1) and taking $v=u_{n+1}$ in (3.1), Noor [23] deduced the inequality (3.7) (which is the inequality (3.15) in [23]). Taking $v=u$ in (3.35), he obtained the following inequality:

$$\begin{aligned}\rho F(\nu_n,g(u))+\langle g(u_{n+1})-g(u_n),g(u)-g(u_{n+1})\rangle&\\+\gamma\|g(u)-g(u_{n+1})\|^2\geq 0.&\end{aligned}\tag{3.43}$$

Applying (3.7) and (3.43) and considering the fact that $F(.,.)$ is partially relaxed strongly g-monotone with constant α, the author derived the inequality (3.16) in [23] as follows:

$$\begin{aligned}
&\langle g(u_{n+1})-g(u_n), g(u)-g(u_{n+1})\rangle \\
&\geq -\rho F(\nu_n, g(u)) - \gamma\|g(u)-g(u_{n+1})\|^2 \\
&\geq \rho\Big(F(\nu_n, g(u)) + F(\nu, g(u_{n+1}))\Big) \\
&\quad -\gamma\|g(u)-g(u_{n+1})\|^2 - \gamma\|g(u_n)-g(u_{n+1})\|^2 \\
&\geq -\alpha\rho\|g(u_n)-g(u_{n+1})\|^2 - \gamma\|g(u)-g(u_{n+1})\|^2 \\
&\quad -\gamma\|g(u_n)-g(u_{n+1})\|^2.
\end{aligned} \tag{3.44}$$

In the end, by combining (3.11) and (3.44), he concluded the required result (3.42). However, unlike the claim in [23], by (3.7) and (3.43) and invoking the definition of partially relaxed strong g-monotonicity of the bifunction F given in part (a) of Definition 5, what obtain is the inequality

$$\begin{aligned}
&\langle g(u_{n+1})-g(u_n), g(u)-g(u_{n+1})\rangle \\
&\geq -\rho F(\nu_n, g(u)) - \gamma\|g(u)-g(u_{n+1})\|^2 \\
&\geq -\rho\Big(F(\nu_n, g(u)) + F(\nu, g(u_{n+1}))\Big) \\
&\quad -\rho\gamma\|g(u)-g(u_{n+1})\|^2 - \gamma\|g(u)-g(u_{n+1})\|^2 \\
&\geq -\alpha\rho\|g(u_n)-g(u_{n+1})\|^2 - (\rho+1)\gamma\|g(u)-g(u_{n+1})\|^2,
\end{aligned} \tag{3.45}$$

not the inequality (3.44). In the meantime, by combining (3.11) and (3.45), we get the inequality

$$\begin{aligned}
&(1-2(\rho+1)\gamma)\|g(u)-g(u_{n+1})\|^2 \\
&\leq \|g(u)-g(u_n)\|^2 - (1-2\alpha\rho)\|g(u_{n+1})-g(u_n)\|^2,
\end{aligned}$$

not the inequality (3.42).

Relying on the above mentioned arguments and considering the fact that Theorem 4 plays an important and key role in the study of the convergence analysis of Algorithm 3, by an argument analogous to the previous one, mentioned for the proof of Theorem 2, we can prove that unlike the claim of the author in [23], using essentially the technique of Theorem 2 (that is, [23, Theorem 3.2], one cannot study the convergence analysis of Algorithm 3.

In order to overcome these difficulties, we now present the correct versions of the problem (3.3), Algorithm 3 and Theorem 4.

Let T, F, g and γ be the same as in RMNEP (3.19). For given $u \in K$ and $\nu \in T(u)$, we consider the problem of finding $w \in K$ such that

$$\rho F(\nu, g(v)) + \langle w-u, v-w\rangle + \rho\gamma\|g(v)-g(w)\|^2 \geq 0, \qquad \forall v \in K, \tag{3.46}$$

where $\rho > 0$ is a constant. If $w = u$, then it goes without saying that w is a solution of RMNEP (3.19). This fact enables us to suggest a predictor-corrector method for solving RMNEP (3.19) as follows.

Algorithm 4. Let T, F, g and γ be the same as in RMNEP (3.19). For given $u_0 \in K$ and $\nu_0 \in T(u_0)$, define the iterative sequences $\{u_n\}$ and $\{\nu_n\}$ by the iterative schemes

$$\rho F(\nu_n, g(v)) + \langle u_{n+1} - u_n, v - u_{n+1}\rangle + \rho\gamma\|g(v) - g(u_{n+1})\|^2 \geq 0, \quad \forall v \in K, \tag{3.47}$$

$$\nu_n \in T(u_n) : \|\nu_{n+1} - \nu_n\| \leq M(T(u_{n+1}), T(u_n)), \tag{3.48}$$

where $\rho > 0$ is a constant and $n = 0, 1, 2, \ldots$.

In order to establish the strong convergence of the sequences generated by Algorithm 4, we need the following definition.

Definition 8. Let $T : K \to CB(\mathcal{H})$ be a multivalued operator and let $g : K \to K$ be a nonlinear operator. The bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is said to be

(a) *g-monotone* with respect to T if

$$F(w_1, g(u_2)) + F(w_2, g(u_1)) \leq 0, \quad \forall u_1, u_2 \in K, w_1 \in T(u_1), w_2 \in T(u_2);$$

(b) *r-strongly g-monotone* with respect to T if there exists a constant $r > 0$ such that, for all $u_1, u_2 \in K, w_1 \in T(u_1), w_2 \in T(u_2)$, we have

$$F(w_1, g(u_2)) + F(w_2, g(u_1)) \leq -r\|g(u_1) - g(u_2)\|^2;$$

(c) *partially ς-strongly g-monotone* with respect to T if there exists a constant $\varsigma > 0$ such that, for all $u_1, u_2, z \in K, w_1 \in T(u_1), w_2 \in T(u_2)$, we have

$$F(w_1, g(u_2)) + F(w_2, g(z)) \leq -\varsigma\|g(z) - g(u_2)\|^2;$$

(d) *partially ζ-relaxed g-monotone* with respect to T of type (I) if there exists a constant $\zeta > 0$ such that, for all $u_1, u_2, z \in K, w_1 \in T(u_1), w_2 \in T(u_2)$, we have

$$F(w_1, g(u_2)) + F(w_2, g(z)) \leq \zeta\|z - u_1\|^2;$$

(e) *partially (ϱ, ϖ)-mixed relaxed and strongly g-monotone with respect to T of type* (I) if there exist two constants $\varrho, \varpi > 0$ such that, for all $u_1, u_2, z \in K, w_1 \in T(u_1), w_2 \in T(u_2)$, we have

$$F(w_1, g(u_2)) + F(w_2, g(z)) \leq \varrho\|z - u_1\|^2 - \varpi\|g(z) - g(u_2)\|^2.$$

It should be remarked that if $z = u_1$, then partially strong g-montonicity with respect to T and partially mixed relaxed and strong g-monotonicity with respect to T of type (I) of the bifunction F reduce to strong g-monotonicity with respect to T, and partially relaxed g-monotonicity with respect to T of type (I) reduces to g-monotonicity.

The following result plays a crucial role in the study of the convergence analysis of Algorithm 4.

Proposition 2. *Let T, F, g and γ be the same as in* RMNEP (3.19) *and let $u \in K$, $\nu \in T(u)$ be the solution of* RMNEP (3.19). *Moreover, assume that $\{u_n\}$ and $\{\nu_n\}$ are the sequences generated by Algorithm 4. If the bifunction F is partially $(\alpha, 2\rho\gamma)$-mixed relaxed and strongly g-monotone with respect to T of type* (I)*, then*

$$\|u-u_{n+1}\|^2 \leq \|u-u_n\|^2-(1-2\alpha\rho)\|u_{n+1}-u_n\|^2, \qquad \forall n \geq 0. \tag{3.49}$$

Proof. Since $u \in K$ and $\nu \in T(u)$ are the solution of RMNEP (3.19), it follows that (u,ν) satisfies (3.24). Taking $v = u_{n+1}$ in (3.24), we obtain

$$\rho F(\nu, g(u_{n+1}))+\rho\gamma\|g(u_{n+1})-g(u)\|^2 \geq 0. \tag{3.50}$$

Setting $v = u$ in (3.47), we get

$$\rho F(\nu_n, g(u))+\langle u_{n+1}-u_n, u-u_{n+1}\rangle+\rho\gamma\|g(u)-g(u_{n+1})\|^2 \geq 0. \tag{3.51}$$

By combining (3.50) and (3.51) and considering the fact that the bifunction F is partially $(\alpha, 2\gamma)$-mixed relaxed and strongly g-monotone with respect to T of type (I), we deduce that

$$\begin{aligned}&\langle u_{n+1}-u_n, u-u_{n+1}\rangle\\ &\geq -\rho F(\nu_n, g(u))-\rho\gamma\|g(u)-g(u_{n+1})\|^2\\ &\geq -\rho\Big(F(\nu_n, g(u))+F(\nu, g(u_{n+1}))\Big)-2\rho\gamma\|g(u)-g(u_{n+1})\|^2\\ &\geq -\alpha\rho\|u_{n+1}-u_n\|^2.\end{aligned} \tag{3.52}$$

Applying (3.29) and (3.52), we conclude that for all $n \geq 0$,

$$\|u-u_n\|^2-\|u-u_{n+1}\|^2-\|u_{n+1}-u_n\|^2 \geq -2\alpha\rho\|u_{n+1}-u_n\|^2,$$

whence we obtain

$$\|u-u_{n+1}\|^2 \leq \|u-u_n\|^2-(1-2\alpha\rho)\|u_{n+1}-u_n\|^2,$$

the required result (3.49). □

The paper is closed by the next assertion that provides us the required conditions under which the iterative sequences generated by Algorithm 4 converges strongly to a solution of RMNEP (3.19).

Theorem 5. *Let $\mathcal{H}$ be a finite dimensional real Hilbert space and let $g : K \to K$ be a continuous surjective mapping. Assume that the bifunction $F : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is continuous in the first argument and the operator $T : K \to C(\mathcal{H})$ is M-Lipschitz continuous with constant δ. Suppose further that all the conditions of Proposition 2 hold and* RMNEP$(T,F,g,K) \neq \emptyset$. *If $\rho \in (0, \frac{1}{2\alpha})$, then the iterative sequences $\{u_n\}$ and $\{\nu_n\}$ generated by Algorithm 4 converge strongly to $\hat{u} \in K$ and $\hat{\nu} \in T(\hat{u})$, respectively, and $(\hat{u}, \hat{\nu})$ is a solution of* RMNEP (3.19).

Proof. Since RMNEP$(T,F,g,K) \neq \varnothing$, we can take $u \in K$ and $v \in T(u)$ as a solution of RMNEP (3.19). In light of the inequality (3.49), we deduce that the sequence $\{\|u_n - u\|\}$ is nonincreasing and so the sequence $\{u_n\}$ is bounded. In the meanwhile, by (3.49), it follows that

$$\sum_{n=0}^{\infty}(1-2\alpha\rho)\|u_{n+1}-u_n\|^2 \leq \|u-u_0\|^2,$$

which guarantees $\|u_{n+1} - u_n\| \to 0$, as $n \to \infty$. Let $\hat{u}$ be a cluster point of the sequence $\{u_n\}$. Considering the fact that the sequence $\{u_n\}$ is bounded, there exists a subsequence $\{u_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \to \hat{u}$, as $i \to \infty$. By a similar proof as in Theorem 1, we conclude that $\{v_{n_i}\}$ is a Cauchy sequence in $\mathcal{H}$ and $v_{n_i} \to \hat{v}$, for some $\hat{v} \in \mathcal{H}$, as $i \to \infty$. Moreover, $\hat{u} \in K$ and $\hat{v} \in T(\hat{u})$ are the solution of RMNEP (3.19), and the sequences $\{u_n\}$ and $\{v_n\}$ have exactly one cluster point $\hat{u}$ and $\hat{v}$, respectively, and the proof is now complete. □

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