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APPROXIMATION BY q-DURRMEYER - STANCU POLYNOMIALS IN COMPACT DISKS IN THE CASE OF $q > 1$

M. KARA

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Abstract. In this paper, the order of simultaneous approximation and Voronovskaja-type results with quantitative estimate for complex q-Kantorovich polynomials $(q > 0)$ attached to analytic functions on compact disks are obtained. In particular, it is proved that for functions analytic in $\{z \in \mathbb{C} : |z| < R\}$, $R > q$, the rate of approximation by the q-Durrmeyer - Stancu operators $(q > 1)$ is of order q^{-n} versus $1/n$ for the classical q-Durrmeyer - Stancu operators.

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1. Introduction

For each integer $k \geq 0$, the q-integer $[k]_q$ and the q-factorial $[k]_q!$ are defined by

$$[k]_q := \begin{cases} \dfrac{1-q^k}{1-q} & \text{if} \quad q \in R^+ \backslash \{1\}, \\ k & \text{if} \quad q = 1 \end{cases} \quad \text{for } k \in N \quad \text{and} \quad [0]_q = 0,$$

$$[k]_q! := [1]_q \, [2]_q \ldots [k]_q \quad \text{for } k \in \mathbb{N} \quad \text{and} \quad [0]_q! = 1.$$

For integers $0 \leq k \leq n$, the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q! \, [n-k]_q!}.$$

For fixed $1 \neq q > 0$, we denote the q-derivative $D_q f(z)$ of f by

$$D_q f(z) = \begin{cases} \frac{f(qz) - f(z)}{(q-1)z}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}$$

This paper is in final form and no version of it will be submitted for publication elsewhere.

The q-analogue of integration in the interval $[0,A]$ is defined by

$$\int_0^A f(t)d_qt := A(1-q)\sum_{n=0}^{\infty} f(Aq^n)q^n, \quad 0<q<1.$$

For $0<q^{-1}<1$ we have

$$B_{q^{-1}}(m,n) = \frac{[m-1]_{q^{-1}}!\,[n-1]_{q^{-1}}!}{[m+n-1]_{q^{-1}}!}$$

In [2], Durrmeyer introduced the following Bernstein-Durrmeyer operator and [8] obtained simultaneous and ordinary approximation for these operator

$$D_n(f;x) = (n+1)\sum_{k=0}^{n} p_{n,k}(q;x)\int_0^1 p_{n,k}(t)\,f(t)dt, \quad x\in[0,1] \tag{1.1}$$

where

$$p_{n,k}(x) = \left(\begin{array}{c} n \\ k \end{array}\right) x^k(1-x)^{n-k}.$$

For $0\le\alpha\le\beta$, Stancu [13] introduced the Bernstein-Stancu operator as follows:

$$B_{n,\alpha,\beta}(f;x) = \sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)p_{n,k}(x). \tag{1.2}$$

In [4] Gupta defined q-Durrmeyer type operators:

$$D_{n,q}(f;x) = [n+1]_q\sum_{k=0}^{n} q^{-k}p_{n,k}(q;x)\int_0^1 f(t)p_{n,k}(t)\,d_qt, \quad x\in[0,1] \tag{1.3}$$

where

$$p_{n,k}(q;x) = \left[\begin{array}{c} n \\ k \end{array}\right]_q x^k\prod_{j=0}^{n-k-1}(1-q^jx).$$

In [4] and [3] (Gupta and Finta) some local and global direct results for different q-analogues of Durrmeyer operator is studied. In [6], Gupta and Wang defined the following $q-$Durrmeyer operator and estimate the rate of convergence for these operator;

$$M_{n,q}(f;x) \tag{1.4}$$

$$= [n+1]_q\sum_{k=1}^{n} q^{1-k}p_{n,k}(q;x)\int_0^1 f(t)p_{n,k-1}(q;qt)\,d_qt + f(0)p_{n,0}(q;x), \quad x\in[0,1]$$

Also Gupta and Finta [5] studied some direct results for the certain q-Durrmeyer operator(1). In [1], Agarwal and Gupta introduced and studied complex case of $q-$Durrmeyer operators (1). In [12], Gal, Gupta and Mahmudov obtained some approximation properties for the following q-Durrmeyer operators;

$$M_{n,q}(f;z)$$
$$= [n]_q \sum_{k=1}^{n} q^{1-k} p_{n,k}(q;x) \int_0^1 f(t) p_{n-1,k-1}(q;qt)\, d_q t + f(0) p_{n,0}(q;z), \quad z \in \mathbb{C}$$

In [9] Mahmudov and Gupta introduced the complex genuine Durrmeyer -Stancu operators as follows:

$$U_n^{\alpha,\beta}(f;z) = p_{n,0}(z) f\left(\frac{\alpha}{n+\beta}\right) + p_{n,n}(z) f\left(\frac{n+\alpha}{n+\beta}\right)$$
$$+(n-1)\sum_{k=1}^{n-1} p_{n,k}(z) \int_0^1 p_{n-2,k-1}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt.$$

where α, β are given two real parameters satisfying the condition $0 \le \alpha \le \beta$.In [11] , M. Ren, X. Zeng and L. Zeng introduced the following complex Stancu-type Durrmeyer operators and obtained the approximation properties of these operators.

$$M_n^{(\alpha,\beta)}(f;z) = p_{n,0}(z) f\left(\frac{\alpha}{n+\beta}\right) + n\sum_{k=1}^{n} p_{n,k}(z) \int_0^1 p_{n-1,k-1}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt.$$

In[7], Gupta and Karslı considered Stancu type generalization as follows

$$M_n^{(\alpha,\beta)}(f;z) = p_{n,0}(z) f\left(\frac{\alpha}{n+\beta}\right) + (n+1)\sum_{k=1}^{n} p_{n,k}(z) \int_0^1 p_{n,k-1}(t) f\left(\frac{nt+\alpha}{n+\beta}\right) dt.$$

Recently, Mahmudov (2014) [10] introduced and obtained the following approximation results of the complex q-Durrmeyer type operator in the case of $q > 1$.

$$\mathfrak{D}_n(f;z) = p_{n,0}(q;q^2 z) f(0)$$
$$+ [n+1]_{q^{-1}} \sum_{k=1}^{n} q^{k-1} p_{n,k}(q;q^2 z) \int_0^1 p_{n,k-1}(q^{-1};q^{-1}t) f\left(q^{k-n} t\right) d_{1/q} t.$$

In the present paper, for $q > 1$,we introduce the complex q-Durrmeyer-Stancu operators as follows

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z) = f\left(\frac{\alpha}{[n]_q+\beta}\right) p_{n,0}\left(q;q^2 z\right) \tag{1.5}$$

$$+\left[n+1\right]_{q^{-1}}\sum_{k=1}^{n}q^{k-1}p_{n,k}\left(q;q^{2}z\right)\int_{0}^{1}p_{n,k-1}(q^{-1};q^{-1}t)f\left(\frac{q^{k-n-2}\left[n\right]_{q}t+\alpha}{\left[n\right]_{q}+\beta}\right)d_{\frac{1}{q}}t.$$

Notice that The case $\alpha=\beta=0$ and $q>1$, studied in [10].

Let $\mathbb{D}_R$ be a disc $\mathbb{D}_R:=\{z\in\mathbb{C}:|z|<R\}$ in the complex plane $\mathbb{C}$. Denote by $H(\mathbb{D}_R)$ the space of all analytic functions on $\mathbb{D}_R$. For $f\in H(\mathbb{D}_R)$ we assume that $f(z)=\sum_{m=0}^{\infty}a_m z^m$ for all $z\in\mathbb{D}_R$. The norm $\|f\|_r:=\max\{|f(z):|z|\le r|\}$. We define $e_m(z)=z^m$ for all $m\in\mathbb{N}\cup\{0\}$.

The paper is organized as follows. In Section 2 we give some auxilary results, in particular we prove a which includes some properties of the complex $q-$Durrmeyer-Stancu operators in the case $q>1$. In Section 3 we study approximation properties (Theorem 1), Voronovskaja type quantitative estimation (Theorem 2) and exact order of approximation (Theorem 3).

2. Auxiliary results

In this section we find relation between the moments of $q-$Durrmeyer operators $\mathfrak{D}_{n,q}(e_j;z)$ and $q-$Durrmeyer-Stancu operators $\mathfrak{D}_{n,q}^{(\alpha,\beta)}(e_m;z)$.

Lemma 1 ([10]). *For $m=0,1,2$, we have*

$$\mathfrak{D}_{n,q}(e_0;z)=1,\quad \mathfrak{D}_{n,q}(e_1;z)=\frac{q^2[n]_q}{[n+2]_q}z,$$

$$\mathfrak{D}_{n,q}(e_2;z)=\frac{q^2[n]_q(1+q)}{[n+3]_q[n+2]_q}z+\frac{q^3[n]_q([n]_q-1)}{[n+3]_q[n+2]_q}z^2,$$

where

$$\mathfrak{D}_{n,q}(e_m;z)=\frac{[n+1]_q!}{[n+m+1]_q!}\sum_{k=0}^{n}p_{n,k}(q;q^2z)\frac{[k+m-1]_q!}{[k-1]_q!}. \tag{2.1}$$

In Lemma 2, we define the moments for complex q-Durrmeyer-Stancu operators $(q>1)$.

Lemma 2. *For all $m,n\in\mathbb{N}_0, z\in\mathbb{C}$ and $0\le\alpha\le\beta$, we have the following recurrence relation*

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}(e_m;z)=\sum_{j=0}^{m}\binom{m}{j}\frac{[n]_q^j\alpha^{m-j}}{([n]_q+\beta)^m}\mathfrak{D}_{n,q}(e_j;z),$$

where $e_m(z)=z^m$.

Proof. By using 2.1 and simple computation,
For $m=0$,

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}(e_0;z)$$

$$= [n+1]_{q^{-1}} \sum_{k=1}^{n} q^{k-1} p_{n,k}(q;q^2 z) \int_0^1 p_{n,k-1}(q^{-1};q^{-1}t) d_{1/q}t + p_{n,0}(q;q^2 z)$$

$$= \mathfrak{D}_{n,q}(e_0;z) = 1.$$

Secondly, we calculate the first order monomial

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}(e_1;z) = \frac{\alpha}{\left([n]_q+\beta\right)} p_{n,0}(q;q^2 z)$$

$$+ [n+1]_{q^{-1}} \sum_{k=1}^{n} q^{k-1} p_{n,k}(q;q^2 z) \int_0^1 p_{n,k-1}(q^{-1};q^{-1}t) \left(\frac{q^{k-n-2} t \, [n]_q + \alpha}{[n]_q + \beta} \right) d_{1/q}t$$

$$= \frac{[n]_q}{[n]_q+\beta} \left\{ [n+1]_{q^{-1}} \sum_{k=1}^{n} q^{k-1} p_{n,k}(q;q^2 z) \int_0^1 q^{k-n-2} t p_{n,k-1}(q^{-1};q^{-1}t) d_{1/q}t \right\}$$

$$+ \frac{\alpha}{\left([n]_q+\beta\right)}$$

$$\left\{ [n+1]_{q^{-1}} \sum_{k=1}^{n} q^{k-1} p_{n,k}(q;q^2 z) \int_0^1 p_{n,k-1}(q^{-1};q^{-1}t) d_{1/q}t + p_{n,0}(q;q^2 z) \right\}$$

$$= \frac{[n]_q}{[n]_q+\beta} \mathfrak{D}_{n,q}(e_1;z) + \frac{\alpha}{\left([n]_q+\beta\right)} \mathfrak{D}_{n,q}(e_0;z)$$

$$= \frac{[n]_q}{\left([n]_q+\beta\right)} \frac{q^2 \, [n]_q}{[n+2]_q} z + \frac{\alpha}{[n]_q+\beta}.$$

Finally, For $m = 2$,

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}(e_2;z) = \frac{\alpha^2}{\left([n]_q+\beta\right)^2} p_{n,0}(q;q^2 z)$$

$$+ \left\{ [n+1]_{q^{-1}} \sum_{k=1}^{n} q^{k-1} p_{n,k}(q;q^2 z) \int_0^1 p_{n,k-1}(q^{-1};q^{-1}t) \frac{\left(q^{k-n-2} t \, [n]_q + \alpha\right)^2}{\left([n]_q+\beta\right)^2} d_{1/q}t \right\}$$

$$= \frac{[n]_q^2}{\left([n]_q+\beta\right)^2} \mathfrak{D}_{n,q}(e_2;z) + \frac{2\,[n]_q\,\alpha}{\left([n]_q+\beta\right)^2} \mathfrak{D}_{n,q}(e_1;z) + \frac{\alpha^2}{\left([n]_q+\beta\right)^2} \mathfrak{D}_{n,q}(e_0;z)$$

$$= \frac{[n]_q^2}{\left([n]_q+\beta\right)^2} \left\{ \frac{q^2\,[n]_q\,(1+q)}{[n+3]_q\,[n+2]_q} z + \frac{q^3\,[n]_q\left([n]_q-1\right)}{[n+3]_q\,[n+2]_q} z^2 \right\}$$

$$+\frac{2[n]_q\alpha}{([n]_q+\beta)^2}\frac{q^2[n]_q}{[n+2]_q}z+\frac{\alpha^2}{([n]_q+\beta)^2}.$$

□

Next lemma estimation of the moments $\mathfrak{D}_{n,q}^{(\alpha,\beta)}(e_m;z)$ in the compact disks $\overline{\mathbb{D}_R}$.

Lemma 3. *For all $m,n\in\mathbb{N}_0$, $z\in\overline{\mathbb{D}_R}$, $0\le\alpha\le\beta$ and $r\ge 1$, we have*

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}(e_m;z)\le(q^2r)^m.$$

Proof. Using the inequality $\left|\mathfrak{D}_{n,q}(e_j;z)\right|\le(q^2r)^j$ ([10])

$$\begin{aligned}\left|\mathfrak{D}_{n,q}^{(\alpha,\beta)}(e_m;z)\right|&\le\sum_{j=0}^{m}\binom{m}{j}\frac{[n]_q^j\,\alpha^{m-j}}{([n]_q+\beta)^m}\left|\mathfrak{D}_{n,q}(e_j;z)\right|\\&\le(q^2r)^m\sum_{j=0}^{m}\binom{m}{j}\frac{[n]_q^j\,\alpha^{m-j}}{([n]_q+\beta)^m}\\&=(q^2r)^m\left(\frac{[n]_q+\alpha}{[n]_q+\beta}\right)^m\le(q^2r)^m.\end{aligned}$$

□

Remark 1. By simple computation, we have

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}((e_1-e_0);z)=\frac{\alpha-([2]_q+\beta)}{[n]_q+\beta}z.$$

3. Approximation by complex $q-$Durrmeyer-Stancu polynomials

We start with the following quantitative estimates of the convergence for complex q-Durrmeyer-Stancu operators attached to an analytic function in a disk of radius $R>1$ and center 0.

Theorem 1. *Let $0\le\alpha\le\beta, 1<q<R<\infty$ and $1\le r<\frac{R}{q^3}$. For all $z\in\mathbb{D}_r$ and $n\in\mathbb{N}$, we have the following inequality*

$$\left|\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-f(z)\right|\le\left(\frac{1+q^2r+\alpha+\beta}{[n]_q}\right)\sum_{m=1}^{\infty}|a_m|\,m\,(m+1)\left(q^3r\right)^m.$$

where $f\in H(\mathbb{D}_R)$.

Proof.

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-f(z)=\sum_{j=0}^{m}\binom{m}{j}\frac{[n]_q^j\,\alpha^{m-j}}{([n]_q+\beta)^m}\left|\mathfrak{D}_{n,q}(e_j;z)-(e_j;z)\right|$$

$$+\sum_{j=0}^{m}\binom{m}{j}\frac{[n]_q^j\,\alpha^{m-j}}{\left([n]_q+\beta\right)^m}z^j-z^m.$$

Using the estimation

$$\left|\mathfrak{D}_{n,q}(e_j;z)-(e_j;z)\right|\le\frac{1+q^2r}{[n+2]_q}j(j+1)(q^3r)^{j-1}.$$

(See [10]). Then we can easily obtain

$$\begin{aligned}\left|\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-f(z)\right|&\le\frac{1+q^2r}{[n+2]_q}m(m+1)(q^3r)^{m-1}+\left(\frac{[n]^m}{([n]+\beta)^m}-1\right)z^m\\&\quad+\sum_{j=0}^{m-1}\binom{m-1}{j}\frac{m}{m-j}\frac{[n]_q^j\,\alpha^{m-j}}{\left([n]_q+\beta\right)^m}z^j\\&\le\frac{1+q^2r}{[n+2]_q}m(m+1)(q^3r)^{m-1}+\frac{m\beta}{[n]_q+\beta}r^m+\frac{m\alpha}{[n]_q}r^{m-1}\\&\le\frac{\left(1+q^2r+\alpha+\beta\right)}{[n]_q}m(m+1)(q^3r)^{m-1}.\end{aligned}$$

□

The next theorem gives Voronovskaja type result in compact disks, for complex q-Kantorovich operators attached to an analytic function in $\mathbb{D}_R$, $R>1$ and center 0. In orders to formulate the results we introduce

$$L_q(f;z)=\frac{(1-q^2z)D_qf(z)-(1-z)D_{q^{-1}}f(z)}{q-1}.$$

Theorem 2. *Let $0\le\alpha\le\beta$ and $1\le r<\frac{R}{q4}$. Then for all $z\in\mathbb{D}_r$ and $n\in\mathbb{N}$, we have the following inequality*

$$\begin{aligned}&\left|\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-f(z)-\frac{\alpha-\left([2]_q+\beta\right)z}{[n]+\beta}-\frac{1}{[n+2]_q}L_q(f;z)\right|\\&\le\frac{11\left(1+q^2r\right)^2}{[n+2]_q^2}\sum_{m=1}^{\infty}(m+1)^4(q^4r)^m\\&+\frac{\left(\alpha^2+\beta^2+\beta\left([2]_q+\beta\right)\right)}{([n]+\beta)^2}\sum_{m=2}^{\infty}|a_m|m(m-1)\left(q^4r\right)^m\\&+\frac{\left(1+q^2r\right)\left(\alpha+[2]_q+\beta\right)}{([n]+\beta)[n+2]_q}\sum_{m=1}^{\infty}|a_m|m^2(m+1)\left(q^4r\right)^m,\end{aligned}$$

where $f\in H(\mathbb{D}_R)$.

Proof. We immediately obtain

$$
\begin{aligned}
&\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-f(z)-\frac{\alpha-\left([2]_q+\beta\right)z}{[n]+\beta}f^{'}(z)-\frac{1}{[n+2]_q}L_q(f;z)\\
&=\mathfrak{D}_{n,q}(f;z)-f(z)-\frac{1}{[n+2]_q}L_q(f;z)+\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-\mathfrak{D}_{n,q}(f;z)\\
&\quad-\frac{\alpha-\left([2]_q+\beta\right)z}{[n]+\beta}f^{'}(z)\\
&=\sum_{m=1}^{\infty}a_m\left(\mathfrak{D}_{n,q}(e_m;z)-e_m(z)-\frac{1}{[n+2]_q}L_q(e_m;z)\right)\\
&+\sum_{m=1}^{\infty}a_m\left(\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-\mathfrak{D}_{n,q}(f;z)-\frac{\alpha-\left([2]_q+\beta\right)z}{[n]+\beta}mz^{m-1}\right)
\end{aligned}
$$

For the first estimate, using the Theorem 3 in [10], namely

$$
\left|\mathfrak{D}_{n,q}(f;z)-f(z)-\frac{1}{[n+2]_q}L_q(f;z)\right|\leq\frac{11\left(1+q^2r\right)^2}{[n+2]_q^2}\sum_{m=1}^{\infty}(m+1)^4(q^4r)^m.
$$

For the second series, we rewrite it as follows.

$$
\begin{aligned}
&\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-\mathfrak{D}_{n,q}(f;z)-\frac{\alpha-\left([2]_q+\beta\right)z}{[n]+\beta}f^{'}(z)\\
&=\sum_{j=0}^{m-1}\binom{m}{j}\frac{[n]_q^j\,\alpha^{m-j}}{([n]+\beta)^m}D_{n,q}(e_j;z)+\frac{[n]_q^m}{([n]+\beta)^m}D_{n,q}(e_m;z)-D_{n,q}(e_m;z)\\
&\qquad-\left[\frac{\alpha-\left([2]_q+\beta\right)z}{[n]+\beta}\right]mz^{m-1}\\
&=\sum_{j=0}^{m-1}\binom{m}{j}\frac{[n]_q^j\,\alpha^{m-j}}{([n]+\beta)^m}D_{n,q}(e_j;z)+\left(\frac{[n]_q^m}{([n]+\beta)^m}-1\right)D_{n,q}(e_m;z)\\
&\qquad-\left[\frac{\alpha-\left([2]_q+\beta\right)z}{[n]+\beta}\right]mz^{m-1}\\
&=\sum_{j=0}^{m-2}\binom{m}{j}\frac{[n]_q^j\,\alpha^{m-j}}{([n]+\beta)^m}D_{n,q}(e_j;z)+\frac{m\,[n]_q^{m-1}\,\alpha}{([n]+\beta)^m}D_{n,q}(e_{m-1};z)\\
&-\sum_{j=0}^{m-1}\binom{m}{j}\frac{[n]_q^j\,\beta^{m-j}}{([n]+\beta)^m}D_{n,q}(e_m;z)-\left[\frac{\alpha-\left([2]_q+\beta\right)z}{[n]+\beta}\right]mz^{m-1}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=0}^{m-2} \binom{m}{j} \frac{[n]_q^j \alpha^{m-j}}{([n]+\beta)^m} D_{n,q}(e_j;z) + \frac{m\,[n]_q^{m-1}\alpha}{([n]+\beta)^m}\left(D_{n,q}(e_{m-1};z) - z^{m-1}\right) \\
&- \sum_{j=0}^{m-2} \binom{m}{j} \frac{[n]_q^j \beta^{m-j}}{([n]+\beta)^m} D_{n,q}(e_m;z) + \left(\frac{[n]_q^{m-1}}{([n]+\beta)^{m-1}} - 1\right)\frac{\alpha}{[n]+\beta} m z^{m-1} \\
&\qquad + \frac{m\,[n]_q^{m-1}\left([2]_q+\beta\right)}{([n]+\beta)^m}\left(z^m - D_{n,q}(e_m;z)\right) \\
&\qquad + \left(1 - \frac{[n]_q^{m-1}}{([n]+\beta)^{m-1}}\right)\left[\frac{\left([2]_q+\beta\right)z}{[n]+\beta}\right] m z^{m-1} \\
&\qquad := \sum_{k=1}^{6} I_k .
\end{aligned}
$$

We use the inequality

$$
1 - \frac{[n]_q^k}{([n]+\beta)^k} \le \sum_{j=1}^{k}\left(1 - \frac{[n]_q}{[n]_q+\beta}\right) = \frac{k\beta}{[n]_q+\beta}
$$

to estimate I_1, I_2, I_3, I_4, I_5 and I_6.

$$
\begin{aligned}
|I_1| &\le \left| \sum_{j=0}^{m-2} \binom{m}{j} \frac{[n]_q^j \alpha^{m-j}}{([n]+\beta)^m} D_{n,q}(e_j;z) \right| \\
&\le \sum_{j=0}^{m-2} \binom{m}{j} \frac{[n]_q^j \alpha^{m-j}}{([n]+\beta)^m} \left|D_{n,q}(e_j;z)\right| \\
&= \sum_{j=0}^{m-2} \binom{m-2}{j} \frac{m(m-1)}{(m-j-1)(m-j)} \frac{[n]_q^j \alpha^{m-j}}{([n]+\beta)^m} \left|D_{n,q}(e_j;z)\right| \\
&\le \frac{m(m-1)}{2} \frac{\alpha^2}{([n]+\beta)^2} \left(q^2 r\right)^{m-2},
\end{aligned}
$$

$$
\begin{aligned}
|I_2| &\le \frac{m\,[n]_q^{m-1}\alpha}{([n]+\beta)^m} \left|D_{n,q}(e_{m-1};z) - z^{m-1}\right| \\
&\le \frac{m\,[n]_q^{m-1}\alpha}{([n]+\beta)^m} \frac{\left(1+q^2 r\right)}{[n+2]_q} (m-1)m\left(q^3 r\right)^{m-2} \\
&\le \frac{m^2(m-1)\alpha\left(1+q^2 r\right)}{([n]+\beta)\,[n+2]_q} \left(q^3 r\right)^{m-2},
\end{aligned}
$$

$$|I_3| \leq \sum_{j=0}^{m-2} \begin{pmatrix} m \\ j \end{pmatrix} \frac{[n]_q^j \beta^{m-j}}{([n]+\beta)^m} \left|D_{n,q}(e_m;z)\right| \leq \frac{m(m-1)\beta^2}{([n]+\beta)^2} \left(q^2 r\right)^m,$$

$$|I_4| \leq \left(\frac{[n]_q^{m-1}}{([n]+\beta)^{m-1}} - 1\right) \frac{\alpha}{[n]+\beta} m z^{m-1} \leq \frac{m(m-1)\alpha\beta}{\left([n]_q+\beta\right)^2} r^{m-1},$$

$$|I_5| \leq \frac{m\,[n]_q^{m-1}\left([2]_q+\beta\right)}{([n]+\beta)^m} \left|z^m - D_{n,q}(e_m;z)\right|$$

$$\leq \frac{m^2(m+1)\left([2]_q+\beta\right)\left(1+q^2 r\right)}{([n]+\beta)\,[n+2]_q} (q^3 r)^{m-1},$$

$$|I_6| \leq \frac{m\,(m-1)\,\beta\left([2]_q+\beta\right)}{([n]+\beta)^2} r^m.$$

Then we get

$$\left|\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z) - \mathfrak{D}_{n,q}(f;z) - \frac{\alpha - \left([2]_q+\beta\right)z}{[n]+\beta} f'(z)\right|$$

$$\leq \frac{m(m-1)}{2} \frac{\alpha^2}{([n]+\beta)^2} \left(q^2 r\right)^{m-2} + \frac{m^2\,(m-1)\,\alpha\left(1+q^2 r\right)}{([n]+\beta)\,[n+2]_q} \left(q^3 r\right)^{m-2}$$

$$+ \frac{m(m-1)\beta^2}{([n]+\beta)^2} \left(q^2 r\right)^m + \frac{m(m-1)\alpha\beta}{\left([n]_q+\beta\right)^2} r^{m-1}$$

$$+ \frac{m^2(m+1)\left([2]_q+\beta\right)\left(1+q^2 r\right)}{([n]+\beta)\,[n+2]_q} (q^3 r)^m + \frac{m\,(m-1)\,\beta\left([2]_q+\beta\right)}{([n]+\beta)^2} r^m$$

$$\leq \frac{m(m-1)\left(\alpha^2+\beta^2+\beta\left([2]_q+\beta\right)\right)}{([n]+\beta)^2} \left(q^4 r\right)^m$$

$$+ \frac{m^2\,(m+1)\left(1+q^2 r\right)\left(\alpha+[2]_q+\beta\right)}{([n]+\beta)\,[n+2]_q} \left(q^4 r\right)^m.$$

□

In the following theorem, for q-Durrmeyer-Stancu polynomials, we obtain the exact order of approximation.

Theorem 3. *Let* $1 < q < R$, $1 \leq r < \frac{R}{q^2}$ *(or* $0 < q \leq 1$, $1 \leq r < R$*) and* $f \in H(\mathbb{D}_R)$*. If* f *is not a constant function then the estimate*

$$\left\|\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f) - f\right\|_r \geq \frac{1}{[n+1]_q} C_{r,q}(f), \quad n \in \mathbb{N},$$

holds, where the constant $C_{r,q}(f)$ *depends on* f, q *and* r *but is independent of* n.

Proof of Theorem 3. For all $z \in \mathbb{D}_R$ and $n \in \mathbb{N}$ we get

$$\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z) - f(z) = \frac{1}{[n+2]_q}$$

$$\left\{ \begin{array}{c} \frac{[n+2]_q}{[n]+\beta}\left(\alpha - \left([2]_q+\beta\right)\right) f'(z) + L_q(f;z) \\ +[n+2]_q\left(\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z) - f(z) - \frac{\alpha-([2]_q+\beta)z}{[n]+\beta} f'(z) - \frac{1}{[n+2]_q} L_q(f;z)\right) \end{array} \right\}.$$

We apply

$$\|F+G\|_r \geq |\|F\|_r - \|G\|_r| \geq \|F\|_r - \|G\|_r$$

to get

$$\left\|\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f) - f\right\|_r \geq \frac{1}{[n+2]_q}$$

$$\left\{ \begin{array}{c} \left\|\frac{[n+2]_q}{[n]+\beta}\left(\alpha e_0 - \left([2]_q+\beta\right)e_1\right) f'\right) + L_q(f)\right\|_r \\ -[n+2]_q\left\|\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f) - f - \frac{\alpha e_0-([2]_q+\beta)e_1}{[n]+\beta} f' - \frac{1}{[n+2]_q} L_q(f;z)\right\|_r \end{array} \right\}.$$

Because by hypothesis f is not a constant in $\mathbb{D}_R$, it follows

$$\left\|\frac{n+2}{[n]+\beta}\left(\alpha e_0 - \left([2]_q+\beta\right)e_1\right) f'\right) + L_q(f)\right\|_r > 0.$$

Indeed, assuming the contrary it follows that

$$\frac{n+2}{[n]+\beta}\left(\alpha e_0 - \left([2]_q+\beta\right)e_1\right) f'\right) + L_q(f) = 0$$

for all $z \in \overline{\mathbb{D}}_R$ that is

$$\frac{n+2}{[n]+\beta}\sum_{m=1}^{\infty} a_m\left(\alpha - ([2]+\beta)z\right) m z^{m-1}$$

$$+\sum_{m=1}^{\infty} a_m\left(\frac{[m]_q - m}{q-1} + q^{-1}\frac{[m]_{q^{-1}} - m}{q^{-1}-1}\right) z^{m-1}\left(1-q^2 z\right)$$

$$-\sum_{m=1}^{\infty} a_m \frac{[m]_{q^{-1}}}{q-1} z^{m-1}(q^2-1)z = 0$$

$$\frac{[n+2]_q}{[n]+\beta}\sum_{m=1}^{\infty} a_m \alpha m z^{m-1} - \frac{[n+2]_q}{[n]+\beta}\sum_{m=1}^{\infty} a_m\left([2]+\beta\right) m z^m$$

$$+\sum_{m=1}^{\infty} a_m\left(\frac{[m]_q - m}{q-1} + q^{-1}\frac{[m]_{q^{-1}} - m}{q^{-1}-1}\right) z^{m-1}$$

$$+\sum_{m=1}^{\infty} a_m\left(\frac{[m]_q-m}{q-1}+q^{-1}\frac{[m]_q-m}{q^{-1}-1}\right)z^m q^2$$

$$-\sum_{m=1}^{\infty} a_m\frac{[m]_{q^{-1}}}{q-1}z^m\left(q^2-1\right)=0$$

$$\left(\frac{[n+2]_q\,\alpha}{[n]+\beta}\right)a_1+\frac{[n+2]_q}{[n]+\beta}\sum_{m=1}^{\infty}a_{m+1}\alpha\,(m+1)\,z^m-\frac{[n+2]_q}{[n]+\beta}\sum_{m=1}^{\infty}a_m\,([2]+\beta)\,mz^m$$

$$+\sum_{m=1}^{\infty}a_{m+1}\left(\frac{[m+1]_q-(m+1)}{q-1}+q^{-1}\frac{[m+1]_{q^{-1}}-(m+1)}{q^{-1}-1}\right)z^m$$

$$+\sum_{m=1}^{\infty}a_m\left(\frac{[m]_q-m}{q-1}+q^{-1}\frac{[m]_q-m}{q^{-1}-1}\right)z^m q^2-\sum_{m=1}^{\infty}a_m\frac{[m]_{q^{-1}}}{q-1}z^m\left(q^2-1\right)=0$$

So,

$$a_1=0$$

$$a_{m+1}\left(\frac{n+2}{[n]+\beta}\alpha\,(m+1)+\left(\frac{[m+1]_q-(m+1)}{q-1}+q^{-1}\frac{[m+1]_{q^{-1}}-(m+1)}{q^{-1}-1}\right)\right)$$

$$=a_m\left(\frac{n+2}{[n]+\beta}\,([2]+\beta)\,m-\left(\frac{[m]_q-m}{q-1}+q^{-1}\frac{[m]_q-m}{q^{-1}-1}\right)q^2+\frac{[m]_{q^{-1}}}{q-1}\left(q^2-1\right)\right)$$

$$a_{m+1}=\frac{\left(\frac{n+2}{[n]+\beta}\alpha\,(m+1)+\left(\frac{[m+1]_q-m+1}{q-1}+q^{-1}\frac{[m+1]_{q^{-1}}-(m+1)}{q^{-1}-1}\right)\right)}{\left(\frac{n+2}{[n]+\beta}\,([2]+\beta)\,m-\left(\frac{[m]_q-m}{q-1}+q^{-1}\frac{[m]_q-m}{q^{-1}-1}\right)q^2+\frac{[m]_{q^{-1}}}{q-1}\left(q^2-1\right)\right)}$$

for all $z\in\overline{\mathbb{D}}_R\backslash\{0\}$. Thus $a_m=0$, $m=1,2,3,\ldots$Thus, f is constant, which is contradiction with the hypothesis.

Now, by Theorem 2 we have

$$[n+2]_q\left|\mathfrak{D}_{n,q}^{(\alpha,\beta)}(f;z)-f(z)-\frac{\alpha e_0-\left([2]_q+\beta\right)e_1}{[n]+\beta}mz^{m-1'}-\frac{1}{[n+2]_q}L_q(f;z)\right|$$

$$\leq\frac{11\left(1+q^2r\right)^2}{[n+2]_q^2}\sum_{m=1}^{\infty}(m+1)^4\left(q^4r\right)^m$$

$$+\frac{\left(\alpha^2+\beta^2+\beta\left([2]_q+\beta\right)\right)}{([n]+\beta)^2}\sum_{m=2}^{\infty}|a_m|\,m(m-1)\left(q^4r\right)^m$$

$$+\frac{\left(1+q^2r\right)\left(\alpha+[2]_q+\beta\right)}{([n]+\beta)\,[n+2]_q}\sum_{m=1}^{\infty}|a_m|\,m^2\,(m+1)\left(q^4r\right)^m\to 0\text{ as }n\to\infty.$$

Moreover,

$$\left\| \frac{[n+2]_q}{[n]+\beta} \left(\alpha e_0 - \left([2]_q + \beta \right) e_1 \right) f^{'} \right) + L_q(f) \right\|_r \to \left\| q^2 \left(\alpha e_0 - \left([2]_q + \beta \right) e_1 \right) f^{'} \right) + L_q(f) \right\|_r$$

Consequently, there exists n_1 (depending only on f and r) such that for all $n \geq n_1$ we have

$$\left\| \frac{[n+2]_q}{[n]+\beta} \left(\alpha e_0 - \left([2]_q + \beta \right) e_1 \right) f^{'} \right) + L_q(f) \right\|_r - [n+2]_q \left\| \mathfrak{D}_{n,q}^{(\alpha,\beta)}(f) - f - \frac{\alpha e_0 - \left([2]_q + \beta \right) e_1}{[n]+\beta} f^{'} - \frac{1}{[n+2]_q} L_q(f;z) \right\|_r \geqslant \frac{1}{2} \left\| \frac{[n+2]_q}{[n]+\beta} \left(\alpha e_0 - \left([2]_q + \beta \right) e_1 \right) f^{'} \right) + L_q(f) \right\|_r$$

which implies

$$\left\| \mathfrak{D}_{n,q}^{(\alpha,\beta)}(f) - f \right\|_r \geq \frac{1}{[n+2]_q} \frac{1}{2} \left\| q^2 \left(\alpha e_0 - \left([2]_q + \beta \right) e_1 \right) f^{'} \right) + L_q(f) \right\|_r, \quad \text{for all } n \geq n_1.$$

For $1 \leq n \leq n_1 - 1$ we have

$$\left\| \mathfrak{D}_{n,q}^{(\alpha,\beta)}(f) - f \right\|_r \geq \frac{1}{[n+2]_q} \left([n+2]_q \left\| \mathfrak{D}_{n,q}^{(\alpha,\beta)}(f) - f \right\|_r \right) = \frac{1}{[n+1]_q} M_{r,n}(f) > 0,$$

which finally implies that

$$\left\| \mathfrak{D}_{n,q}^{(\alpha,\beta)}(f) - f \right\|_r \geq \frac{1}{[n+1]_q} C_{r,q}(f),$$

for all n, with

$$C_{r,q}(f) = \min \left\{ M_{r,1}(f), \ldots, M_{r,n_1-1}(f), \frac{1}{2} \left\| q^2 \left(\alpha e_0 - \left([2]_q + \beta \right) e_1 \right) f^{'} \right) + L_q(f) \right\|_r \right\}.$$

□

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Author’s address

M. Kara
Eastern Mediterranean University, Gazimagusa,TRNC, Mersin 10, Turkey
E-mail address: mustafa.kara@emu.edu.tr